

LOOP GROUP METHODS FOR THE NON-ABELIAN HODGE CORRESPONDENCE ON A 4-PUNCTURED SPHERE

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ABSTRACT. The non-abelian Hodge correspondence is a real analytic map between the moduli space of stable Higgs bundles and the deRham moduli space of irreducible flat connections mediated by solutions to the self-duality equations. In this paper we construct self-duality solutions for strongly parabolic $\mathfrak{sl}(2, \mathbb{C})$ Higgs fields on a 4-punctured sphere with parabolic weights $t \sim 0$ using complex analytic methods. We identify the rescaled limit hyper-Kähler moduli space \mathcal{M}_t at $t = 0$ to be the completion of the nilpotent orbit in $\mathfrak{sl}(2, \mathbb{C})$ modulo a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action, equipped with the Eguchi-Hanson metric. Our methods and computations are based on the twistor approach to the self-duality equations using Deligne and Simpson's λ -connections interpretation. By construction we can compute the Taylor expansions of the holomorphic symplectic form ϖ_t on \mathcal{M}_t at $t = 0$ which turn out to have closed form expressions in terms of multiple polylogarithms (MPLs). The geometric properties of \mathcal{M}_t lead to some identities of certain MPLs which we believe deserve further investigations.

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1. INTRODUCTION

Hitchin's self-duality equations [29] on a degree zero and rank two hermitian vector bundle $V \rightarrow \Sigma$ over a compact Riemann surface Σ are equations on a pair (∇, Ψ) consisting of a special unitary connection ∇ and an (trace free) endomorphism valued $(1, 0)$ -form Ψ satisfying

$$(1) \quad \bar{\partial}^\nabla \Psi = 0 \quad \text{and} \quad F^\nabla + [\Psi, \Psi^*] = 0.$$

These equations are equivalent to the flatness of the whole associated family of connections

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$$(2) \quad \nabla^\lambda = \nabla + \lambda^{-1}\Psi + \lambda\Psi^*$$

parametrized by $\lambda \in \mathbb{C}^*$ – the spectral parameter. Solutions to (1) give rise to equivariant harmonic maps from the Riemann surface into the hyperbolic 3-space $\mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$ by considering the Higgs field Ψ as the $(1, 0)$ -part of the differential of the harmonic map [13]. An application of a classical result by Eells-Sampson [15] then shows the existence of a unique solution in each homotopy class of equivariant maps, i.e., when prescribing the (totally reducible) monodromy of the harmonic map [13].

As first recognized by Hitchin [29], the moduli space of solutions to the self-duality equations (modulo gauge transformations) \mathcal{M}_{SD} has two very different incarnations as complex analytic spaces. Firstly as the moduli space of Higgs bundles \mathcal{M}_{Higgs} , i.e., as the moduli space of polystable pairs $(\bar{\partial}_V, \Psi)$ satisfying $\bar{\partial}_V \Psi = 0$, and secondly as the moduli space of totally reducible flat connections \mathcal{M}_{dR} . The non-abelian Hodge correspondence is the map between \mathcal{M}_{Higgs} and \mathcal{M}_{dR} obtained through solutions to Hitchin's equations. Both complex analytic spaces \mathcal{M}_{Higgs} and \mathcal{M}_{dR} induce anti-commuting complex structures on \mathcal{M}_{SD} turning it into a hyper-Kähler manifold when equipped with its natural L^2 -metric. Since the mapping is mediated by the harmonic map this correspondence is not explicit and it is not possible to see every facet of the geometry within one framework only.

Every hyper-Kähler manifold can be described using complex analytic data (subject to additional reality conditions) via twistor theory [30]. For \mathcal{M}_{SD} , the twistor space has been identified with the so-called Deligne-Hitchin moduli space \mathcal{M}_{DH} introduced by Deligne and Simpson [43]. The space \mathcal{M}_{DH} is obtained by gluing the moduli space of λ -connections on the Riemann surface with the moduli space of λ -connections on the complex conjugate Riemann surface, and the associated family of flat connections (2) naturally extends to a special real holomorphic section – a twistor line. The main subject of the paper is to construct twistor lines and the hyper-Kähler structure of \mathcal{M}_{SD} ‘*entirely complex analytically, so we bypass the nonlinear elliptic theory necessary to define the harmonic metrics*’¹.

The self-duality equations (1) generalize to punctured Riemann surfaces by imposing first order poles of the Higgs fields and a growth condition of the harmonic metric determined by their parabolic weights, called tameness, see Simpson [42]. In the following, we restrict to *strongly* parabolic case, where the Higgs fields have nilpotent residues adapted to the parabolic filtration. Then the associated family of flat connections has (up to conjugation) the same local monodromy at each puncture [42, table on page 720] for all spectral parameters $\lambda \in \mathbb{C}^*$. The moduli space of solutions has again a hyper-Kähler structure which was first studied by Konno [32]. The complex structure I denotes hereby the one of the moduli space of (polystable) strongly parabolic Higgs fields, and the complex structure J denotes that of the moduli space of logarithmic connections with prescribed conjugacy class of their local monodromy.

In this paper, we study the simplest non-trivial case, where the underlying Riemann surface is a 4-punctured sphere and restrict to Fuchsian systems, i.e., logarithmic connections on the trivial holomorphic (rank 2) bundle. On the Higgs side, we assume that the parabolic Higgs bundles are strongly parabolic with the same parabolic weights $\pm t$ with $t \in (0, \frac{1}{4})$ at each of the four singular points. The moduli space of self-duality equations is then denoted by \mathcal{M}_t . For $t \rightarrow 0$ the space of polystable parabolic Higgs pairs bounded by tC (for some fixed $C > 0$) degenerates to the point given by the trivial parabolic Higgs pair. When rescaling by t its blow-up limit $\frac{1}{t}\mathcal{M}_t$ at $t = 0$ is given by the moduli space of parabolic Higgs fields Φ on the trivial holomorphic bundle. Using an implicit theorem argument we construct for $t \sim 0$ twistor lines in the (parabolic) Deligne-Hitchin moduli space near this singular limit using complex analytic methods, i.e., we construct solutions for which the Higgs field $\Psi \sim t\Phi$ is small enough. The length of Φ is measured using its L^2 -norm, or equivalently by the energy $\mathcal{E}(\nabla, \Phi)$ of the equivariant harmonic map corresponding to the Higgs pair.

