Equivalence between the spectral and the finite elements matrices

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Abstract

In this paper, we prove the spectral equivalence of the mass matrix for a Legendre–Gauss–Lobatto method with the mass matrix of finite elements method; we also prove analogous results on rigidity matrices. For this purpose, we establish some asymptotic formulae for Legendre polynomials and for the roots of their derivatives.

Keywords: Preconditioner; Spectral method; Finite elements method; Mass matrix; Rigidity matrix; Legendre polynomials

1. Introduction

A well-known preconditioner for spectral methods uses finite differences or low degree finite elements on the nodes of the spectral method. This is a very efficient process, which has been validated numerically by Orszag [1], Deville and Mund [2], Canuto and Quarteroni [3] and others … but for which the theory was lacking.

In this paper, we first show the spectral equivalence of the rigidity matrix K_S for a spectral Legendre–Gauss–Lobatto method on the interval $[-1,1]$ with Dirichlet conditions with the rigidity matrix K_F of P_1 finite elements method. In a second part, we prove the analogous equivalence between the mass matrix M_S for the spectral method and the mass matrix M_F for the finite elements method. We finally establish that the norm of $M_F^{-1}M_S$ considered as an operator from the discrete Sobolev space $\mathbb{H}_N^1$ to itself is bounded from below and from above independently on N. All these results can be generalised to a square with Dirichlet conditions with the tensored matrices $1\otimes K+K\otimes 1$ and $1\otimes M+M\otimes 1$.

2. Definitions and expressions of the mass and rigidity matrices

Denote by L_N the Legendre polynomial of degree N and by $-1=\xi_0<\xi_1<\cdots<\xi_N=1$ the roots of $(1-X^2)L'_N$. Proposition I.4.5 of Bernardi and Maday [4] gives us the existence of non-negative numbers ρ_k, $0\le k\le N$ such that for all polynomials Φ of degree at most equal to $2N-1$,

$$\int_{-1}^{1}\Phi(x)\,\mathrm{d}x=\sum_{k=0}^{N}\Phi(\xi_k)\rho_k.$$

Bernardi and Maday [4] establish an exact formula for ρ_k which is

$$\rho_k=\frac{2}{N(N+1)L_N^2(\xi_k)},\quad 0\le k\le N. \tag{1}$$

The collocation method on the knots ξ_k is defined by the data of a basis, the Lagrange basis built on those knots and denoted by l_k for $0\le k\le N$, and the data of an inner product defined by

$$(u,v)_N=\sum_{k=0}^{N}u(\xi_k)v(\xi_k)\rho_k.$$

Thus, the coefficients of the mass matrix M_S for the spectral method are given by

$$(M_S)_{i,j}=(l_i,l_j)_N=\delta_{i,j}\rho_j,\quad 1\le i,j\le N-1$$

and those of the rigidity matrix K_S by

$$(K_S)_{i,j}=(l'_i,l'_j)_N=\sum_{k=0}^{N}\rho_k l'_i(\xi_k)l'_j(\xi_k).$$

Let us now define the matrices for the finite elements method. We denote by Φ_k, $1\le k\le N-1$ the hat functions centred on the knots ξ_k which span the space of P_1 finite elements. The coefficients of the mass matrix M_F are obtained by mass lumping, i.e. by approaching the integral of $\Phi_i\Phi_j$ for $1\le i,j\le N-1$ using the trapezoid rule. Thus, M_F is diagonal and

$$(M_F)_{i,i}=\frac{\xi_{i+1}-\xi_{i-1}}{2},\quad 1\le i\le N-1.$$

The rigidity matrix K_F is tridiagonal and its coefficient $(K_F)_{i,j}$ is the integral of $\Phi'_i\Phi'_j$.

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3. Equivalence between the rigidity matrices

To prove the spectral equivalence between the rigidity matrices, we use some operators defined as follows: α_N^S maps a vector r of $\mathbb{R}^{N-1}$ to the polynomial which interpolates the values r_k at the knots ξ_k, $1 \le k \le N-1$; conversely, β_N collocates, i.e. if p is continuous on $[-1,1]$ with $p(-1) = p(1) = 0$, $\beta_N p$ is the vector of values $p(\xi_k)$, $1 \le k \le N-1$; α_N^F is the operator analogous to α_N^S for the finite elements method, i.e. $\alpha_N^F r$ is the piecewise affine interpolation of r_k at ξ_k.

To prove the spectral equivalence of K_F with K_S is equivalent to prove the equivalence in H_0^1-norm of the operators $\alpha_N^F \beta_N$ and $\alpha_N^S \beta_N$. This is first shown for functions belonging to $H_0^1 \cap H^3$ and the final result is obtained by density of $H_0^1 \cap H^3$ in H_0^1.

We suppose that the converse is true and we construct a sequence u_N of $H_0^1 \cap H^3$ such that

$$u_N \longrightarrow u_\infty \text{ strongly in } H^2, \quad |\alpha_N^S \beta_N u_N|_1 = 1 \quad \text{and}$$
$$|\alpha_N^F \beta_N u_N|_1 \longrightarrow 0 \quad \text{as } N \longrightarrow \infty.$$

We use Theorem III.1.19 of Bernardi and Maday [4] which shows that $\alpha_N^S \beta_N$ is a continuous operator independently on N and that the difference $\alpha_N^S \beta_N u - u$ for $u \in H^2$ is bounded by $1/N$; we establish the analogous result on $\alpha_N^F \beta_N$ and thus both sequences $\alpha_N^S \beta_N u_N$ and $\alpha_N^F \beta_N u_N$ converge to u_∞. Therefore, we infer that $u_\infty = 0$ and $|u_\infty|_1 = 1$.

