A minimal surface with one limit end and unbounded curvature

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Abstract : we construct an embedded minimal surface in euclidean space which has infinitely many catenoidal ends, one limit end, and unbounded Gaussian curvature.

1 Introduction

The main goal of this paper is to prove the following theorem.

Theorem 1 *There exists a complete, properly embedded minimal surface in euclidean space $\mathbb{R}^3$ which has unbounded Gaussian curvature. It has infinite genus, infinitely many catenoid type ends, and one limit end.*

In my knowledge, it is the first example with unbounded Gaussian curvature, and also the first one with one limit end.

Let me first argue why all examples of complete embedded minimal surfaces known so far have bounded Gaussian curvature. The vast majority of examples either have finite total curvature, or are periodic and have finite total curvature in the quotient by the period. The surface, or its quotient by the period in the periodic case, is then diffeomorphic to a compact surface punctured at a finite number of points which correspond to the ends. Moreover, the ends are asymptotically flat, so the Gaussian curvature tends to zero at the punctures so is a bounded function on the surface.

A few examples do not have finite total curvature in any quotient, like the genus one helicoid [5] and the Riemann examples with handles constructed in [4]. These two examples have finite genus. By a recent result of Meeks, Perez and Ros [10], a properly embedded minimal surface with finite genus has bounded curvature. We also know a few examples which have infinite genus : the Saddle Towers with infinitely many ends [7] and the quasi-periodic examples constructed in [8]. Both are proven to have bounded Gaussian curvature in these papers.

Next we recall the definition of a limit end from [1]. For any connected manifold M, the set of ends $\mathcal{E}$ has a natural topology that makes $\mathcal{E}$ into a compact space. The limit points in $\mathcal{E}$ are by definition the limit ends of M. For example, the Riemann minimal examples have infinitely many ends and two limit ends.

Collin, Kusner, Meeks and Rosenberg [1] have proven that a properly embedded minimal surface with infinitely many ends has at most two limit ends. Meeks, Perez and Ros [10] have proven that a properly embedded minimal surface with infinitely many ends and finite genus must have precisely two limit ends. Therefore, an example with one limit end must have infinite genus. No example with one limit end was known, so it seems interesting to construct an example to illustrate the theory.

Let me point out that the existence of such an example is not completely unexpected. Indeed, at least heuristically, one can imagine how to construct one by inductively desingularizing a family of suitable catenoids. However, we don't have a general enough desingularization theorem at our disposal yet, and there are fantastic technicalities in trying to carry out such a construction. So the purpose of this paper is to contruct an example using another idea, in a somewhat more economical way.

Another remark is that if we relax the embeddedness condition, then there are plenty of known complete, immersed minimal surfaces with unbounded Gaussian curvature. For example, the example of Nadirashvili [11] of a complete minimal immersion in a ball certainly has unbounded curvature. Embeddedness is a strong constraint on the geometry of minimal surfaces.

Heuristically, our example is constructed inductively as follows. Start with the catenoid and stack a plane on top of it. Glue a finite number of catenoidal necks in between. After this first step one gets a Costa Hoffman Meeks surface with three ends. Then iterate this process infinitely many times, increasing the number of ends by one at each step. What we need to carry on this construction is a theorem which, from a minimal surface with n ends, produces a minimal surface with one more end. This theorem is the main result of this paper and is stated in the next section.

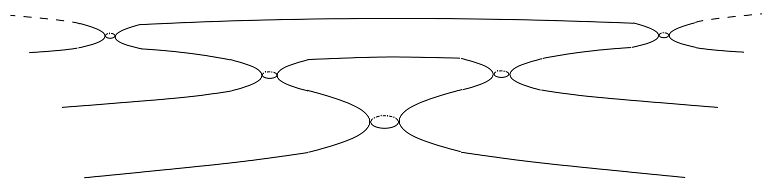

Figure 1: A sketch of the surface we get after two steps. Only two catenoidal necks have been represented at each level for clarity.

2 Main result

Given an embedded minimal surface M of finite total curvature in $\mathbb{R}^3$ with n catenoidal ends, we denote $c_1(M), c_2(M), \cdots, c_n(M)$ the logarithmic growths of its ends, ordered from bottom to top.

Theorem 2 *Let M be an embedded minimal surface in $\mathbb{R}^3$ with finite total curvature, genus G and n horizontal catenoidal ends with logarithmic growths satisfying*

$$c_1(M) < c_2(M) \cdots < c_{n-1}(M) < 0 < c_n(M).$$

Assume that M has a vertical plane of symmetry and non-degenerate Weierstrass Representation.

Consider an integer $m \geq 2$ such that

$$m - 1 > \frac{c_n(M)}{|c_{n-1}(M)|}. \tag{1}$$

There exists a one parameter family of embedded minimal surfaces $(M_t)_{0<t<\varepsilon}$ which has the following properties:

1. *M_t has finite total curvature, genus $G + m - 1$ and $n + 1$ catenoidal ends, whose logarithmic growths satisfy*

$$c_1(M_t) < c_2(M_t) \cdots < c_n(M_t) < 0 < c_{n+1}(M_t).$$

2. *$(M_t)_{0<t<\varepsilon}$ converges smoothly to M on compact subsets of $\mathbb{R}^3$ when $t \to 0$.*
3. *M_t has a vertical plane of symmetry and has non-degenerate Weierstrass Representation.*
4. *The limit of the logarithmic growth of the ends of M_t are*

$$\lim_{t\to 0} c_k(M_t) = \begin{cases} c_k(M) \text{ if } 1 \leq k \leq n-1 \\ \frac{-1}{m-1} c_n(M) \text{ if } k = n \\ \frac{m}{m-1} c_n(M) \text{ if } k = n+1 \end{cases} \tag{2}$$

5. *The maximum of the absolute value of the Gaussian curvature on M_t is greater than $\dfrac{(m-1)^2}{2c_n(M)^2}$.*

This theorem will be proven in section 5. The definition of "non-degenerate Weierstrass Representation" will be given in section 4. In particular, the catenoid has non-degenerate Weierstrass Representation.

Heuristically, M_t is constructed by stacking a horizontal plane on top of M and gluing m catenoidal neck placed on a circle in between. When $t \to 0$, the catenoids drift off to infinity, which explains the second point of the theorem. The necks have waist radius approximately $\frac{c_n(M)}{m-1}$, which explains the last point of the theorem. Observe that condition (1) and point 4 ensures that $c_{n-1}(M_t) < c_n(M_t)$ as required.

As far as technique goes, the proof of theorem 2 follows the lines of [15], using Weierstrass representation and opening nodes. The new feature is that we allow the minimal surface M as a building block in this construction, whereas in the former construction we only used catenoids. To do this, we need to develop the notion that M has non-degenerate Weierstrass Representation, which we will do in section 4.

3 Proof of theorem 1

In this section, we prove theorem 1 as a consequence of theorem 2. We construct inductively a sequence of minimal surfaces $(S_n)_{n\geq 2}$, an increasing sequence of balls $(B_n)_{n\geq 2}$ and a sequence of positive numbers $(C_n)_{n\geq 2}$ with the following properties:

1. Each S_n is an embedded minimal surface of finite total curvature with n catenoidal ends satisfying
$$c_1(S_n) < c_2(S_n) < \cdots < c_{n-1}(S_n) < 0 < c_n(S_n)$$
and with a vertical plane of symmetry and non-degenerate Weierstrass Representation.
2. For all $\ell \geq k \geq 2$, one has
$$k-2 < \sup_{S_\ell \cap B_k} |K| < C_k \qquad \text{and} \qquad \text{Area}(S_\ell \cap B_k) < C_k \tag{3}$$
where K denotes the Gaussian curvature.

The process is initiated with S_2 equal to the standard catenoid, $B_2 = B(0,2)$ and C_2 a suitable constant. Take $n \geq 2$ and assume that S_ℓ, B_ℓ and C_ℓ have been constructed for all $\ell \leq n$, so that (3) is satisfied for all $2 \leq k \leq \ell \leq n$. We apply theorem 2 with $M = S_n$ and $m = m_n$ chosen large enough so that condition (1) is satisfied and $\frac{(m-1)^2}{2c_n(S_n)^2} > n-1$. The output of the theorem is a family of minimal surfaces $(M_t)_{0<t<\varepsilon}$ which converges to S_n on each B_k for $k \leq n$. Hence we can choose t small enough so that $S_{n+1} = M_t$ satisfies (3) for all $k \leq n$. By the last item of theorem 2, there are points on S_{n+1} where $|K| > n-1$. We take a ball B_{n+1} large enough to contain one such point, and containing B_n. Then we can choose a constant C_{n+1} so that S_{n+1} satisfies (3) for $k = n+1$ and we are done.

For each $k \geq 2$, the sequence $(S_n \cap B_k)_{n\geq k}$ has uniform curvature and area estimate, so has a subsequence which converges smoothly by a standard compactness result (theorem 4.2.1 in [13]). By a diagonal process, the sequence $(S_n)_{n\geq 2}$ has a subsequence which converges smoothly on each B_k, to a complete embedded minimal surface S_∞. Now for all $k \geq 2$, $\sup_{S_\infty \cap B_k} |K| \geq k-2$, so S_∞ has unbounded Gaussian curvature and the theorem is proven. □

Remark 1 *All the catenoidal ends of S_∞ have negative logarithmic growth.*

In the above argument, we have chosen the sequence $(m_n)_{n\geq 2}$ so that the limit surface S_∞ has unbounded Gaussian curvature, but can we choose it so that S_∞ has bounded Gaussian curvature ?