¹C. Simpson in [43, Section 4]

Theorem 1. *For $t \sim 0$ fixed let $C > 0$ and consider Higgs fields with $\mathcal{E}(d, \Phi) < C$. Then, there exists $\varepsilon > 0$, depending only on the constant C , such that twistor lines s_t can be constructed with $s_t(\lambda = 0) = t\Phi$ using complex analytic methods only. In particular, the method allows to explicitly compute the Taylor expansions of the twistor lines in t .*

In fact, we find two different families of real holomorphic sections s_t^\pm of the Deligne-Hitchin moduli space for a given initial Higgs field Φ . Only one of them s_t^+ turn out to be twistor lines, while the other family s_t^- corresponds to higher solutions as introduced in [22]. By construction Theorem 1 also allows us to study the Taylor expansions of the geometric structures on \mathcal{M}_t , such as its hyper-Kähler structure, explicitly. In particular, the (rescaled) limit metric for $t \rightarrow 0$ can be identified to be the Eguchi-Hanson metric g_{EH} modulo a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action. For rational $t = \frac{l}{2k}$, we can take a suitable k -fold cover Σ_k of the four punctured sphere (depending on the value of t), such that the (pull-back of) logarithmic connections on the sphere are gauge equivalent to symmetric (smooth) connections on the compact Riemann surface Σ_k of genus $k - 1 = O(\frac{1}{t})$. The moduli space of equivariant solutions to the self-duality equations \mathcal{M}_{SD}^l on Σ_k then converge (on every compact subset) to a corresponding compact subset of the Eguchi-Hanson space modulo the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action as $k \rightarrow \infty$. More precisely,

Theorem 2. *Let $C > 0$ and $l \in \mathbb{N}^{>0}$ be fixed. Consider the compact subspaces*

$$\mathcal{C}_k^l := \{[\nabla, \Phi] \in \mathcal{M}_{SD}^l(\Sigma_k) \mid \mathcal{E}(\nabla, \Phi) \leq C\}$$

of \mathcal{M}_{SD}^l with the induced hyper-Kähler metric g_k . Then for $k \rightarrow \infty$, we obtain smooth convergence

$$(\mathcal{C}_k^l, g_k) \longrightarrow (\{v \in T^*\mathbb{CP}^1 \mid \mathcal{E}_{EH}(v) \leq C\}, 32\pi l g_{EH})/(\mathbb{Z}_2 \times \mathbb{Z}_2).$$

The main advantage of the approach is that the higher order terms of the metric expansion are also accessible. We have

Theorem 3. *The hyper-Kähler metric of the moduli space \mathcal{M}_t of strongly parabolic Higgs bundles on the 4-punctured sphere is real analytic in its weight t and there exist an explicit algorithm to computing its Taylor expansion at the trivial parabolic Higgs pair in $t = 0$. More precisely, when computing the Taylor expansion of the twistor lines in t at the trivial connection, the n -th order coefficients are polynomials in λ of degree $n + 1$ and can be expressed explicitly in terms of multiple polylogarithms of depth and weight at most $n + 1$.*

Also the second family of solutions yields a hyper-Kähler metric, the n -th order coefficients of the two metric expansions differ only by $(-1)^n$.

It would be interesting to compare our results to those of [18], where the moduli space is studied for fixed weights and large Higgs fields in the regular locus, as well as in the context of current developments on hyper-Kähler 4-manifolds (see e.g. [9, 10, 11]), and in view of the different ways of obtaining hyper-Kähler structures on moduli spaces (see e.g. [5] and the references therein).

The paper is organized as follows. In Section 2 we introduce Higgs bundles, the associated moduli spaces as well as its hyper-Kähler structure for compact Riemann surfaces. Their complex geometric properties are encoded in the Deligne-Hitchin moduli space, see [43], and we discuss twistor lines as special real holomorphic sections induced by self-duality solutions. Furthermore, we identify the twisted holomorphic symplectic structure on the Deligne-Hitchin moduli space with (a version of) Goldman's symplectic structure. We extend this discussion to the case of punctured surfaces and strongly parabolic Higgs pairs in Section 3, and explain the relationship between the compact and punctured case for rational weights. Thereafter, we give the ansatz in terms of Fuchsian systems for constructing real holomorphic sections through an implicit function theorem argument and loop group methods in Section 4 and prove Theorem 1 in Section 5. To find appropriate coordinates, we first consider the 4-fold covering of the Higgs bundles moduli space \mathcal{M}_{Higgs} and we rescale it by the factor $\frac{1}{t}$. The limit space at $t = 0$ is then given by the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ (with coordinates (u, v)) at the origin. To perform the implicit theorem argument, we thus first consider the regular case in Theorem

8 and then when $(u, v) \rightarrow (0, 0)$ in Theorem 9. It turns out that in the appropriate $(u, v) \rightarrow (0, 0)$ limit the Higgs field vanishes, and the corresponding sections s_0^+ are flat unitary connections, which are twistor lines. Due to the fact that twistor lines form a connected component of the space of real holomorphic sections, all constructed sections must be twistor lines as well. Interestingly, we find a Lax pair type equation describing the deformation given by the implicit function theorem. In Section 6, we compute the limit non-abelian Hodge correspondence at $t = 0$, and identify the rescaled limit hyper-Kähler metric to be Eguchi-Hanson. One advantage of our construction is that we can compute Taylor expansion of all geometric quantities. As an example, we compute first order derivatives of the parameters in Section 6.3 and put them together to obtain the first order expansion of the non-abelian Hodge correspondence, the twisted holomorphic symplectic structure and the hyper-Kähler metric. In the last Section 7 we analyse the structure of the higher order derivatives of the twistor sections with respect to t . As in [27] for minimal surfaces in the 3-sphere, the n -th order derivatives of the parameters are polynomial in λ of order $(n + 1)$. Since the twisted holomorphic symplectic form is a Laurent polynomial of degree one in λ , evaluating it on meromorphic 1-forms which represent tangent vectors to $\mathcal{M}(t)$ yields infinitely many cancelations for higher order terms in λ leading to identities for some iterated integrals, called Ω -values, which can be expressed in terms of multiple polylogarithms. In Section 7.3 we give some identities of depth three Ω -values obtained from this idea which are non-trivial in the sense that they cannot be derived from the shuffle and stuffle relations alone.

2. PRELIMINARIES

In this preliminary section, we recall basic results about the moduli spaces of solutions to Hitchin's self-duality equations with a focus on the twistorial description of its hyper-Kähler structure. Unless stated otherwise, the underlying Riemann surface Σ is assumed to be compact throughout the section.