This proves that the rigidity matrices K_S and K_F are spectrally equivalent.

4. Equivalence between the mass matrices

We first prove that the mass matrices M_S and M_F are spectrally equivalent. Since they are both diagonal, it is sufficient to show that the coefficients

$$\sigma_k = \frac{2\rho_k}{\xi_{k+1} - \xi_{k-1}}, \quad 1 \le k \le N-1 \tag{2}$$

of the diagonal of $M_F^{-1} M_S$ are bounded from above and below independently on N. For this purpose, we use estimates on ρ_k with respect to ξ_k of Bernardi and Maday [4] and estimates on $\eta_k = \arccos \xi_k$ from Theorems 6.21.2 and 6.21.3 of Szegö [5].

The last but not least thing to prove is that $M_F^{-1} M_S$ is bounded from below and from above independently on N as an operator from the discrete Sobolev space $\mathbb{H}_1^N$, composed of the vectors U of $\mathbb{R}^{N-1}$ such that $\alpha_N^F U \in H_0^1$, to itself. We define the norm over $\mathbb{H}_1^N$ as $||U||_{\mathbb{H}_N^1} = |\alpha_N^F U|_1$ and setting $U_0 = U_N = 0$, it is equal to:

$$\|U\|_{\mathbb{H}_N^1} = \left(\sum_{k=0}^{N-1} \frac{|U_{k+1} - U_k|^2}{\xi_{k+1} - \xi_k} \right)^{1/2}.$$

Let us set

$$N' = \left\lfloor \frac{N-1}{2} \right\rfloor \quad \text{and}$$

$$\mu_k = \frac{2 - |\xi_k| - |\xi_{k+1}|}{\xi_{k+1} - \xi_k} \left(\sqrt{\sigma_{k+1}} - \sqrt{\sigma_k} \right)^2.$$

We prove that a vector of $\mathbb{H}_1^N$ is discrete Hölder continuous and using the previous result on σ_k, it is thus sufficient to show that $\sum_{k=0}^{N'} \mu_k$ is bounded independently on N. We recall that not only the sum index depends on N but also the elements ξ_k and σ_k.

For this purpose, we study the expression μ_k for three different areas: k bounded, i.e. $k \in [0, K]$; k belonging to $[\lambda N, N/2]$ with $0 < \lambda < 1/2$ and finally k belonging to $[K, \lambda N]$. The global method is the same for these three areas: we have or establish an asymptotic formula for $L'_N(\cos\theta)$ which gives an asymptotic formula for $\eta_k = \arccos(\xi_k)$. Using or proving an asymptotic formula for $L_N(\cos\theta)$, we calculate $L_N(\xi_k)$ and thanks to Eq. (1) we obtain an asymptotic formula for ρ_k. We deduce a formula for σ_k from Eq. (2) and therefore a formula for μ_k. What differs in the three areas is the way we obtain the formulae for L'_N and L_N.

For $k \in [\lambda N, N/2]$, we use asymptotic formulae for $L'_N(\cos\theta)$ and $L_N(\cos\theta)$ given by Theorems 8.21.10 and 8.21.4 of Szegö [5]. Their remainders are uniform for $\theta \in [\varepsilon, \pi/2]$ for all $\varepsilon > 0$ and since η_k is equivalent to $(\pi/4 + k\pi)/(N + 3/2)$, we can only use those formulae for $k \in [\lambda N, N/2]$. We deduce the asymptotic formula for ξ_k from the formula of $L'_N(\cos\theta)$ using a convenient version of the implicit functions theorem given by Lemma 7.1. of De Mottoni and Schatzman [6].

For $k \in [0, K]$, we use Theorems 8.1.1 and 8.1.2 of Szegö [5] which say that uniformly on a bounded interval, $L_N(\cos(z/N))$ is equivalent to $J_0(z)$ and $L'_N(\cos(z/N))$ to $2N J_1(z)/z$ where J_0 and J_1 are Jacobi functions. Therefore, η_k is equivalent to z_k/N where z_k is the kth positive zero of J_1 and the implicit functions theorem of [6] gives us an asymptotic formula for z_k.

At last, for $k \in [K, \lambda N]$, we use integral representation of $L_N(\cos\theta)$ and $L'_N(\cos\theta)$ given by formula 4.10.3 of Szegö [5]. To calculate an asymptotic for $L_N(\cos\theta)$ and $L'_N(\cos\theta)$, we generalise the method of stationary phase explained in chapter 7.7 of Hörmander [7] and the convenient variable for the asymptotic is not $1/N$ but $N \sin(z/N)^{-1}$. The same implicit functions theorem of [6] enables us to find a formula for ξ_k and therefore for μ_k to complete the proof.

5. Conclusions

On the one hand, our proof validates practical knowledge which has been available and widely used since 1980.

It is interesting to notice that formulae and estimates which are available in the literature of orthogonal polynomials, such as Szegö [5], can be extended for the needs of numerical analysis. On the other hand, the results of this article are an essential step for the analysis of preconditioned Runge–Kutta methods used for the integration of diffusion equations. Besides, in a paper in preparation, we have already proved some results of unconditional stability, convergence and order for the Richardson extrapolation of the residual smoothing scheme. This is a scheme introduced by Averbuch et al. [8] to integrate numerically parabolic equations with a self-adjoint operator A and a preconditioner B of A; the scheme is time-explicit in A and time-implicit in B, and one of our examples is to obtain A from a spectral Gauss–Lobatto–Legendre method and B from the finite elements method. Therefore, this article enables us to estimate the consistency error of the residual smoothing scheme in that case and to conclude the complete study of the properties of that scheme.

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