The sequence $(m_n)_{n\geq 2}$ must be chosen so that condition (1) is satisfied at each step. Using formula (2), we have

$$c_n(S_{n+1}) \simeq \frac{-1}{m_n - 1} c_n(S_n)$$

$$c_{n+1}(S_{n+1}) \simeq \frac{m_n}{m_n - 1} c_n(S_n)$$

where $\simeq$ means that it can be chosen arbitrarily close by taking t small enough. So condition (1) reads as $m_{n+1} - 1 > m_n$. Take an arbitrary sequence $(m_n)_{n\geq 2}$ satisfying

$$m_2 \geq 3 \quad \text{and} \quad \forall n \geq 2, \quad m_{n+1} \geq m_n + 2 \tag{4}$$

By the above process, we obtain a sequence of minimal surface $(S_n)_{n\geq 2}$ which converges to an embedded minimal surface S_∞ with infinitely many catenoidal ends. By induction, we have $m_n \geq 2n - 1$ and

$$c_n(S_n) \simeq \prod_{i=2}^{n-1} \frac{m_i}{m_i - 1} \leq \prod_{i=2}^{n-1} \left(1 + \frac{1}{2i-2}\right) = O(\sqrt{n}).$$

Hence, $\lim \frac{c_n(S_n)}{m_n - 1} = 0$. By the last item of theorem 2, this means that whatever the choice of the sequence $(m_n)_{n\geq 2}$ satisying (4), the minimal surface S_∞ will have unbounded Gaussian curvature.

Also, we have

$$\forall n \geq 2, \quad c_n(S_\infty) \simeq \frac{-1}{m_n - 1} \prod_{i=2}^{n-1} \frac{m_i}{m_i - 1}$$

so depending on the choice of the sequence $(m_n)_{n\geq 2}$, the series $\sum c_n(S_\infty)$ can be convergent or divergent.

4 Non-degenerate Weierstrass Representation

Let M be an embedded minimal surface in $\mathbb{R}^3$ with genus G and n horizontal catenoidal ends. Let (Σ, g, ϕ_3) be its Weierstrass Representation. Here Σ is a compact Riemann surface, the Gauss map $g : \Sigma \to \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a meromorphic function and the height differential ϕ_3 is a meromorphic 1-form on Σ with n simple poles which we call $q_1, \cdots, q_n$. These points correspond to the ends of M and are called the punctures. The degree of the Gauss map is $d = G + n - 1$. Define

$$\phi_1 = \frac{1}{2}(g^{-1} - g)\phi_3, \quad \phi_2 = \frac{i}{2}(g^{-1} + g)\phi_3.$$

Our minimal surface M is parametrized on $\Sigma \setminus \{q_1, \cdots, q_n\}$ by

$$z \mapsto \operatorname{Re} \int_{z_0}^{z} (\phi_1, \phi_2, \phi_3). \tag{5}$$

We assume that M has a vertical plane of symmetry. Without loss of generality we assume that M is symmetric with respect to the plane $x_2 = 0$. On Σ, this symmetry corresponds to a antiholomorphic involution σ such that $g \circ \sigma = \overline{g}$ and $\sigma^* \phi_3 = \overline{\phi_3}$. Moreover, σ fixes the punctures $q_1, \cdots, q_n$.

Definition 1 *We say that the triple (Σ, g, ϕ_3) is σ-symmetric if there exists a antiholomorphic involution $\sigma : \Sigma \to \Sigma$ satisfying $g \circ \sigma = \overline{g}$ and $\sigma^* \phi_3 = \overline{\phi_3}$.*

Let us pretend we would like to deform M, keeping the vertical plane of symmetry. In the following sections, we count how many parameters are available for σ-symmetric deformation of the Weierstrass data and how many equations need to be solved. "Non-degenerate Weierstrass Representation" means that the jacobian matrix of equations with respect to parameters has maximal rank.

4.1 Deformation of Weierstrass data

4.1.1 The Gauss map

We assume that we are given a holomorphic deformation $(\Sigma_a, g_a)_{a \in \mathcal{A}}$ of (Σ, g). Here, the parameter space $\mathcal{A}$ is an open set in $\mathbb{C}^N$ and the number N of parameters should be at least $4G + n - 3$.

There are several ways to define the meaning of the sentence : (Σ_a, g_a) depends holomorphically on $a \in \mathcal{A}$. We adopt the following algebraic geometric point of view. We assume that we are given a complex manifold $\mathcal{M}$ of dimension $N + 1$, a proper holomorphic submersion $\pi : \mathcal{M} \to \mathcal{A}$ and a meromorphic function $\mathcal{G} : \mathcal{M} \to \overline{\mathbb{C}}$. We define $\Sigma_a = \pi^{-1}(a)$, this is a compact Riemann surface. We define g_a to be the restriction of $\mathcal{G}$ to Σ_a. We assume that the polar divisor $\mathcal{G}^{-1}(\infty)$ is transverse to the fibers of π so that g_a is a meromorphic function on Σ_a. We also assume that there exists $\underline{a} \in \mathcal{A}$ such that $(\Sigma_{\underline{a}}, g_{\underline{a}})$ is isomorphic to (Σ, g), and we identify them.

Recall that our initial Riemann surface $\Sigma = \Sigma_{\underline{a}}$ admits a anti-holomorphic involution σ such that $g\circ\sigma = \overline{g}$. We assume that it extends to a anti-holomorphic involution $\sigma_{\mathcal{M}}$ of $\mathcal{M}$, and that there exists a anti-holomorphic involution $\sigma_{\mathcal{A}}$ of $\mathcal{A}$ such that $\pi \circ \sigma_{\mathcal{M}} = \sigma_{\mathcal{A}} \circ \pi$ and $\mathcal{G} \circ \sigma_{\mathcal{M}} = \overline{\mathcal{G}}$. We define $\mathcal{A}_\sigma = \{a \in \mathcal{A} \,|\, \sigma_{\mathcal{A}}(a) = a\}$. This is a real analytic manifold of dimension N. For $a \in \mathcal{A}_\sigma$, we call $\sigma_a : \Sigma_a \to \Sigma_a$ (or more simply σ) the restriction of $\sigma_{\mathcal{M}}$. This is a antiholomorphic involution of Σ_a such that $g_a \circ \sigma_a = \overline{g_a}$.

All this structure is what we call a σ-symmetric deformation of (Σ, g). We write $(\Sigma_a, g_a)_{a\in\mathcal{A}_\sigma}$ for short.

Remark 2 *In this construction, we do not need to parametrize all deformations of (Σ, g). We only need to have enough parameters. Still, to define "non-degenerate Weierstrass Representation", it would be natural to assume that the family $(\Sigma_a, g_a)_{a\in A}$ parametrizes all deformations of (Σ, g), up to isomorphism of branched coverings.*

Recall that a branched covering $g : \Sigma \to \overline{\mathbb{C}}$ of degree d is simple if each fiber contains at least $d-1$ points (so each fiber contains at most one branch point, and it has branching order 1). The moduli space of simple branched coverings of genus G and degree d is called a Hurwitz space. It has a structure of analytic complex manifold of dimension $2G+2d-2$. Moreover, the branching values of g can be used as local coordinates on this space [3]. This means that if (Σ, g) is simple, then we can parametrize its deformations using $N = 2G + 2d - 2$ complex parameters.

Now if our covering has a branch point of higher branching order, when deforming it, this branch point may split into several smaller order branch points. It is not true anymore that the list of branching values provide local coordinates, contrarily to what is claimed in [9]. Consider for example the following two deformations of the covering $z \mapsto z^4$ of the Riemann sphere :

$$f_t(z) = z^4 + 4tz^3,$$

$$g_t(z) = z^4 + 4t\alpha z^3 + 4t^2\alpha^2 z^2, \qquad \text{with } \alpha^4 = -27.$$

By explicit computations, they have the same branching values close to 0, namely 0, 0 and $-3t$. They are not isomorphic because f_t has a branch point of branching order 2 at the origin and a simple branch point (at $-3t$), whereas g_t has three simple branch points (at 0, $-t\alpha$ and $-2t\alpha$).

Still, it is possible to parametrize locally all deformations of a branched covering with a higher order branch point by $2G+2d-2$ complex numbers. I could not find a reference for this fact. In a previous version of this paper, I explained how to do this, but that made the paper much longer, and also was not needed for our construction.

4.1.2 The height differential

We need to write all candidates for the height differential ϕ_3 on Σ_a. The most natural way to define such a meromorphic 1-form is to prescribe its poles, residues and periods. Recall that on our initial Riemann surface Σ, the height differential has n simple poles at $q_1, \cdots, q_n$, and the Gauss map g has, alternately, a simple zero or a simple pole at these points. For a close enough to $\underline{a}$, there exists $q_1(a), \cdots, q_n(a)$ in Σ_a, depending holomorphically on a, such that g_a has either a simple zero or a simple pole at each $q_i(a)$. We need that the height differential has simples poles at these points.

Consider a canonical homology basis $A_1, \cdots, A_G, B_1, \cdots, B_G$ of Σ. We may extend it to a canonical basis of Σ_a depending continuously on a. Given complex parameters $c = (c_1, \cdots, c_{n-1}) \in \mathbb{C}^{n-1}$ and $\alpha = (\alpha_1, \cdots, \alpha_G) \in \mathbb{C}^G$ we define ϕ_3 as the unique meromorphic 1-form on Σ_a with simple poles at $q_1(a), \cdots, q_n(a)$ and the following residues and A-periods :

$$\mathrm{Res}_{q_i(a)}\phi_3 = -c_i \qquad 1 \leq i \leq n-1,$$

$$\int_{A_i} \phi_3 = 2\pi i \alpha_i \qquad 1 \leq i \leq G.$$

Geometrically, the number c_i represents the logarithmic growth of the end q_i, thanks to the minus sign. The residue at $q_n(a)$ is $c_1 + \cdots + c_{n-1}$ by the Residue theorem.

It is a standard fact that ϕ_3 depends holomorphically on the parameter a, in the following sense : Since $\pi : \mathcal{M} \to \mathcal{A}$ is a submersion, there exists locally a holomorphic function $z : \mathcal{M} \to \mathbb{C}$ such that $(z, \pi) : \mathcal{M} \to \mathbb{C} \times \mathcal{A}$ is a local diffeomorphism. Then the restriction of z to Σ_a is a local coordinate on Σ_a, and in term of this coordinate, we may write $\phi_3 = h(z, a)dz$ where h is a holomorphic function.

Next assume that $a \in \mathcal{A}_\sigma$ so Σ_a admits a antiholomorphic involution σ. We would like ϕ_3 to satisfy $\sigma^*\phi_3 = \overline{\phi_3}$. The involution σ fixes the points $q_1(a), \cdots, q_n(a)$.