2.1. The hyper-Kähler structures and their twistor spaces. The moduli space of irreducible solutions of Hitchin's self-duality equations \mathcal{M}_{SD} has through the non-abelian Hodge correspondence three complex structures $I, J, K = IJ$ which are Kähler with respect to the same Riemannian metric g . In fact there exist a whole \mathbb{CP}^1 worth of complex structures defined by

$$I_\lambda = \frac{1 - |\lambda|^2}{1 + |\lambda|^2} I + \frac{\lambda + \bar{\lambda}}{1 + |\lambda|^2} J + \frac{i(\lambda - \bar{\lambda})}{1 + |\lambda|^2} K$$

for $\lambda \in \mathbb{C} \subset \mathbb{CP}^1$. Within this family we find the complex structure I at $\lambda = 0$, and the complex structure J at $\lambda = 1$. Let

$$\omega_I = -g(., I.) \quad \omega_J = -g(., J.) \quad \text{and} \quad \omega_K = -g(., K.)$$

be the associated Kähler forms so that $h = g + i\omega_L$ is the corresponding hermitian metric for $L \in \{I, J, K\}$. The twistor space of a hyper-Kähler manifold \mathcal{M} , introduced in [30], is the smooth manifold

$$\mathcal{P} := \mathcal{M} \times \mathbb{CP}^1$$

equipped with the (integrable) complex structure

$$\mathbb{I} = (I_\lambda, i),$$

at the point $x \in \mathcal{M}$ and $\lambda \in \mathbb{C} \subset \mathbb{CP}^1$, where i is the standard complex structure of \mathbb{CP}^1 . Furthermore, the twistor space has a natural anti-holomorphic involution \mathcal{T} given by $(x, \lambda) \mapsto (x, -\bar{\lambda}^{-1})$. By construction the twistor space has a holomorphic projection π to \mathbb{CP}^1 and a twisted relative holomorphic symplectic form given by

$$\widehat{\omega} \in H^0(P, \Lambda^2 V^* \otimes \mathcal{O}(2))$$

where $V = \ker d\pi$ (is the complex tangent bundle to the fibers) and $\mathcal{O}(2)$ denotes the dual of the pull-back of the canonical bundle over $\mathbb{C}P^1$. In terms of the Kähler forms the twisted relative holomorphic symplectic form has the following explicit expression

$$(3) \quad \widehat{\omega} = \varpi \otimes \lambda \frac{\partial}{\partial \lambda},$$

where

$$(4) \quad \varpi = \lambda^{-1}(\omega_J + i\omega_K) - 2\omega_I - \lambda(\omega_J - i\omega_K),$$

see [30, Equation (3.87)] or (23) below. Sections of \mathcal{P} are holomorphic maps $s: \mathbb{C}P^1 \rightarrow \mathcal{P}$ with $\pi \circ s = \text{Id}$.

When restricting to $\mathcal{M} = \mathcal{M}_{SD}$, the simplest sections are *constant* sections

$$(5) \quad \lambda \mapsto ([\nabla, \Psi], \lambda)$$

for a solution of the self-duality equations (∇, Ψ) . By dimension count, these twistor lines give rise to an open subspace of the space of real holomorphic sections [30]. The following well known theorem then follows from the completeness of the moduli space of self-duality solutions [29].

Theorem 4. *Twistor lines form an open and closed subset of the space of real holomorphic sections.*

A convenient set-up for studying associated families of flat connections obtained from solutions to self-duality equations is to consider them as real sections of the Deligne-Hitchin moduli space \mathcal{M}_{DH} .

2.2. The Deligne-Hitchin moduli space \mathcal{M}_{DH} . The Deligne-Hitchin moduli space was first introduced by Deligne (see [43, 44]) as a complex analytic way of viewing the associated twistor space of the moduli space of solutions to the self-duality equations. As such it interpolates between the moduli space of Higgs bundles \mathcal{M}_{Higgs} and the moduli space of flat connections \mathcal{M}_{dR} .

Definition 1. *Let Σ be a compact Riemann surface and $\lambda \in \mathbb{C}$ fixed. A (integrable) λ -connection on a \mathcal{C}^∞ -complex vector bundle $V \rightarrow \Sigma$ is a pair $(\bar{\partial}, D)$ consisting of a holomorphic structure $\bar{\partial}$ on V and a linear first order differential operator*

$$D: \Gamma(\Sigma, V) \rightarrow \Omega^{(1,0)}(\Sigma, V)$$

satisfying the λ -Leibniz rule

$$D(fs) = \lambda \partial f \otimes s + fDs,$$

where ∂ is the trivial ∂ -operator for functions f and s is a section of V , and the integrability condition

$$(6) \quad D\bar{\partial} + \bar{\partial}D = 0.$$

Remark 2. *The operators D and $\bar{\partial}$ also act on $(0,1)$ -forms and $(1,0)$ -forms respectively. For $\lambda = 0$ the integrability condition (6) is equivalent to*

$$D = \Psi \in H^0(M, K_\Sigma \text{End}(V))$$

being a holomorphic endomorphism-valued 1-form, and for $\lambda \neq 0$ we have that

$$\nabla = \frac{1}{\lambda}D + \bar{\partial}$$

is a flat connection.

Example 3. *Consider on the hermitian bundle $V \rightarrow \Sigma$ a solution $(\nabla = \partial^\nabla + \bar{\partial}^\nabla, \Psi)$ of the self-duality equations. Then, the pair*

$$(\bar{\partial}^\nabla + \lambda\Psi^*, \lambda\partial^\nabla + \Psi)$$

defines a λ -connection on V for every $\lambda \in \mathbb{C}$ which coincides with the Higgs pair $(\bar{\partial}^\nabla, \Psi)$ at $\lambda = 0$ and with the flat connection $\nabla^1 = \nabla + \Psi + \Psi^$ at $\lambda = 1$.*

Definition 4. *A $\text{SL}(2, \mathbb{C})$ λ -connection is a λ -connection on a rank 2 vector bundle $V \rightarrow \Sigma$, such that the induced λ -connection on the determinant bundle $\Lambda^2 V$ is trivial.*

Definition 5. A $\mathrm{SL}(2, \mathbb{C})$ λ -connection $(\bar{\partial}, D)$ is called *stable*, if every $\bar{\partial}$ -holomorphic subbundle $L \subset V$ with

$$D(\Gamma(\Sigma, L)) \subset \Omega^{(1,0)}(\Sigma, L)$$

is of negative degree and semi-stable if its degree is non-positive. All other λ -connections are called *unstable*. A $\mathrm{SL}(2, \mathbb{C})$ λ -connection is called *polystable* if it is either stable or the direct sum of dual λ -connections on degree zero line bundles.