Definition 2 *Let Σ be a Riemann surface with a antiholomorphic involution σ. We say that a canonical homology basis $\{A_1, \cdots, A_G, B_1, \cdots, B_G\}$ is σ-symmetric if there exists an involution of $\{1, \cdots, G\}$ (also denoted σ) such that for all $1 \leq i \leq G$, $\sigma(A_i) \sim -A_{\sigma(i)}$ and $\sigma(B_i) \sim B_{\sigma(i)}$.*

We assume that we are given a σ-symmetric canonical homology basis. The condition $\sigma^*\phi_3 = \overline{\phi_3}$ is then equivalent to $\alpha_{\sigma(i)} = \overline{\alpha_i}$, for $1 \leq i \leq G$, and $c_i \in \mathbb{R}$ for $1 \leq i \leq n-1$.

At this point we have defined a family of σ-symmetric triples (Σ_a, g_a, ϕ_3) depending on the parameter $X = (a, c, \alpha)$. We write $\underline{X} = (\underline{a}, \underline{c}, \underline{\alpha})$ for the value of X which gives the Weierstrass data of the minimal surface M we were given. We call $\underline{X}$ the central value of the parameter X. We will need the parameter X to be close enough to $\underline{X}$, and we will restrict the parameter space $\mathcal{A}$ accordingly.

4.2 The equations

In this section, we do not write explicitely the dependance of objects on parameters, so we write (Σ, g) for (Σ_a, g_a), being understood that everything depends on parameters. We write $(\underline{\Sigma}, \underline{g}, \underline{\phi_3})$ for the Weierstrass data of the minimal surface we are given (namely, at the central value of the parameters).

In order to be define an immersed minimal surface, the triple (Σ, g, ϕ_3) must satisfy the following conditions :

1. At any zero of the height differential ϕ_3, the Gauss map g needs a zero or a pole, with the same multiplicity.
2. At each puncture $q_1, \cdots, q_n$, the residues of ϕ_1, ϕ_2 and ϕ_3 must be real.
3. For all $\nu = 1, 2, 3$ and $1 \leq i \leq G$, we need $\mathrm{Re} \int_{A_i} \phi_\nu = \mathrm{Re} \int_{B_i} \phi_\nu = 0$.

Point 2 and 3 guarantee that (5) is well defined, and point 1 that it is an immersion. Let us count how many equations we have to solve, taking into account the symmetry.

1. Regarding point 1, the height differential ϕ_3 has $2G+n-2$ zeros, counting multiplicity. Let us first assume that the zeros of $\underline{\phi_3}$ are simple. Then for X close enough to $\underline{X}$, ϕ_3 still has simple zeros, and we may label them $\zeta_1(X), \cdots, \zeta_{2G+n-2}(X)$ so that they depend continuously on X. If $\underline{g}$ has a zero (resp. a pole) at $\zeta_i(\underline{X})$, we need to solve the equation $g(\zeta_i(X)) = 0$ (resp. $g(\zeta_i(X))^{-1} = 0$). By symmetry, the map $(g(\zeta_i(X))^{\pm 1})_{1 \leq i \leq 2G+n-2}$ takes values into a real space of dimension $2G + n - 2$.
2. Regarding point 2, the residues of ϕ_1 and ϕ_3 are already real by symmetry. The residue of ϕ_2 at q_j is $\frac{i}{2} \mathrm{Res}\, g^{-1}\phi_3$ if $g(q_j) = 0$, and $\frac{i}{2} \mathrm{Res}\, g\phi_3$ if $g(q_j) = \infty$. Provided point 1 is satisfied, the only poles of $g\phi_3$ and $g^{-1}\phi_3$ are at the punctures. Applying the Residue Theorem to $g\phi_3$ and $g^{-1}\phi_3$, it suffices to solve the equation $\mathrm{Im}\,\mathrm{Res}_{q_j} \phi_2 = 0$ for $1 \leq j \leq n-2$. Therefore, point 2 counts as $n - 2$ real equations.
3. Regarding point 3, we have by symmetry $\sigma^* \phi_\nu = (-1)^{\nu+1} \overline{\phi_\nu}$ for $\nu = 1, 2, 3$. From this we obtain, for $1 \leq i \leq G$, provided point 2 is satisfied so that all residues are real,

$$\mathrm{Re} \int_{A_{\sigma(i)}} \phi_\nu = (-1)^\nu \mathrm{Re} \int_{A_i} \phi_\nu$$

$$\mathrm{Re} \int_{B_{\sigma(i)}} \phi_\nu = (-1)^{\nu+1} \mathrm{Re} \int_{B_i} \phi_\nu.$$

   So the map $(\mathrm{Re} \int_{A_i} \phi_\nu, \mathrm{Re} \int_{B_i} \phi_\nu)_{1 \leq i \leq G, 1 \leq \nu \leq 3}$ takes values into a real space of dimension $3G$.

4. In case $\underline{\phi_3}$ has a zero of multiplicity $k \geq 2$ at some point ζ, the problem is that this zero may split into several zeros of smaller multiplicity when we deform the Weierstrass data. Let us assume for example that $\underline{g}$ has a zero (of multiplicity k) at ζ. We introduce a local coordinate z on Σ_a as in section 4.1.2. In term of this coordinate, we may write $g_a = f(z,a)$ where f is a holomorphic function (specifically, $f = \mathcal{G} \circ (z,\pi)^{-1}$). By the Weierstrass Preparation Theorem, we can write locally

$$g_a = f_1(z,a)(z^k + \sum_{i=0}^{k-1} \mu_i(a) z^i)$$

where the function f_1 does not vanish and the coefficients μ_i are holomorphic functions of a. The polynomial $z^k + \sum \mu_i(a) z^i$ is called the Weierstrass Polynomial of g_a, it depends on the choice of the local coordinate z. In the same way, we can write locally

$$\phi_3 = f_2(z,X)(z^k + \sum_{i=0}^{k-1} \nu_i(X) z^i) dz$$

where f_2 does not vanish and the coefficients $\nu_i(X)$ are holomorphic functions of the parameter $X = (a, c, \alpha)$. We need to solve the k equations $\mu_i(a) = \nu_i(X)$ for $0 \leq i \leq k-1$, which ensure that g_a and ϕ_3 have locally the same zeros. These equations replace the k equations that we had in the case of simple zeros.

Let us write all these equations as $\mathcal{F}(X) = 0$. The map $\mathcal{F}$ takes values into a real space of dimension $5G + 2n - 4$.

Definition 3 *We say that M has non-degenerate Weierstrass Representation if there exists a family of σ-symmetric deformations $(g_a)_{a \in \mathcal{A}}$ of $\underline{g}$ such that the differential of $\mathcal{F}$ with respect to the parameter X at $\underline{X}$ is onto.*

For example, the catenoid has non-degenerate Weierstrass Representation, because in this case, $G = 0$ and $n = 2$, so there are no equations to solve.

Remark 3 *As was already pointed out in remark 2, one can parametrize all σ-symmetric deformations of $\underline{g}$ using $2G + 2d - 2$ real parameters. In this case, the total number of real parameters is $5G + 3n - 5$. A stronger, and more natural, definition of "non-degenerate Weierstrass Representation" is to require that the differential of $\mathcal{F}$ is onto for this choice of parameters. Equivalently, the kernel of the differential of $\mathcal{F}$ has dimension $n - 1$. By a straightforward application of the implicit function theorem, if M has non-degenerate Weierstrass Representation in this stronger sense, it can be deformed into a family of non-congruent minimal surfaces depending on $n - 1$ real parameters, the expected dimension for the moduli space of embedded minimal surfaces of finite total curvature with n catenoidal ends.*

5 Proof of theorem 2

Without loss of generality, we may assume by scaling that the logarithmic growth of the top end of M is 1, and by choice of orientation that the Gauss map has a zero at the top end. To construct the family of minimal surfaces $(M_t)_{0<t<\varepsilon}$, we write down candidates for its Weierstrass Representation, depending on enough parameters. We define the Riemann surface and the Gauss map by opening nodes. The height differential ϕ_3 is defined by prescribing periods and residues. The equations are solved using the implicit function theorem, using the fact that M has non-degenerate Weierstrass Representation.

5.1 Opening nodes

5.1.1 The Riemann surface with nodes

Since M has non-degenerate Weierstrass Representation, we are given a holomorphic family $(\Sigma_a, g_a)_{a\in\mathcal{A}}$. Let $q_n(a) \in \Sigma_a$ be the point which corresponds to the top end. We have $g_a(q_n(a)) = 0$ by our choice of orientation.

Consider two copies of the complex plane, denoted $\mathbb{C}^-$ and $\mathbb{C}^+$. We choose m distinct, non-zero points $p_1^-, \cdots, p_m^-$ in $\mathbb{C}^-$ and m distinct points $p_1^+, \cdots, p_m^+$ in $\mathbb{C}^+$. Consider the disjoint union $\Sigma_a \cup \mathbb{C}^- \cup \mathbb{C}^+$. Identify the point $q_n(a) \in \Sigma_a$ with the point 0 in $\mathbb{C}^-$. For $1 \leq i \leq m$, identify the point $p_i^- \in \mathbb{C}^-$ with the point $p_i^+ \in \mathbb{C}^+$. This defines a Riemann surfaces with $m+1$ nodes.

5.1.2 Coordinates in a neighborhood of the nodes

We define two meromorphic functions g^- on $\mathbb{C}^-$ and g^+ on $\mathbb{C}^+$ by

$$g^-(z) = \frac{-1}{z} + \sum_{i=1}^{m} \frac{\beta_i^-}{z - p_i^-}$$

$$g^+(z) = \sum_{i=1}^{m} \frac{\beta_i^+}{z - p_i^+}.$$

Here $\beta_1^-, \cdots, \beta_m^-$ and $\beta_1^+, \cdots, \beta_m^+$ are non-zero complex parameters. We write $\beta^\pm = (\beta_1^\pm, \cdots, \beta_m^\pm)$ and $p^\pm = (p_1^\pm, \cdots, p_m^\pm)$.

Since g_a has a simple zero at $q_n(a)$ and $g^\pm$ have simple poles, we can fix a small number $0 < \epsilon < 1$ such that $v_0^- := g_a$ is a diffeomorphism from a small neighborhood U_0^- of $q_n(a)$ to the disk $D(0, \epsilon)$, $v_0^+ := 1/g^-$ is a diffeomorphism from a small neighborhood U_0^+ of 0 in $\mathbb{C}^-$ to $D(0, \epsilon)$ and for each $i = 1, \cdots, m$, $v_i^\pm := 1/g^\pm$ is a diffeomorphism from a small neighborhood $U_i^\pm$ of $p_i^\pm$ in $\mathbb{C}^\pm$ to $D(0, \epsilon)$.

5.1.3 Opening nodes

Consider a complex parameter t such that $0 < |t| < \epsilon^2$. We remove the disk $|v_0^-| < \frac{t}{\epsilon}$ from U_0^- and the disk $|v_0^+| < \frac{t}{\epsilon}$ from U_0^+. We identify the point $z \in U_0^- \subset \Sigma_a$ with the point $z' \in U_0^+ \subset \mathbb{C}^-$ such that $v_0^-(z)v_0^+(z') = t$. This is equivalent to

$$g_a(z) = tg^-(z'). \tag{6}$$

For each $1 \leq i \leq m$, we remove the disks $|v_i^-| < \frac{t^2}{\epsilon}$ from U_i^- and $|v_i^+| < \frac{t^2}{\epsilon}$ from U_i^+. We identify the point $z \in U_i^- \subset \mathbb{C}^-$ with the point $z' \in U_i^+ \subset \mathbb{C}^+$ such that $v_i^-(z)v_i^+(z') = t^2$. This is equivalent to

$$tg^-(z) = \frac{1}{tg^+(z')}. \tag{7}$$

This defines a Riemann surface of genus $\widetilde{G} = G + m - 1$. We call it $\widetilde{\Sigma}_{a,b}$, where $b = (p^+, p^-, \beta^+, \beta^-, t) \in \mathbb{C}^{4m+1}$ denotes the collection of the new parameters. We compactify it by adding the points at infinity in $\mathbb{C}^-$ and $\mathbb{C}^+$.

By a slight abuse of language, we will denote by $\Sigma_a \subset \widetilde{\Sigma}_{a,b}$, $\mathbb{C}^- \subset \widetilde{\Sigma}_{a,b}$ and $\mathbb{C}^+ \subset \widetilde{\Sigma}_{a,b}$ the domains Σ_a, $\mathbb{C}^-$ and $\mathbb{C}^+$ minus the disks that were removed when opening nodes.

5.1.4 The Gauss map

The point of choosing the coordinates $v_i^\pm$ in this particular way, is that by construction the following function is well defined on $\widetilde{\Sigma}_{a,b}$: let

$$\widetilde{g}_{a,b}(z) = \begin{cases} g_a(z) & \text{if } z \in \Sigma_a \\ tg^-(z) & \text{if } z \in \mathbb{C}^- \\ \dfrac{1}{tg^+(z)} & \text{if } z \in \mathbb{C}^+ \end{cases}$$

This is a well defined meromorphic function on $\widetilde{\Sigma}_{a,b}$ by equations (6) and (7).

5.1.5 Central value of the parameters

The parameter t is in the punctured disk of radius ϵ^2. The parameters $p^\pm$ and $\beta^\pm$ are in a small neighborhood of a central value denoted $\underline{p}^\pm$ and $\underline{\beta}^\pm$ given by

$$\underline{p}_i^+ = \omega^{-i}, \quad \underline{p}_i^- = \omega^i, \qquad 1 \leq i \leq m,$$

$$\underline{\beta}_i^+ = \underline{\beta}_i^- = \frac{1}{m-1}, \qquad 1 \leq i \leq m,$$

where $\omega = e^{2\pi i/m}$ is a primitive m-th root of unity. We denote by $\mathcal{B} \subset \mathbb{C}^{4m+1}$ the domain of the parameter b. At the central value of the parameters, we have

$$g^-(z) = \frac{-1}{z} + \frac{1}{m-1}\sum_{i=1}^{m} \frac{1}{z - \omega^i} = \frac{-1}{z} + \frac{mz^{m-1}}{(m-1)(z^m - 1)}$$

$$g^+(z) = \frac{1}{m-1}\sum_{i=1}^{m}\frac{1}{z-\omega^i} = \frac{mz^{m-1}}{(m-1)(z^m-1)}.$$

Proposition 1 *The family $(\widetilde{\Sigma}_{a,b}, \widetilde{g}_{a,b})_{(a,b)\in\mathcal{A}\times\mathcal{B}}$ depends holomorphically on (a,b), in the sense explained in section 4.1.1.*

To prove this proposition, we simply have to define a complex manifold $\widetilde{\mathcal{M}}$, a holomorphic submersion $\widetilde{\pi} : \widetilde{\mathcal{M}} \to \mathcal{A}\times\mathcal{B}$ and a meromorphic function $\widetilde{\mathcal{G}} : \widetilde{\mathcal{M}} \to \overline{\mathbb{C}}$, such that $\widetilde{\Sigma}_{a,b} = \widetilde{\pi}^{-1}(a,b)$ and $\widetilde{g}_{a,b} = \widetilde{\mathcal{G}}$ restricted to $\widetilde{\Sigma}_{a,b}$. We give the details in appendix B. □

5.1.6 Symmetry

Next we restrict the parameters (a,b) so that $\widetilde{\Sigma}_{a,b}$ admits a antiholomorphic involution. We define $\delta : \mathbb{C}^m \to \mathbb{C}^m$ by

$$\delta(z_1, z_2, \cdots, z_m) = (\overline{z_{m-1}}, \overline{z_{m-2}}, \cdots, \overline{z_1}, \overline{z_m}).$$

Observe that the central values $\underline{p}^\pm$ and $\underline{\beta}^\pm$ are invariant under δ. We define $\sigma_{\mathcal{B}}$ on $\mathcal{B}$ by

$$\sigma_{\mathcal{B}}(p^+, p^-, \beta^+, \beta^-, t) = (\delta(p^+), \delta(p^-), \delta(\beta^+), \delta(\beta^-), \overline{t}).$$

Let $\mathcal{B}_\sigma = \{b \in \mathcal{B} \,|\, \sigma_{\mathcal{B}}(b) = b\}$. This is a real space of dimension $4m+1$. If $(a,b) \in \mathcal{A}_\sigma \times \mathcal{B}_\sigma$, then $g_a \circ \sigma = \overline{g_a}$ and $g^\pm(\overline{z}) = \overline{g^\pm(z)}$. Consequently, the map $\widetilde{\sigma} : \widetilde{\Sigma}_{a,b} \to \widetilde{\Sigma}_{a,b}$ given by

$$\widetilde{\sigma}(z) = \begin{cases} \sigma(z) & \text{if } z \in \Sigma_a \\ \overline{z} \text{ if } z \in \mathbb{C}^\pm \end{cases}$$

is well defined on $\widetilde{\Sigma}_{a,b}$ and satisfies $\widetilde{g}_{a,b} \circ \widetilde{\sigma} = \overline{\widetilde{g}_{a,b}}$.

5.2 The height differential

As in section 4.1.2, we define the height differential on $\widetilde{\Sigma}_{a,b}$ by prescribing periods and residues, so we need to define a canonical homology basis of $\widetilde{\Sigma}_{a,b}$. Recall that its genus is $\widetilde{G} = G + m - 1$. The cycles $A_1, \cdots, A_G, B_1, \cdots, B_G$ on Σ_a define us $2G$ cycles on $\widetilde{\Sigma}_{a,b}$ by inclusion. For $1 \leq i \leq m-1$, let A_{G+i} be the homology class of the circle $C(p_i^+, \epsilon)$ with the positive orientation. This circle is homologous to the circle $C(p_i^-, \epsilon)$ with the negative orientation. Also one has $\sigma(A_{G+i}) = -A_{G+m-i}$.

For $1 \leq i \leq \frac{m}{2}$, we define B_{G+i} as the composition of the following paths :

1. a path from the point $v_m^+ = \epsilon$ to the point $v_i^+ = -\epsilon$ in $\mathbb{C}^+$,
2. the segment from $v_i^+ = -\epsilon$ to $v_i^+ = -\frac{t^2}{\epsilon}$,
3. a path from the point $v_i^- = -\epsilon$ to the point $v_m^- = \epsilon$ in $\mathbb{C}^-$,

4. the segment from $v_m^- = \epsilon$ to $v_m^- = \frac{t^2}{\epsilon}$.

For $\frac{m}{2} < i \leq m-1$, we define B_{G+i} as $\sigma(B_{G+m-i})$. Then $A_1, \cdots, A_{\widetilde{G}}, B_1, \cdots, B_{\widetilde{G}}$ is a σ-symmetric canonical homology basis of $\widetilde{\Sigma}_{a,b}$.

In point 1 and 3, the paths may be choosen to depend continuously on parameters, and must avoid all disks around the nodes. Actually in case m is even and $i = \frac{m}{2}$, some care must be taken when choosing these paths : we use the upper half of the circle of radius 2 and connect it to the endpoints using real segments. This way, $B_{G+\frac{m}{2}} - \sigma(B_{G+\frac{m}{2}})$ is homologous to the circle of radius 2 in $\mathbb{C}^+$ minus the circle of radius 2 in $\mathbb{C}^-$, which are both null-homotopic in $\widetilde{\Sigma}_{a,b}$ (because we compactified by adding the points at infinity).