For $\lambda \neq 0$, every $\bar{\partial}$ -holomorphic and D -invariant line subbundle $L \subset V$ must be parallel with respect to the flat connection $\nabla = \frac{1}{\lambda}D + \bar{\partial}$. Therefore, the degree of L is 0 and the λ -connection $(\bar{\partial}, D)$ is semi-stable. Moreover, $(\bar{\partial}, D)$ is stable if and only if the flat connection $\nabla = \frac{1}{\lambda}D + \bar{\partial}$ is irreducible. The situation is different at $\lambda = 0$ and we need to restrict to polystable λ -connections to obtain a well-behaved moduli space.

Definition 6. The Hodge moduli space $\mathcal{M}_{Hod} = \mathcal{M}_{Hod}(\Sigma)$ is the space of all polystable $\mathrm{SL}(2, \mathbb{C})$ λ -connections on $V = \Sigma \times \mathbb{C}^2 \rightarrow \Sigma$ modulo gauge transformations, i.e., \mathcal{M}_{Hod} consists of gauge classes of triples $(\lambda, \bar{\partial}, D)$ for $\lambda \in \mathbb{C}$ and $(\bar{\partial}, D)$ a polystable λ -connection. The gauge-equivalence class of $(\lambda, \bar{\partial}, D)$ is denoted by

$$[\lambda, \bar{\partial}, D] \in \mathcal{M}_{Hod}$$

or by

$$[\lambda, \bar{\partial}, D]_{\Sigma} \in \mathcal{M}_{Hod}(\Sigma)$$

to emphasis its dependence on the Riemann surface.

The Hodge moduli space admits a natural holomorphic map

$$\pi_{\Sigma}: \mathcal{M}_{Hod} \longrightarrow \mathbb{C}; \quad [\lambda, \bar{\partial}, D] \longmapsto \lambda$$

whose fiber at $\lambda = 0$ is the (polystable) Higgs moduli space \mathcal{M}_{Higgs} , and at $\lambda = 1$ it is the deRham moduli space of flat (and totally reducible) $\mathrm{SL}(2, \mathbb{C})$ -connections \mathcal{M}_{dR} . We consider both spaces as complex analytic spaces endowed with their natural complex structures I and J , respectively.

The next step is then to compactify the λ -plane \mathbb{C} to \mathbb{CP}^1 . For a given Riemann surface Σ let $\bar{\Sigma}$ be the Riemann surface with conjugate complex structure. As differentiable manifolds we have $\Sigma \cong \bar{\Sigma}$ and thus their deRham moduli spaces of flat $\mathrm{SL}(2, \mathbb{C})$ -connections are naturally isomorphic. Then the two Hodge moduli spaces $\mathcal{M}_{Hod}(\Sigma)$ and $\mathcal{M}_{Hod}(\bar{\Sigma})$ can be glued together via Deligne gluing [44]

$$\mathcal{G}: \mathcal{M}_{Hod}(\Sigma) \setminus \pi_{\Sigma}^{-1}(0) \longrightarrow \mathcal{M}_{Hod}(\bar{\Sigma}) \setminus \pi_{\bar{\Sigma}}^{-1}(0); \quad [\lambda, \bar{\partial}, D]_{\Sigma} \longmapsto [\frac{1}{\lambda}, \frac{1}{\lambda}D, \frac{1}{\lambda}\bar{\partial}]_{\bar{\Sigma}}$$

along $\lambda \in \mathbb{C}^*$ to give Deligne-Hitchin moduli space

$$\mathcal{M}_{DH} = \mathcal{M}_{Hod}(\Sigma) \cup_{\mathcal{G}} \mathcal{M}_{Hod}(\bar{\Sigma}).$$

The natural fibration π_{Σ} on \mathcal{M}_{Hod} extends holomorphically to the whole Deligne-Hitchin moduli space to give $\pi: \mathcal{M}_{DH} \rightarrow \mathbb{CP}^1$ whose restriction to $\mathcal{M}_{Hod}(\bar{\Sigma})$ is $1/\pi_{\bar{\Sigma}}$.

Remark 7. Note that the Deligne gluing map \mathcal{G} maps stable λ -connections over Σ to stable $\frac{1}{\lambda}$ -connections on $\bar{\Sigma}$. Hence, it maps the smooth locus of $\mathcal{M}_{Hod}(\Sigma)$ (consisting of stable λ -connections) to the smooth locus of $\mathcal{M}_{Hod}(\bar{\Sigma})$, and thus \mathcal{M}_{DH} is equipped with a complex manifold structure at all stable points. Moreover, since λ -connections for $\lambda \in \mathbb{C}^*$ can be reinterpreted as flat connections, we can identify

$$\mathbb{C}^* \times \mathcal{M}_{dR}(\Sigma) = \mathcal{M}_{Hod}(\Sigma) \setminus \pi_{\Sigma}^{-1}(0).$$

Definition 8. A section of \mathcal{M}_{DH} is a holomorphic map

$$s: \mathbb{CP}^1 \longrightarrow \mathcal{M}_{DH}$$

such that $\pi \circ s = \mathrm{Id}$.

Example 9. *The associated family of flat connections ∇^λ (2) to a solution of the self-duality equations (1) gives rise to a section of $\mathcal{M}_{DH} \rightarrow \mathbb{CP}^1$ via*

$$(7) \quad s(\lambda) = [\lambda, \bar{\partial}^\nabla + \lambda \Psi^*, \lambda \bar{\partial}^\nabla + \Psi]_\Sigma \in \mathcal{M}_{Hod}(\Sigma) \subset \mathcal{M}_{DH}.$$

When identifying the Deligne-Hitchin moduli space with the twistor space $\mathcal{P} \rightarrow \mathbb{CP}^1$ of the hyper-Kähler space \mathcal{M}_{SD} (at the smooth points), the section given by (7) is identified with the ‘constant’ twistor line (5), see [43].

Definition 10. *A section s of \mathcal{M}_{DH} is called stable, if the λ -connection $s(\lambda)$ is stable for all $\lambda \in \mathbb{C}^*$ and if the Higgs pairs $s(0)$ on Σ and $s(\infty)$ on $\bar{\Sigma}$ are stable.*

It follows from Hitchin [29] and Donaldson [13] that every stable point in \mathcal{M}_{DH} uniquely determines a twistor line. Therefore, a twistor line s is already stable if $s(\lambda_0)$ is stable for some $\lambda_0 \in \mathbb{C}$. Moreover, twistor lines are in one-to-one correspondence with self-duality solutions (1). The following characterization of twistor lines as particular *negative real holomorphic sections* of \mathcal{M}_{DH} is useful to decide when certain real sections give rise to global solutions to the self-duality equations.