Let ∞^- and ∞^+ denote the point at infinity in $\mathbb{C}^-$ and $\mathbb{C}^+$. The punctures, corresponding to the $n+1$ catenoidal ends, are at $q_1, \cdots, q_{n-1}, \infty^-$ and ∞^+. We define the height differential $\widetilde{\phi}_3$ on $\widetilde{\Sigma}_{a,b}$ as in section 4.1 by prescribing its A-periods and its residues at all punctures but one. Actually, by the residue theorem, prescribing the residue at ∞^+ is equivalent to prescribe the period on the circle $C(p_m^+, \epsilon)$. So we define $\widetilde{\phi}_3$ on $\widetilde{\Sigma}_{a,b}$ as the unique meromorphic 1-form with simple poles at the punctures with the following residues and periods :

$$\int_{A_j} \widetilde{\phi}_3 = 2\pi i \alpha_j, \qquad 1 \leq j \leq G$$

$$\text{Res}_{q_i} \widetilde{\phi}_3 = -c_i, \qquad 1 \leq i \leq n-1,$$

$$\int_{C(p_j^+, \epsilon)} \widetilde{\phi}_3 = 2\pi i \gamma_j, \qquad 1 \leq j \leq m$$

The central values of the new parameter $\gamma = (\gamma_1, \cdots, \gamma_m)$ is given by

$$\underline{\gamma}_j = \frac{1}{m-1}, \quad 1 \leq j \leq m.$$

Regarding symmetry, we assume that the parameters $c = (c_1, \cdots, c_{n-1})$ and $\alpha = (\alpha_1, \cdots, \alpha_G)$ are as in section 4.1.2, and that the parameter γ satisfies $\delta(\gamma) = \gamma$, where δ is defined in section 5.1.6. This ensures that $\widetilde{\sigma}^* \widetilde{\phi}_3 = \overline{\widetilde{\phi}_3}$.

Next, by the Residue Theorem in $\mathbb{C}^+$ we have

$$\text{Res}_{\infty^+} \widetilde{\phi}_3 = -\sum_{j=1}^{m} \gamma_j, \tag{8}$$

and by the Residue Theorem in $\widetilde{\Sigma}_{a,b}$ we have

$$\text{Res}_{\infty^-} \widetilde{\phi}_3 = \sum_{j=1}^{n-1} c_j + \sum_{j=1}^{m} \gamma_j. \tag{9}$$

Proposition 2 *The differential $\widetilde{\phi}_3$ depends analytically on all parameters. Moreover, it extends analytically at $t = 0$, with value*

$$\widetilde{\phi}_3 = \begin{cases} \phi_3 & \text{in } \Sigma_a, \\ -\sum_{i=1}^{n-1} c_i \frac{dz}{z} - \sum_{i=1}^{m} \frac{\gamma_i}{z - p_i^-} dz & \text{in } \mathbb{C}^-, \\ \sum_{i=1}^{m} \frac{\gamma_i}{z - p_i^+} dz & \text{in } \mathbb{C}^+. \end{cases}$$

In particular, at the central value of the parameters, we have $\widetilde{\phi}_3 = -g^- dz$ in $\mathbb{C}^-$ and $\widetilde{\phi}_3 = g^+ dz$ in $\mathbb{C}^+$ when $t = 0$.

Proof : the analytic dependance on parameters is standard since we have a holomorphic family of Riemann surfaces. For the analytic extension at $t = 0$ we may fix the value of all parameters but t. It is proven in [6] that $\widetilde{\phi}_3$ extends at $t = 0$ to a regular differential on the noded Riemann surface, namely a meromorphic differential which has simple poles at the nodes. The residues at the nodes are determined by the prescribed periods. This gives the claimed formula for $\widetilde{\phi}_3$ at $t = 0$. The last point comes from the fact that $\underline{\gamma} = \underline{\beta}^- = \underline{\beta}^+$ and $\underline{c}_1 + \cdots + \underline{c}_{n-1} = -1$, since we assumed that the logarithmic growth of the top end of M is 1. □

5.3 The equations

In this section, we do not write explicitely the dependance of objects on parameters, so we write $(\widetilde{\Sigma}, \widetilde{g})$ for $(\widetilde{\Sigma}_{a,b}, \widetilde{g}_{a,b})$. Let $\widetilde{X} = (a, c, \alpha, p^+, p^-, \beta^+, \beta^-, \gamma)$ be the collection of all parameters but t. We write $\underline{\widetilde{X}}$ for the central value of the parameter $\widetilde{X}$. As in section 4.2, in order to define an immersed minimal surface, the triple $(\widetilde{\Sigma}, \widetilde{g}, \widetilde{\phi}_3)$ must satisfy the following conditions :

1a. At each zero of $\widetilde{\phi}_3$ in $\Sigma \subset \widetilde{\Sigma}$, g needs a zero or a pole, with the same multiplicity.

1b. At each zero of $\widetilde{\phi}_3$ in $\mathbb{C}^-$, g^- needs a zero with the same multiplicity.

1c. At each zero of $\widetilde{\phi}_3$ in $\mathbb{C}^+$, g^+ needs a zero with the same multiplicity.

2. At each puncture $q_1, \cdots q_{n-1}, \infty^-, \infty^+$, the residues of ϕ_1 and ϕ_2 must be real.

3. $\text{Re} \int_{A_i} \widetilde{\phi}_\nu = \text{Re} \int_{B_i} \widetilde{\phi}_\nu = 0 \quad$ for $1 \leq i \leq G + m - 1$, $\nu = 1, 2, 3$.

In the following points, we study how to write these conditions as equations which extend smoothly at $t = 0$.

1. Let us see first the equations which concern only the restriction of the Weierstrass data to the domain Σ of $\widetilde{\Sigma}$. For $(t, \widetilde{X})$ close to $(0, \underline{\widetilde{X}})$, $\widetilde{\phi}_3$ has

$2G+n-2$ zeros in Σ, which we may write as $\zeta_i(t,\widetilde{X})$ for $1 \le i \le 2G+n-2$, depending continously on $(t,\widetilde{X})$. We define

$$\widehat{\mathcal{F}}_1(t,\widetilde{X}) = \left((g(\zeta_i))^{\pm 1}\right)_{1\le i\le 2G+n-2}$$

where the exponent ± 1 is chosen as in point 1 of section 4.2. In case of multiple zeros, this definition is adapted as in point 4 of the same section. Also define

$$\widehat{\mathcal{F}}_2(t,\widetilde{X}) = (\operatorname{Im}\operatorname{Res}_{q_i}\widetilde{\phi}_2)_{1\le i\le n-2}$$

$$\widehat{\mathcal{F}}_3(t,\widetilde{X}) = \left(\operatorname{Re}\int_{A_i}\widetilde{\phi}_\nu, \operatorname{Re}\int_{B_i}\widetilde{\phi}_\nu\right)_{1\le i\le G, 1\le\nu\le 3}.$$

Let $\widehat{\mathcal{F}}(t,\widetilde{X}) = (\widehat{\mathcal{F}}_1(t,\widetilde{X}), \widehat{\mathcal{F}}_2(t,\widetilde{X}), \widehat{\mathcal{F}}_3(t,\widetilde{X}))$. When $t=0$, we have, $\widetilde{\phi}_3 = \phi_3$ in Σ by proposition 2 and $\widetilde{g} = g$ in Σ by definition, so $\widehat{\mathcal{F}}(0,\widetilde{X}) = \mathcal{F}(X)$, where the map $\mathcal{F}$ was defined in section 4.2. The non-degeneracy hypothesis will take care of this equation.

2. Next we consider the equations which concern the restriction of the Weierstrass data to $\mathbb{C}^-$ and $\mathbb{C}^+$, starting with point 1b. When $(t,\widetilde{X}) = (0,\underline{\widetilde{X}})$, $\widetilde{\phi}_3 = -g^- dz$ in $\mathbb{C}^-$ by proposition 2, so $\widetilde{\phi}_3$ has m simple zeros in $\mathbb{C}^-$. Hence, for $(t,\widetilde{X})$ close enough to $(0,\underline{\widetilde{X}})$, $\widetilde{\phi}_3$ has m simple zeros in $\mathbb{C}^-$ which we may label $\xi_1(t,\widetilde{X}),\cdots,\xi_m(t,\widetilde{X})$ so that $\xi_{m+1-i} = \overline{\xi_i}$. Let $\mathcal{Z}^-(t,\widetilde{X}) = (g^-(\xi_1),\cdots,g^-(\xi_m))$. (The letter $\mathcal{Z}$ stands for "zero"). This is an analytic function of all parameters.

3. When $(t,\widetilde{X}) = (0,\underline{\widetilde{X}})$, $\widetilde{\phi}_3 = g^+ dz$ in $\mathbb{C}^+$ has a zero of multiplicity $m-1$ at the origin. As in point 4 of section 4.2 in the case of a multiple zero, we introduce the Weierstrass polynomials of g^+ and $\widetilde{\phi}_3$ in a neighborhood of 0 in $\mathbb{C}^+$ in term of the standard z coordinate. We need that these two polynomials have the same coefficients, this gives us $m-1$ equations which we write as $\mathcal{Z}^+(t,\widetilde{X}) = 0$.

4. Regarding point 2, as explained in point 2 of section 4.2, it suffices to ensure that the residues of $\widetilde{\phi}_2$ are real at all punctures where the Gauss map has a pole, except one, and all punctures where the Gauss map has a zero, except one. Provided the equation $\widehat{\mathcal{F}}_2(t,\widetilde{X}) = 0$ is solved, it suffices to ensure that the residue of $\widetilde{\phi}_2$ at ∞^+ is real. This will be taken care of in the next point.

5. Regarding the A-periods in point 3, we define the vertical A-periods as

$$\mathcal{V}_j^A(t,\widetilde{X}) = \operatorname{Re}\int_{C(p_j^+,\epsilon)}\widetilde{\phi}_3 = -2\pi\operatorname{Im}(\gamma_j), \qquad 1\le j\le m-1,$$

and the renormalised horizontal A-periods as

$$\mathcal{H}_j^A(t,\widetilde{X}) = \frac{1}{2\pi i t}\left(\operatorname{Re}\int_{C(p_j^+,\epsilon)}\widetilde{\phi}_1 + i\operatorname{Re}\int_{C(p_j^+,\epsilon)}\widetilde{\phi}_2\right), \qquad 1\le j\le m.$$

Let $\mathcal{V}^A = (\mathcal{V}^A_1, \cdots, \mathcal{V}^A_{m-1})$ and $\mathcal{H}^A = (\mathcal{H}^A_1, \cdots, \mathcal{H}^A_m)$. By the Residue theorem in $\mathbb{C}^+$, the equation $\mathcal{H}^A(t, \widetilde{X}) = 0$ implies that the residue of $\widetilde{\phi}_2$ at ∞^+ is real as required in the previous point.

The function $\mathcal{H}^A$ extends analytically at $t = 0$ by the following computation :

$$\begin{aligned} \mathcal{H}^A_j(t, \widetilde{X}) &= \frac{1}{4\pi i t}\left(\overline{\int_{C(p_j^+,\epsilon)} \widetilde{g}^{-1}\widetilde{\phi}_3} - \int_{C(p_j^+,\epsilon)} \widetilde{g}\widetilde{\phi}_3\right) \\ &= \frac{1}{4\pi i t}\left(\overline{\int_{C(p_j^+,\epsilon)} \widetilde{g}^{-1}\widetilde{\phi}_3} + \int_{C(p_j^-,\epsilon)} \widetilde{g}\widetilde{\phi}_3\right) \\ &= \frac{1}{4\pi i}\left(\overline{\int_{C(p_j^+,\epsilon)} g^+\widetilde{\phi}_3} + \int_{C(p_j^-,\epsilon)} g^-\widetilde{\phi}_3\right) \end{aligned} \tag{10}$$

6. By lemma 1 in [15], the function $\int_{B_{G+i}} \widetilde{\phi}_3 - (\gamma_i - \gamma_m)\log t^2$ extends to an analytic function of all parameters at $t = 0$. We make the change of variable $t = \exp(\frac{-1}{\tau^2})$ where τ is a real parameter in a neighborhood of zero. We define the renormalised vertical B-periods as

$$\mathcal{V}^B_i(\tau, \widetilde{X}) = \tau^2 \mathrm{Re}\int_{B_{G+i}} \widetilde{\phi}_3, \qquad 1 \le i \le m-1.$$

   It extends as a smooth function of the parameters τ and $\widetilde{X}$ at $\tau = 0$, with value

$$\mathcal{V}^B_i(0, \widetilde{X}) = -2\mathrm{Re}(\gamma_i - \gamma_m).$$

   We define $\mathcal{V}^B = (\mathcal{V}^B_1, \cdots, \mathcal{V}^B_{m-1})$.

7. We define the renormalised horizontal B-periods as

$$\mathcal{H}^B_j(\tau, \widetilde{X}) = t\left(\mathrm{Re}\int_{B_{G+j}} \widetilde{\phi}_1 + i\mathrm{Re}\int_{B_{G+j}} \widetilde{\phi}_2\right), \qquad 1 \le j \le m-1$$

   By lemma 2 in [15], it extends to a smooth function of the parameters τ and $\widetilde{X}$ at $\tau = 0$ with value

$$\mathcal{H}^B_i(0, \widetilde{X}) = \overline{\frac{1}{2}\int_{p_i^-}^{p_m^-} \frac{\widetilde{\phi}_3}{g^-}} - \frac{1}{2}\int_{p_m^+}^{p_i^+} \frac{\widetilde{\phi}_3}{g^+}. \tag{11}$$

   We define $\mathcal{H}^B = (\mathcal{H}^B_1, \cdots, \mathcal{H}^B_{m-1})$

Let $\widetilde{\mathcal{F}}(\tau, \widetilde{X})$ be the collection of the equations that we have to solve, namely

$$\widetilde{\mathcal{F}} = (\widehat{\mathcal{F}}, \mathcal{Z}^-, \mathcal{Z}^+, \mathcal{V}^A, \mathcal{H}^A, \mathcal{V}^B, \mathcal{H}^B).$$

Because of the symmetries, the map $\mathcal{F}$ takes values into a certain real vector space. Let us determine this space and compute its dimension.

The map $\widehat{\mathcal{F}}$ takes values in the same space as the map $\mathcal{F}$ in section 4.2. The map $\mathcal{Z}^-$ satisfies $\mathcal{Z}^-_{m+1-i} = \overline{\mathcal{Z}^-_i}$ for $1 \leq i \leq m$. This defines a real subspace of dimension m of $\mathbb{C}^m$. The map $\mathcal{Z}^+$ takes values into $\mathbb{R}^{m-1}$ because the Weierstrass polynomials have real coefficients by symmetry. The map $\mathcal{V}^A$ satisfies $\mathcal{V}^A_i = -\mathcal{V}^A_{m-i}$ for $1 \leq i \leq m-1$. This defines a space of dimension $[\frac{m-1}{2}]$ of $\mathbb{R}^{m-1}$. The map $\mathcal{H}^A$ satisfies $\mathcal{H}^A_{m-i} = \overline{\mathcal{H}^A_i}$ for $1 \leq i \leq m-1$ and $\mathcal{H}^A_m \in \mathbb{R}$. This defines a real subspace of dimension m of $\mathbb{C}^m$. The map $\mathcal{V}^B$ satisfies $\mathcal{V}^B_{m-i} = \mathcal{V}^B_i$ for $1 \leq i \leq m-1$. This defines a subspace of dimension $[\frac{m}{2}]$ of $\mathbb{R}^{m-1}$. The map $\mathcal{H}^B$ satisfies $\mathcal{H}^B_{m-i} = \overline{\mathcal{H}^B_i}$ for $1 \leq i \leq m-1$. This defines a real subspace of dimension $m-1$ of $\mathbb{C}^{m-1}$.

Everything together, the map $\widetilde{\mathcal{F}}$ takes values into a real vector space of dimension

$$5G + 2n + 5m - 7 = 5\widetilde{G} + 2(n+1) - 4$$

which is the expected dimension in the definition of non-degenerate Weierstrass Representation for a minimal surface of genus $\widetilde{G}$ and $n+1$ ends.

5.4 Solving the equations

We want to solve the equation $\widetilde{\mathcal{F}}(\tau, \widetilde{X})$ to get $\widetilde{X}$ as an implicit function of τ.

Proposition 3 *We have $\widetilde{\mathcal{F}}(0, \underline{\widetilde{X}}) = 0$, and the partial differential of $\widetilde{\mathcal{F}}$ with respect to $\widetilde{X}$ at $(0, \underline{\widetilde{X}})$ is onto.*

Proof : we make a change of parameters so that the partial differential is triangular by blocks. Let

$$\begin{aligned} \gamma_i &= \gamma_m + \dot{\gamma}_i, \qquad 1 \leq i \leq m-1, \\ \beta^-_i &= \gamma_i + \dot{\beta}^-_i, \qquad 1 \leq i \leq m, \\ \beta^+_i &= \gamma_i + \dot{\beta}^+_i, \qquad 1 \leq i \leq m, \\ p^+_i &= \overline{p^-_i} + \dot{p}^+_i, \qquad 1 \leq i \leq m. \end{aligned}$$

We write $\dot{\gamma} = (\dot{\gamma}_1, \cdots, \dot{\gamma}_{m-1})$, $\dot{\beta}^{\pm} = (\dot{\beta}^{\pm}_1, \cdots, \dot{\beta}^{\pm}_m)$ and $\dot{p}^+ = (\dot{p}^+_1, \cdots, \dot{p}^+_m)$. The central value of each of these new parameters is 0. Now the parameters are a, α, c, γ_m, $\dot{\gamma}$, $\dot{\beta}^-$, $\dot{\beta}^+$, p^-, $\dot{p}^+$ and τ. In the following points, we evaluate the partial differential at the central value of each equation with respect to all these parameters except τ. In all this discussion, the parameter τ is equal to 0 so $t = 0$. The height differential $\widetilde{\phi}_3$ is explicitely given by proposition 2.

1. *The partial differential of $\widehat{\mathcal{F}}$ with respect to the parameters (a, α, c) is onto. Moreover, its partial derivative with respect to all other parameters is zero.*

   The first point comes from the non-degeneracy hypothesis, and the second from the fact that when $t = 0$, the restriction of the Weierstrass data to Σ only depends on the parameters a, α and c.

2. *The partial differential of $\mathcal{Z}^-$ with respect to $\dot\beta^-$ is onto. The partial derivatives of $\mathcal{Z}^-$ with respect to all other parameters except c are zero.*

   To see the first statement, we assume that all parameters except $\dot\beta^-$ have their central value. Observe that the noded Riemann surface $\underline{\Sigma}$ does not depend on β^-, so the zeros $\xi_1, \cdots, \xi_m$ of $\widetilde{\phi}_3$ are fixed. Hence we have

   $$\frac{\partial \mathcal{Z}_i^-}{\partial \dot\beta_j^-} = \frac{1}{\xi_i - p_j^-}.$$

   If we forget about the symmetries, so $\dot\beta^-$ and $\mathcal{Z}^-$ are in $\mathbb{C}^m$, then the partial differential is an isomorphism (we recognise a Cauchy determinant) When we impose the symmetries, $\dot\beta^-$ is restricted to a real space of dimension m and $\mathcal{Z}^-$ takes values in a space of the same dimension so the partial differential remains an isomorphism.

   Regarding the second point, assume that $\dot\beta^- = 0$ and $c = \underline{c}$. Then $\widetilde{\phi}_3 = -g^- dz$ in $\mathbb{C}^-$, so they have the same zeros and $\mathcal{Z}^- = 0$.

3. *The partial differential of $\mathcal{Z}^+$ with respect to $\dot\beta^+$ is onto. All other partial derivatives of $\mathcal{Z}^+$ are zero.*

   To see the first point, we fix the value of all parameters except β^+ and we compute $\mathcal{Z}^+$. We have

   $$g^+ = \frac{1}{z^m - 1} \sum_{j=1}^m \beta_j^+ (z^{m-1} + p_j^+ z^{m-2} + \cdots + (p_j^+)^{m-1}).$$

   From this we see that the Weierstrass polynomial of g^+ in a neighborhood of 0 is

   $$z^{m-1} + \frac{\sum_{j=1}^m \beta_j^+ (p_j^+ z^{m-2} + \cdots + (p_j^+)^{m-1})}{\sum_{j=1}^m \beta_j^+}.$$

   Since $\widetilde{\phi}_3$ does not depend on β^+, it has a zero of multiplicity $m-1$ at 0 and its Weierstrass polynomial is z^{m-1}. Hence

   $$\mathcal{Z}_i^+ = \frac{\sum_{j=1}^m \beta_j^+ (p_j^+)^i}{\sum_{j=1}^m \beta_j^+} \qquad \text{for } 1 \le i \le m-1$$

   and

   $$\frac{\partial \mathcal{Z}_i^+}{\dot\beta_j^+} = \frac{m-1}{m} (p_j^+)^i.$$

   We recognise a Van der Monde matrix. From this it follows that the partial differential has a one dimensional kernel, so is onto.