2.3. Automorphisms of the Deligne-Hitchin moduli space. To define a real structure on \mathcal{M}_{DH} we need to look at some natural automorphisms of Deligne-Hitchin moduli space first. For every $\mu \in \mathbb{C}^*$ the (multiplicative) action of μ on \mathbb{CP}^1 has a natural lift to \mathcal{M}_{DH} by

$$\mu([\lambda, \bar{\partial}, D]) = [\mu\lambda, \bar{\partial}, \mu D].$$

Definition 11. *We denote by $N : \mathcal{M}_{DH} \rightarrow \mathcal{M}_{DH}$ the map given by multiplication with $\mu = -1$, namely*

$$[\lambda, \bar{\partial}, D] \mapsto [-\lambda, \bar{\partial}, -D].$$

Furthermore, we have a natural anti-holomorphic automorphism denoted by C .

Definition 12. *Let $C : \mathcal{M}_{DH} \rightarrow \mathcal{M}_{DH}$ be (the continuation of) the map*

$$\tilde{C} : \mathcal{M}_{Hod}(\Sigma) \rightarrow \mathcal{M}_{Hod}(\bar{\Sigma})$$

given by

$$(8) \quad [\lambda, \bar{\partial}, D] \mapsto [\bar{\lambda}, \bar{\bar{\partial}}, \bar{D}]_{\bar{\Sigma}}.$$

To be more concrete, for

$$\bar{\partial} = \bar{\partial}_0 + \eta \quad \text{and} \quad D = \lambda(\partial_0) + \omega$$

where $d = \partial_0 + \bar{\partial}_0$ is the trivial connection, $\eta \in \Omega^{0,1}(\Sigma, \mathfrak{sl}(2, \mathbb{C}))$, and $\omega \in \Omega^{1,0}(\Sigma, \mathfrak{sl}(2, \mathbb{C}))$, we define the complex conjugate on the trivial \mathbb{C}^2 -bundle over $\bar{\Sigma}$ to be

$$\bar{\bar{\partial}} = \partial_0 + \bar{\eta} \quad \text{and} \quad \bar{D} = \bar{\lambda}(\bar{\partial}_0) + \bar{\omega}.$$

The map C covers the map

$$\lambda \in \mathbb{CP}^1 \mapsto \bar{\lambda}^{-1} \in \mathbb{CP}^1.$$

Since C and N commute and both maps are involutive, their composition

$$\mathcal{T} = CN$$

is an involution as well, which covers the fixed-point free involution $\lambda \mapsto -\bar{\lambda}^{-1}$ on \mathbb{CP}^1 .

2.4. Real sections. Consider the anti-holomorphic involution of the associated Deligne-Hitchin moduli space

$$\mathcal{T} = CN: \mathcal{M}_{DH} \longrightarrow \mathcal{M}_{DH}$$

covering the antipodal involution

$$\lambda \longmapsto -\bar{\lambda}^{-1}$$

of \mathbb{CP}^1 . We call a holomorphic section s of \mathcal{M}_{DH} real (with respect to \mathcal{T}) if

$$(9) \quad \mathcal{T}(s(\lambda)) = s(-\bar{\lambda}^{-1})$$

holds for all $\lambda \in \mathbb{CP}^1$.

Example 13. *Twistor lines (7) are real holomorphic sections with respect to \mathcal{T} . Let (∇, Ψ) be a solution of the self-duality equations on Σ with respect to the standard hermitian metric on $\mathbb{C}^2 \rightarrow \Sigma$. Because we are dealing with $\mathfrak{sl}(2, \mathbb{C})$ -matrices, the unitary connection ∇ satisfies*

$$\nabla = \nabla^* = \bar{\nabla} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which is equivalent to

$$\bar{\partial}^\nabla = \bar{\partial}^{\bar{\nabla}} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \partial^\nabla = \bar{\partial}^{\bar{\nabla}} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and analogously we have

$$\Psi = - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{\Psi}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \Psi^* = - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{\Psi} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Therefore, the twistor line (7) satisfies

$$(10) \quad \begin{aligned} \mathcal{T}(s(\lambda)) &= \mathcal{T}([\lambda, \bar{\partial}^\nabla + \lambda \Psi^*, \Psi + \lambda \partial^\nabla]_\Sigma) \\ &= [-\bar{\lambda}^{-1}, \bar{\lambda}^{-1}(\bar{\Psi} + \lambda \bar{\partial}^{\bar{\nabla}}), -\bar{\lambda}^{-1}(\bar{\partial}^{\bar{\nabla}} + \lambda \bar{\Psi}^*)]_\Sigma \\ &= [-\bar{\lambda}^{-1}, \bar{\partial}^{\bar{\nabla}} + \bar{\lambda}^{-1} \bar{\Psi}, -\bar{\Psi}^* - \bar{\lambda}^{-1} \bar{\partial}^{\bar{\nabla}}]_\Sigma \\ &= [(-\bar{\lambda}^{-1}, \bar{\partial}^{\bar{\nabla}} + \bar{\lambda}^{-1} \bar{\Psi}, -\bar{\Psi}^* - \bar{\lambda}^{-1} \bar{\partial}^{\bar{\nabla}}) \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}]_\Sigma \\ &= [(-\bar{\lambda}^{-1}, \bar{\partial}^\nabla - \bar{\lambda}^{-1} \Psi^*, \Psi - \bar{\lambda}^{-1} \partial^\nabla)]_\Sigma = s(-\bar{\lambda}^{-1}). \end{aligned}$$

Let $s(\lambda)$ be a real holomorphic section with $\nabla^\lambda \sim \lambda^{-1} \Psi + \nabla + \dots$ being a lift to the space of flat connections such that $(\bar{\partial}^\nabla, \Psi)$ is stable. For the existence of such lifts see [7, Lemma 2.2]. Then the reality condition (9) gives rise to a holomorphic family of gauge transformations $g(\lambda)$ satisfying

$$\overline{\nabla^{-\bar{\lambda}^{-1}}} = \nabla^\lambda \cdot g(\lambda).$$

Applying this equation twice we obtain

$$\nabla^\lambda \cdot g(\lambda) \overline{g(-\bar{\lambda}^{-1})} = \nabla^\lambda.$$

Because the section s is stable, the connections ∇^λ are irreducible for all $\lambda \in \mathbb{C}^*$. Therefore $g(\lambda) \overline{g(-\bar{\lambda}^{-1})}$ is a constant multiple of the identity for every $\lambda \in \mathbb{C}^*$. By [22, Lemma 1.18] we can choose g to be $\text{SL}(2, \mathbb{C})$ -valued and thus

$$(11) \quad g(\lambda) \overline{g(-\bar{\lambda}^{-1})} = \pm \text{Id}.$$

The sign on the right hand side is independent of the lift ∇^λ of s and is preserved in a connected component of real sections motivating the following definition.