   The second point is clear because if $\dot\beta^+ = 0$ then $\widetilde{\phi}_3 = g^+ dz$ in $\mathbb{C}^+$ so they have the same zeros and $\mathcal{Z}^+ = 0$.

4. *The partial differential of $(\mathcal{V}^A, \mathcal{V}^B)$ with respect to $\dot{\gamma}$ is an isomorphism. All other partial derivatives are zero.*

   Indeed, in term of the new parameters we have $\mathcal{V}_i^A = -2\pi \mathrm{Im}(\dot{\gamma}_i)$ and $\mathcal{V}_i^B = -2\mathrm{Re}(\dot{\gamma}_i)$, so the partial derivative is injective. Because of the symmetry, the domain and target spaces have the same dimension, namely $m-1$.

5. *The partial differential of $\mathcal{H}^B$ with respect to $\dot{p}^+$ is onto. The partial derivatives with respect to all other parameters except $\dot{\beta}^+$, $\dot{\beta}^-$ and c are zero.*

   Indeed, if $\dot{\beta}^+ = 0$, $\dot{\beta}^- = 0$ and $c = \underline{c}$, then $\widetilde{\phi}_3 = -g^- dz$ in $\mathbb{C}^-$ and $\widetilde{\phi}_3 = g^+ dz$ in $\mathbb{C}^+$. By equation (11), we get $\mathcal{H}_i^B = \frac{1}{2}(\dot{p}_m^+ - \dot{p}_i^+)$. The statement readily follows.

6. *The partial differential of $\mathcal{H}^A$ with respect to $(p_1^-, \cdots, p_{m-1}^-, \gamma_m)$ is an isomorphism.*

   To prove this, assume that all parameters but p^- and γ_m have their central value. Then $\widetilde{\phi}_3 = -g^- dz$ in $\mathbb{C}^-$ and $\widetilde{\phi}_3 = g^+ dz$ in $\mathbb{C}^+$, so formula (10) gives

$$\begin{aligned}
\mathcal{H}_i^A &= -\frac{1}{2}\overline{\mathrm{Res}_{p_i^+}(g^+)^2} - \frac{1}{2}\mathrm{Res}_{p_i^-}(g^-)^2 \\
&= -\sum_{j \neq i} \overline{\overline{\frac{\gamma_m^2}{p_i^+ - p_j^+}}} - \sum_{j \neq i} \frac{\gamma_m^2}{p_i^- - p_j^-} + \frac{\gamma_m}{p_i^-} \\
&= -2\sum_{j \neq i} \frac{\gamma_m^2}{p_i^- - p_j^-} + \frac{\gamma_m}{p_i^-}.
\end{aligned}$$

   This implies that

$$\sum_{i=1}^{m} p_i^- \mathcal{H}_i^A = -m(m-1)\gamma_m^2 + m\gamma_m. \qquad (12)$$

   When γ_m has its central value, namely $\frac{1}{m-1}$, the right hand side is zero. When $p_1^-, \cdots, p_m^-$ have their central value, all terms in the left sum are equal, so all are zero. This proves that $\mathcal{H}^A = 0$ at the central value, which can also be verified by explicit computation.

   To prove that the partial differential is an isomorphism, we first forget the symmetry, so that $p_1^-, \cdots, p_m^-$ and γ_m are complex numbers and $\mathcal{H}^A \in \mathbb{C}^m$. We compute, for $1 \leq i, j \leq m-1$

$$\frac{\partial \mathcal{H}_i^A}{\partial p_j^-} = \frac{-2\gamma_m^2}{(p_i^- - p_j^-)^2} \quad \text{if } j \neq i,$$

$$\frac{\partial \mathcal{H}_i^A}{\partial p_i^-} = \sum_{j \neq i} \frac{2\gamma_m^2}{(p_i^- - p_j^-)^2} - \frac{\gamma_m}{(p_i^-)^2}.$$

This square matrix of order $m-1$ is proven to be invertible in appendix A. By differentiating (12) we get

$$\sum_{i=1}^{m} p_i^- \frac{\partial \mathcal{H}_i^A}{\partial \gamma^m} = -m.$$

It follows that the partial differential of $\mathcal{H}^A$ with respect to $(p_1^-, \cdots, p_{m-1}^-, \gamma_m)$ is an isomorphism.

Now if we impose symmetries, $(p_1^-, \cdots, p_{m-1}^-, \gamma_m)$ is restricted to a real space of dimension m, and $\mathcal{H}^A$ takes values into a real space of the same dimension, so the differential remains an isomorphism.

The lemma readily follows from these statements : the matrix of the differential of $(\widehat{\mathcal{F}}, \mathcal{Z}^-, \mathcal{Z}^+, (\mathcal{V}_A, \mathcal{V}_B), \mathcal{H}^B, \mathcal{H}^A)$ with respect to $((\alpha, a, c), \dot{\beta}^-, \dot{\beta}^+, \dot{\gamma}, \dot{p}^+, (p^-, \gamma_m))$ is block-triangular. □

Remark 4 *We can see in point 6 why we imposed the symmetry σ to the whole construction. This symmetry ensures that the parameter γ_m is real, which is required by the period problem. Without assuming symmetry, we would need the parameter γ_m to be a complex number for point 6 to hold.*

Since imposing the symmetry adds a lot of bulk to the construction, let me explain why the construction is hard without the symmetry. If we do not impose the symmetry with respect to a vertical plane, then the minimal surface M can be freely rotated around the vertical axis. This introduces an extra parameter. Determining the value of this parameter depends in a subtle way on the interaction between M and the catenoidal necks.

5.5 Proof of Theorem 2

By proposition 3 and the Implicit Function Theorem, for τ in a neighborhood of 0, there exists a smooth function $\widetilde{X}(\tau)$ such that $\widetilde{\mathcal{F}}(\tau, \widetilde{X}(\tau)) = 0$. For $t > 0$ close to zero, let use write $(\widetilde{\Sigma}_t, \widetilde{g}_t, \widetilde{\phi}_{3,t})$ for the Weierstrass data corresponding to the value $\tau = (-\log t)^{-1/2}$ and $\widetilde{X} = \widetilde{X}(\tau)$ of the parameters. This Weierstrass data defines us a minimal immersion $\widetilde{\psi}_t$ on $\widetilde{\Sigma}_t$ minus the punctures. Let M_t be its image. In the following points we prove that the family $(M_t)_{0<t<\varepsilon}$ has all the properties claimed in Theorem 2.

1. M_t has $n+1$ catenoidal ends at $q_1, \cdots, q_{n-1}$, ∞^- and ∞^+. The logarithmic growths are the opposite of the residue of $\widetilde{\phi}_3$ at these points, so by equations (8) and (9), their limit value when $t \to 0$ are $c_1, \cdots, c_{n-1}, 1 - \frac{m}{m-1}$ and $\frac{m}{m-1}$. Since we have scaled M so that $c_n(M) = 1$, this gives formula (2).

2. *M_t converges to M on compact subsets of $\mathbb{R}^3$.*

   This follows from the fact that $\widetilde{g}_t$ converges to g on Σ and $\widetilde{\phi}_{3,t}$ converges to ϕ_3 on compact subsets of Σ minus the punctures, where (Σ, g, ϕ_3) is the Weierstrass data of M.

3. *M_t has non-degenerate Weierstrass Representation.*

   Indeed, the differential of $\widetilde{\mathcal{F}}$ with respect to $\widetilde{X}$ at $(\tau, \widetilde{X}(\tau))$ remains onto for τ close to 0.

4. *M_t is embedded.*

   To prove this statement, we study the asymptotic behavior of $\widetilde{\psi}_t$ on each of the domains Σ, $\mathbb{C}^-$ and $\mathbb{C}^+$ when $t \to 0$. On $\mathbb{C}^+$ we have

$$\lim_{t\to 0} t\widetilde{\phi}_1 = -\frac{dz}{2}, \quad \lim_{t\to 0} t\widetilde{\phi}_2 = i\frac{dz}{2}, \quad \lim_{t\to 0} \widetilde{\phi}_3 = \frac{1}{m-1}\sum_{i=1}^{m} \frac{dz}{z-\omega^i}.$$

   Define $\widehat{\psi}_t$ on $\mathbb{C}^+$ as the composition of $\widetilde{\psi}_t - \widetilde{\psi}_t(0)$ with the affine transformation $(x_1, x_2, x_3) \mapsto (-2tx_1, -2tx_2, x_3)$. Then

$$\lim_{t\to 0} \widehat{\psi}_t(z) = (\operatorname{Re} z, \operatorname{Im} z, u^+(z))$$

   where u^+ is the harmonic function

$$u^+(z) = \frac{1}{m-1}\sum_{i=1}^{m} \log|z-\omega^i|.$$

   So the image of $\widehat{\psi}_t$ converges to the graph of u^+. For h large enough, the graph of u^+ intersects the plane $x_3 = -h$ in m closed convex curves, so the same is true for the image of $\widehat{\psi}_t$ for t small enough. As a conclusion, we can find a height c_1 (depending on t) such that the image of $\mathbb{C}^+$ by $\widetilde{\psi}_t$ cuts the plane $x_3 = c_1$ in m closed convex curves. We call S^+ the part which is above this plane. The surface S^+ is embedded (as a graph) and has one upward-going catenoidal end.

   In the same way, after horizontal scaling by $-2t$ and vertical translation, the image of $\mathbb{C}^-$ by $\widetilde{\psi}$ converges to the graph of $u^-(\overline{z})$, where

$$u^-(z) = -\log|z| + \frac{1}{m-1}\sum_{i=1}^{m} \log|z-\omega^i|.$$

   For h large enough, the graph of u^- intersects the plane $x_3 = h$ in m closed convex curves and the plane $x_3 = -h$ in two closed convex curves, one inside the other. Again we may find some heights c_2 and c_3, with $c_3 < c_2 < c_1$ such that for t small enough, the image of $\mathbb{C}^-$ by $\widetilde{\psi}_t$ cuts the plane $x_3 = c_2$ in m closed convex curves and the plane $x_3 = c_3$ in two closed convex curves, one inside the other. Let S^- be the part bounded by the m top curves and the inside bottom curve. It is an embedded surface with one downward catenoidal end.