Definition 14. [7, Definition 2.16] *A stable real holomorphic section s of \mathcal{M}_{DH} is called positive or negative depending on the sign of (11).*

Example 15. *Twistor lines are always negative sections. In fact, the associated family of flat connections is a canonical lift of the twistor line to the space of flat connections, and with respect to the standard hermitian structure on \mathbb{C}^2 the gauge*

$$(12) \quad g(\lambda) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

as in (11), is constant in λ and squares to $-Id$.

The space of real sections of \mathcal{M}_{DH} has multiple connected components. A negative real holomorphic section lying in the connected component of the twistor lines must be a twistor line itself by Theorem 4. It corresponds therefore to a global solution of the self-duality equation.

2.5. Loop groups. We briefly introduce basic notions and statements about loop groups which are relevant with regard to the paper. For details we refer to [40] or [14]. Define

- $\Lambda\mathrm{SL}(2, \mathbb{C}) := \{ \text{real analytic maps (loops)} \Phi: \mathbb{S}^1 \longrightarrow \mathrm{SL}(2, \mathbb{C}), \quad \lambda \longmapsto \Phi^\lambda \};$
- $\Lambda\mathfrak{sl}(2, \mathbb{C}) := \{ \text{real analytic maps (loops)} \eta: \mathbb{S}^1 \longrightarrow \mathfrak{sl}(2, \mathbb{C}), \quad \lambda \longmapsto \eta^\lambda \}.$

Then, $\Lambda\mathrm{SL}(2, \mathbb{C})$ is an infinite dimensional Frechet Lie group via pointwise multiplication with $\Lambda\mathfrak{sl}(2, \mathbb{C})$ as its Lie algebra. We consider $\mathbb{S}^1 = \{ \lambda \in \mathbb{C}P^1 \mid \lambda\bar{\lambda} = 1 \}$, and denote

$$(13) \quad \begin{aligned} \Lambda_+\mathrm{SL}(2, \mathbb{C}) &= \{ \Phi \in \Lambda\mathrm{SL}(2, \mathbb{C}) \mid \Phi \text{ extends holomorphically to } \lambda = 0 \} \\ \Lambda_-\mathrm{SL}(2, \mathbb{C}) &= \{ \Phi \in \Lambda\mathrm{SL}(2, \mathbb{C}) \mid \Phi \text{ extends holomorphically to } \lambda = \infty \} \end{aligned}$$

and

$$\Lambda_+\mathfrak{sl}(2, \mathbb{C}) = \{ \eta \in \Lambda\mathfrak{sl}(2, \mathbb{C}) \mid \eta \text{ extends holomorphically to } \lambda = 0 \}.$$

We also denote

$$\Lambda^*\mathrm{SL}(2, \mathbb{C}) = \{ B \in \Lambda_-\mathrm{SL}(2, \mathbb{C}) \mid B(\infty) = Id \}.$$

The following classical theorem, due to Birkhoff, Grothendieck and others, is essential for our method, see [40] for details and [14] for the loop group method for harmonic maps from simply connected 2-dimensional domains into positively curved symmetric spaces.

Theorem 5. *There is an open and dense subset $\mathcal{U} \subset \Lambda\mathrm{SL}(2, \mathbb{C})$ called the big cell such that every $g \in \mathcal{U}$ admits a factorization*

$$g = g^+ g^-$$

with $g^+ \in \Lambda_+\mathrm{SL}(2, \mathbb{C})$ and $g^- \in \Lambda^\mathrm{SL}(2, \mathbb{C})$. This splitting is unique and depends holomorphically on $g \in \mathcal{U}$. The pair (g^+, g^-) is called the Birkhoff factorization of g .*

2.6. Reconstruction of self-duality solutions from admissible negative real sections. A section s of the Deligne-Hitchin moduli space has lifts to families of flat connections in both Hodge moduli spaces. Let

$$\tilde{\nabla}^\lambda = \lambda^{-1} \Psi_1 + \nabla + \text{higher order terms in } \lambda$$

be a lift around $\lambda = 0$. Due to s being real, there exists a family of $\mathrm{SL}(2, \mathbb{C})$ gauges $g(\lambda)$ satisfying

$$\overline{\tilde{\nabla}^{-\bar{\lambda}^{-1}}} = \tilde{\nabla}^\lambda \cdot g(\lambda).$$

Assume that the Birkhoff factorization

$$g(\lambda) = g^+(\lambda) g^-(\lambda)$$

exist for every $z \in \Sigma$.

Assume that the section s is negative, i.e.,

$$(14) \quad \overline{g(-\bar{\lambda}^{-1})}^{-1} = \overline{g^-(\bar{\lambda}^{-1})}^{-1} \overline{g^+(-\bar{\lambda}^{-1})}^{-1} = -g(\lambda).$$

Since $\lambda \rightarrow -\bar{\lambda}^{-1}$ interchanges the $()^+$ and $()^-$ parts of the Birkhoff factorization, the uniqueness assertion gives

$$(15) \quad g^+(\lambda) = -\overline{g^-(-\bar{\lambda}^{-1})}^{-1} B^{-1} \quad \text{and} \quad g^-(\lambda) = \overline{Bg^+(-\bar{\lambda}^{-1})}^{-1}$$

for $B = (g^+(\lambda = 0))^{-1}: \Sigma \rightarrow \mathrm{SL}(2, \mathbb{C})$. Combining (14) and (15) we get $\bar{B}B = -\mathrm{Id}$. This implies that B lies in the same conjugacy class as $\delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, i.e., there exist $G: \Sigma \rightarrow \mathrm{SL}(2, \mathbb{C})$ satisfying $\delta = G^{-1}B\bar{G}$. Then

$$\nabla^\lambda = \tilde{\nabla}^\lambda.(g^+(\lambda)G)$$

satisfies $\overline{\nabla^{-\bar{\lambda}^{-1}}} = \nabla^\lambda.\delta$ and is therefore the associated family of a self-duality solution with respect to the standard hermitian metric.

Remark 16. *In this paper we construct real holomorphic sections via loop group methods, i.e., we write down a particular lift of a real section s of the Deligne-Hitchin moduli space around $\lambda = 0$ in terms of a Fuchsian potential η (i.e., a λ -dependent Fuchsian connection 1-form). To obtain the actual harmonic map into the hyperbolic 3-space \mathbb{H}^3 (or the associated self-duality solution), we need thus to first perform a global Birkhoff factorization and then show that the obtained section is negative. This needs further conditions, as examples for which the harmonic map into \mathbb{H}^3 becomes singular and intersects the boundary at infinity of \mathbb{H}^3 exists [22] as well as positive real sections with global Birkhoff factorization that give rise to harmonic maps into the de-Sitter 3-space [7, Theorem 3.4].*

2.7. Goldman's symplectic form on the moduli space of λ -connections. Fix $\lambda \in \mathbb{C}$ and consider the space of λ -connections modulo gauge transformations. Let $(\bar{\partial}, D)$ be a λ -connection. A tangent vector to the (infinite dimensional) space of λ -connection is given by

$$(A, B) \in \Omega^{(0,1)}(\mathfrak{sl}(2, \mathbb{C})) \oplus \Omega^{(1,0)}(\mathfrak{sl}(2, \mathbb{C}))$$

satisfying the linearized compatibility (flatness) condition

$$(16) \quad 0 = \bar{\partial}B + DA.$$

Tangent vectors at $(\bar{\partial}, D)$ which are generated by the infinitesimal gauge transformation $\xi \in \Gamma(\Sigma, \mathfrak{sl}(2, \mathbb{C}))$ are given by

$$(A, B) = (\bar{\partial}\xi, D\xi).$$

The Goldman symplectic structure Ω^λ on the moduli space of λ -connections [16, Section 1] is defined to be

$$(17) \quad \Omega^\lambda((A_1, B_1), (A_2, B_2)) = 4 \int_\Sigma \mathrm{tr}(A_1 \wedge B_2 - A_2 \wedge B_1)$$

for $(A_1, B_1), (A_2, B_2) \in \Omega^{(0,1)}(\mathfrak{sl}(2, \mathbb{C})) \oplus \Omega^{(1,0)}(\mathfrak{sl}(2, \mathbb{C}))$ representing tangent vectors. On a compact Riemann surface, the symplectic structure Ω^λ is gauge invariant, and thus can be computed using arbitrary representatives of the tangent vectors, which makes it well-defined on the moduli space of λ -connections. Since we are not aware of any explicit reference (except for $\lambda = 0$ and $\lambda = 1$), we give the simple proof (which is of course an instance of an infinite-dimensional symplectic reduction for the gauge group action with the curvature as moment map, see [1]).

Lemma 17. *Let $\lambda \in \mathbb{C}$ be fixed. The holomorphic symplectic form Ω^λ is well-defined on the moduli space of λ -connections.*

Proof. By construction, Ω^λ is complex bilinear. Proposition 18 below identifies Ω^λ with the twisted holomorphic symplectic form (4) restricted to the fiber over λ , up to a constant factor. Thus closeness and non-degeneracy follows from [30, Section 3(F)]. It remains to show that Ω^λ descends to the moduli space. A gauge transformation acts on tangent vectors to the space of connections by conjugation, and since the trace is conjugation invariant we obtain

$$\Omega^\lambda((A_1, B_1).g, (A_2, B_2).g) = \Omega^\lambda((A_1, B_1), (A_2, B_2)).$$

Let (A_1, B_1) satisfy $0 = \bar{\partial}B + DA$ and let $(A_2, B_2) = (\bar{\partial}\xi, D\xi)$ be a tangent vector that is also tangent to the gauge orbit. Then,

$$(18) \quad \begin{aligned} \Omega^\lambda((A_1, B_1), (A_2, B_2)) &= 4 \int_{\Sigma} \text{tr}(A_1 \wedge D\xi - \bar{\partial}\xi \wedge B_1) \\ &= -4 \int_{\Sigma} d \text{tr}(A_1 \xi + \xi B_1) - 4 \int_{\Sigma} \text{tr}(DA_1 \xi + \xi \bar{\partial}B_1) = 0. \end{aligned}$$

Therefore, Ω^λ is well-defined on the quotient space (the moduli space of λ -connections). \square

The Goldman symplectic form in fact coincides (up to scaling) with the twisted holomorphic symplectic form (4) on the twistor space $\mathcal{P} = \mathcal{M}_{SD} \times \mathbb{C}P^1$ of the moduli space of self-duality solutions, also compare with [20, Equation (4.10)].

Proposition 18. *Let $\lambda \in \mathbb{C} \subset \mathbb{C}P^1$ be fixed. Then, the fiber $\mathcal{P}_\lambda = \pi^{-1}(\lambda)$ is the moduli space of λ -connections, and*

$$\lambda \varpi|_{\mathcal{P}_\lambda} = \Omega^\lambda$$

is the Goldman symplectic form on the moduli space of λ -connections.

Proof. For $\lambda \in \mathbb{C}$ fixed, it is shown in [44, Theorem 4.2] that $\mathcal{P}_\lambda = \pi^{-1}(\{\lambda\})$ is the moduli space of λ -connections. In order to evaluate the corresponding symplectic forms, we need to first find appropriate representatives for the tangent vectors. To do so, let h be the standard hermitian metric and let (∇, Ψ) be a solution of the self-duality equations. The tangent space of \mathcal{M}_{SD} at $[\nabla, \Psi]$ is given by

$$(\xi, \phi) \in \Omega^{(0,1)}(\mathfrak{sl}(2, \mathbb{C})) \oplus \Omega^{(1,0)}(\mathfrak{sl}(2, \mathbb{C}))$$

satisfying

$$(19) \quad \begin{aligned} 0 &= d^\nabla(\xi - \xi^*) + \frac{1}{2}[\phi \wedge \Psi^*] + \frac{1}{2}[\Psi \wedge \phi^*] \\ 0 &= \bar{\partial}^\nabla \phi + [\xi, \Psi] \end{aligned}$$

modulo infinitesimal gauge deformations (see [29, (6.1)]). These equations mean that we have an infinitesimal deformation

$$t \mapsto (\nabla + t(\xi - \xi^*), \Psi + t\phi)$$

of a solution to the self-duality equation (for the fixed hermitian metric h), and we can choose harmonic representatives which are orthogonal to the (unitary) gauge orbit, which is equivalent to

$$(20) \quad \mathcal{D}^*(\xi, \phi) = 0$$

where

$$\mathcal{D}: \Gamma(\Sigma, \mathfrak{su}(2)) \rightarrow \Omega^{(0,1)}(\Sigma, \mathfrak{sl}(2)) \oplus \Omega^{(1,0)}(\Sigma, \mathfrak{sl}(2)); \quad \psi \mapsto ((d^\nabla \psi)'', [\Psi, \psi])$$

see [29, 19]. Provided (19), (20) hold, the complex structures $I, J, K = IJ$ are given by

$$(21) \quad I(\xi, \phi) = (i\xi, i\phi) \quad J(\xi, \phi) = (i\phi^*, -i\xi^*) \quad \text{and} \quad K(\xi, \phi) = (-\phi^*, \xi^*),$$

see [29, page 109]. Provided (19), (20) hold for (ξ_k, ϕ_k) , $k = 1, 2$, the Hitchin metric on \mathcal{M}_{SD} (see [29, (6.2)]) is defined to be

$$g((\xi_1, \phi_1), (\xi_2, \phi_2)) := 2i \int_{\Sigma} \text{tr}(\xi_1^* \wedge \xi_2 + \xi_2^* \wedge \xi_1 + \phi_1 \wedge \phi_2^* + \phi_2 \wedge \phi_1^*).$$