   Finally, since the top end of M is catenoidal, we may find some height $c_4 < c_3$ such that the image of Σ by $\widetilde{\psi}_t$ cuts the plane $x_3 = c_4$ in one closed convex curves (and what is above is an annulus). Let S be the part which is below this plane. It is embedded because M is.

The pieces S, S^- and S^+ are disjoint. (For S^- and S, this uses the maximum principle and the fact that the logarithmic growth of the end of S^- is larger than the logarithmic growth of the top end of S). Each component of the complementary set in M_t of $S \cup S^- \cup S^+$ is a minimal annulus bounded by two closed convex curves in parallel planes. By a theorem of Shiffman [14], such an annulus is fibered by horizontal curves. It follows that M_t is embedded.

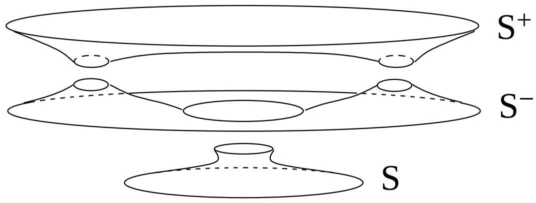

Figure 2: The pieces S^+, S^- and S (in case M is a catenoid).

5. *For t small enough, the maximum of $|K|$ on M_t is greater than $\frac{1}{2}(m-1)^2$.*

Indeed, assume by contradiction that this is not true. Then we can find a sequence $(t_n)_n$ converging to zero such that the Gaussian curvature on M_{t_n} is bounded by $\frac{1}{2}(m-1)^2$. Let A_n be the image of the annulus bounded by the circles $|v_1^+| = \epsilon$ and $|v_1^-| = \epsilon$ in M_{t_n}. Translate A_n so that the point where the Gauss map is one is at the origin. The images of the boundary circles are close to circles of radius $O(\frac{1}{t_n})$, so for any ball $B(0,R)$, $A_n \cap B(0,R)$ is properly embedded in $B(0,R)$ for n large enough. As we have uniform Gaussian curvature and area estimate (by the monotonicity formula) for A_n, a subsequence of $(A_n)_n$ converges smoothly on compact subsets of $\mathbb{R}^3$ to a complete embedded minimal annulus ([13], theorem 4.2.1) hence a catenoid. As the flux of A_n converges to $(0, 0, \frac{2\pi}{m-1})$, the limit catenoid has waist radius $\frac{1}{m-1}$. Since the maximum of the Gaussian curvature on this catenoid is $(m-1)^2$, we have a contradiction. $\square$

A A matrix computation

Lemma 1 *Consider an integer $m \geq 2$. Let $\omega = e^{2\pi i/m}$. Define $p_i = \omega^i$ for $1 \leq i \leq m$. Consider the order $m-1$ square matrix A defined by*

$$a_{ii} = \frac{m-1}{(p_i)^2} - \sum_{\substack{1 \leq j \leq m \\ j \neq i}} \frac{2}{(p_i - p_j)^2}$$

$$a_{ij} = \frac{2}{(p_i - p_j)^2} \qquad \text{if } j \neq i$$

Then A is invertible.

Proof. We prove that A has dominant diagonal. The proof relies on the following elementary observation : if $z \in \mathbb{C}$ is such that $|z| = 1$, then

$$1 - 2\mathrm{Re}\frac{1}{(1+z)^2} = \frac{2}{|1+z|^2}$$

We have

$$(p_i)^2 a_{ii} = \sum_{j=1}^{m-1} \left(1 - \frac{2}{(1-\omega^j)^2}\right).$$

Since this is a real number,

$$(p_i)^2 a_{ii} = \sum_{j=1}^{m-1} \mathrm{Re}\left(1 - \frac{2}{(1-\omega^j)^2}\right) = \sum_{j=1}^{m-1} \frac{2}{|1-\omega^j|^2}$$

$$|a_{ii}| = \sum_{j=1}^{m-1} \frac{2}{|1-\omega^j|^2} = \sum_{\substack{1 \leq j \leq m-1 \\ j \neq i}} |a_{ij}| + \frac{2}{|p_i - 1|^2}.$$

Hence $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ so A is invertible by Hadamard theorem. □

B Proof of proposition 1

We need to define a complex manifold $\widetilde{\mathcal{M}}$, a holomorphic submersion $\widetilde{\pi} : \widetilde{\mathcal{M}} \to \mathcal{A} \times \mathcal{B}$, a meromorphic function $\mathcal{G} : \widetilde{\mathcal{M}} \to \overline{\mathbb{C}}$ and a antiholomorphic involution $\widetilde{\sigma}_{\mathcal{M}} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$. To define $\widetilde{\mathcal{M}}$ we consider the disjoint union

$$(\mathcal{M} \times \mathcal{B}) \cup (\mathbb{C}^- \times \mathcal{A} \times \mathcal{B}) \cup (\mathbb{C}^+ \times \mathcal{A} \times \mathcal{B})$$

and we make some identifications.

Recall that $b = (p^+, p^-, \beta^+, \beta^-, t)$. We write $t(b)$ for the last component of the vector b, and $g^{\pm}(z, b)$ to emphasize that $g^{\pm}(z)$ also depends on the components $\beta^{\pm}$ and $p^{\pm}$ of b. Let $\mathcal{U}_0^-$ be the component of

$$\{(z, b) \in \mathcal{M} \times \mathcal{B} \,:\, |\mathcal{G}(z)| < \epsilon\}$$

containing $\{q_n\} \times \mathcal{B}$, and let

$$\mathcal{V}_0^- = \{(z,b) \in \mathcal{U}_0^- \ : \ |\mathcal{G}(z)| < \frac{|t(b)|}{\epsilon}\}.$$

In the same way let $\mathcal{U}_0^+$ be the component of

$$\{(z,a,b) \in \mathcal{M}^- \times \mathcal{A} \times \mathcal{B} \, | \ : |g^-(z,b)| > \frac{1}{\epsilon}\}$$

containing $\{0\} \times \mathcal{A} \times \mathcal{B}$. Let

$$\mathcal{V}_0^+ = \{(z,a,b) \in \mathcal{U}_0^+ \ : \ |t(b)g^-(z,b)| > \epsilon\}.$$

Finally define

$$\mathcal{W} = \{(z,a,b) \in \mathbb{C} \times \mathcal{A} \times \mathcal{B} \ : \ \frac{|t(b)|}{\epsilon} < |z| < \epsilon\}.$$

Then the applications $\varphi_0^\pm : \mathcal{U}_0^\pm \setminus \mathcal{V}_0^\pm \to \mathcal{W}$ defined by

$$\varphi_0^-(z,b) = (\mathcal{G}(z), \pi(z), b)$$

$$\varphi_0^+(z,a,b) = (t(b)g^-(z,b), a, b)$$

are holomorphic and bijective, hence biholomorphic. Remove the domains $\mathcal{V}_0^-$ from $\mathcal{M} \times \mathcal{A}$ and $\mathcal{V}_0^+$ from $\mathbb{C}^- \times \mathcal{A} \times \mathcal{B}$. Identify the point $(z,b) \in \mathcal{U}_0^- \setminus \mathcal{V}_0^-$ with the point $(z',a,b) \in \mathcal{U}_0^+ \setminus \mathcal{V}_0^+$ such that $\varphi_0^-(z,a) = \varphi_0^+(z',a,b)$. This defines a complex manifold. We do the same kind of definitions and identifications for the other nodes. We call $\widetilde{\mathcal{M}}$ the resulting complex manifold.

The holomorphic submersion $\widetilde{\pi} : \widetilde{\mathcal{M}} \to \mathcal{A} \times \mathcal{B}$ is defined by

$$\begin{cases} \widetilde{\pi}(z,b) = (\pi(z), b) \text{ on } \mathcal{M} \times \mathcal{B} \\ \widetilde{\pi}(z,a,b) = (a,b) \text{ on } \mathbb{C}^\pm \times \mathcal{A} \times \mathcal{B} \end{cases}$$

The meromorphic function $\widetilde{\mathcal{G}} : \widetilde{\mathcal{M}} \to \overline{\mathbb{C}}$ is defined by

$$\begin{cases} \widetilde{\mathcal{G}}(z,b) = \mathcal{G}(z) \text{ on } \mathcal{M} \times \mathcal{B} \\ \widetilde{\mathcal{G}}(z,a,b) = t(b)g^-(z,b) \text{ on } \mathbb{C}^- \times \mathcal{A} \times \mathcal{B} \\ \widetilde{\mathcal{G}}(z,a,b) = \frac{1}{t(b)g^+(z,b)} \text{ on } \mathbb{C}^+ \times \mathcal{A} \times \mathcal{B} \end{cases}$$

The anti-holomorphic involution $\widetilde{\sigma}_{\mathcal{M}}$ is defined by

$$\begin{cases} \widetilde{\sigma}_{\mathcal{M}}(z,b) = (\sigma_{\mathcal{M}}(z), \sigma_{\mathcal{B}}(b)) \text{ on } \mathcal{M} \times \mathcal{B} \\ \widetilde{\sigma}_{\mathcal{M}}(z,a,b) = (\overline{z}, \sigma_{\mathcal{A}}(a), \sigma_{\mathcal{B}}(b)) \text{ on } \mathbb{C}^\pm \times \mathcal{A} \times \mathcal{B} \end{cases}$$

By construction, these maps are well defined on $\widetilde{\mathcal{M}}$ and have the desired properties, namely $\widetilde{\mathcal{G}} \circ \widetilde{\sigma}_{\mathcal{M}} = \overline{\widetilde{\mathcal{G}}}$ and $\widetilde{\pi} \circ \widetilde{\sigma}_{\mathcal{M}} = \widetilde{\sigma}_{\mathcal{A}} \circ \widetilde{\pi}$, where $\widetilde{\sigma}_{\mathcal{A}}(a,b) = (\sigma_{\mathcal{A}}(a), \sigma_{\mathcal{A}}(b))$. Moreover, $\widetilde{\pi}^{-1}(a,b) = \widetilde{\Sigma}_{a,b}$ and the restriction of $\widetilde{\mathcal{G}}$ to $\widetilde{\Sigma}_{a,b}$ is $\widetilde{g}_{a,b}$. □

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