Thus

$$\begin{aligned}
\omega_I((\xi_1, \phi_1), (\xi_2, \phi_2)) &= -g((\xi_1, \phi_1), (i\xi_2, i\phi_2)) \\
&= 2 \int_{\Sigma} \text{tr}(\xi_1^* \wedge \xi_2 - \xi_2^* \wedge \xi_1 - \phi_1 \wedge \phi_2^* + \phi_2 \wedge \phi_1^*) \\
\omega_J((\xi_1, \phi_1), (\xi_2, \phi_2)) &= 2 \int_{\Sigma} \text{tr}(\xi_1^* \wedge \phi_2^* - \phi_2 \wedge \xi_1 + \phi_1 \wedge \xi_2 - \xi_2^* \wedge \phi_1^*) \\
\omega_K((\xi_1, \phi_1), (\xi_2, \phi_2)) &= -2i \int_{\Sigma} \text{tr}(-\xi_1^* \wedge \phi_2^* - \phi_2 \wedge \xi_1 + \phi_1 \wedge \xi_2 + \xi_2^* \wedge \phi_1^*)
\end{aligned}$$

which gives

$$(\omega_J + i\omega_K)(\xi_1, \phi_1), (\xi_2, \phi_2) = -4 \int_{\Sigma} \text{tr}(\phi_2 \wedge \xi_1 - \phi_1 \wedge \xi_2).$$

Recall that twistor lines corresponding to the self duality solution (∇, Ψ) are given by

$$\lambda \mapsto [\lambda, \bar{\partial}^{\nabla} + \lambda\Psi^*, \Psi + \lambda\bar{\partial}^{\nabla}]_{\Sigma} \in \mathcal{M}_{Hod} \subset \mathcal{M}_{DH}$$

or equivalently, by the associated family of flat connections

$$(22) \quad \lambda \in \mathbb{C}^* \mapsto \nabla + \lambda^{-1}\Psi + \lambda\Psi^*.$$

Thus, the tangent vectors $X = (\xi_1, \phi_1)$ and $Y = (\xi_2, \phi_2)$ correspond to sections of the normal bundle to the twistor line in \mathcal{M}_{DH} given by

$$X : \lambda \mapsto [0, \xi_1 + \lambda\phi_1^*, \phi_1 - \lambda\xi_1^*]$$

and

$$Y : \lambda \mapsto [0, \xi_2 + \lambda\phi_2^*, \phi_2 - \lambda\xi_2^*].$$

Therefore, when fixing $\lambda \in \mathbb{C}$, we obtain from (17)

$$\begin{aligned}
(23) \quad \Omega^{\lambda}(X, Y) &= \Omega^{\lambda}((\xi_1 + \lambda\phi_1^*, \phi_1 - \lambda\xi_1^*), (\xi_2 + \lambda\phi_2^*, \phi_2 - \lambda\xi_2^*)) \\
&= (\omega_J + i\omega_K)(X, Y) - 2\lambda\omega_I(X, Y) - \lambda^2(\omega_J - i\omega_K)(X, Y).
\end{aligned}$$

□

3. STRONGLY PARABOLIC HIGGS BUNDLES

In this section we generalise the setup in section 2 to punctured Riemann surfaces. We introduce strongly parabolic Higgs fields, and explain their relationship to Higgs bundles on compact surfaces when their parabolic weights are rational. We will restrict to the $\mathfrak{sl}(2, \mathbb{C})$ -case, which we refer to as *trace-free*. In particular, the holomorphic bundles are of rank 2, have trivial determinant bundle, and the corresponding Higgs fields are trace-free.

3.1. Parabolic structures and logarithmic connections. Let Σ be a compact Riemann surface, $p_1, \dots, p_n \in \Sigma$ be pairwise distinct and $\Sigma^0 = \Sigma \setminus \{p_1, \dots, p_n\}$. Let $V = \mathbb{C}^2 \times \Sigma \rightarrow \Sigma$ be the trivial C^{∞} -complex vector bundle over Σ of rank two and $d = \bar{\partial}^0 + \bar{\partial}^0$ be the decomposition of the deRham differential d on Σ into its $(1, 0)$ and $(0, 1)$ parts. Furthermore, let $\bar{\partial} = \bar{\partial}^0 + \gamma$, $\gamma \in \Gamma(\Sigma, \bar{K}_{\Sigma}\mathfrak{sl}(2, \mathbb{C}))$ be a holomorphic structure on V , and consider the corresponding holomorphic rank 2 vector bundle $\mathcal{V} := (V, \bar{\partial})$. Since γ is trace free, the determinant bundle $\Lambda^2 V$ of \mathcal{V} is holomorphically trivial. A holomorphic $\text{SL}(2, \mathbb{C})$ frame $F = (s, t)$ on an open set $U \subset \Sigma$ is given by two holomorphic sections $s, t \in H^0(U, \mathcal{V})$ such that $s \wedge t = 1$.

Definition 19. A (trace-free) parabolic structure \mathcal{P} on the holomorphic bundle \mathcal{V} with parabolic divisor $D = p_1 + \dots + p_n$ is given by a collection of quasiparabolic lines, i.e., 1-dimensional vector subspaces $\ell_j \subset V_{p_j}$, together with parabolic weights $\alpha_j \in [0, \frac{1}{2}]$ for $j = 1, \dots, n$.

For a general definition of parabolic structures see for example [35, 42], and for the trace-free convention see [39].

The *parabolic degree* of a holomorphic subbundle L of \mathcal{V} on Σ is defined by

$$\text{pdeg}(L) = \deg(L) + \sum_j (-1)^{\sigma_j} \alpha_j, \quad \text{with} \quad \sigma_j = \begin{cases} 0 & \text{if } L_{p_j} = \ell_j \\ 1 & \text{if } L_{p_j} \neq \ell_j \end{cases}.$$

Two different parabolic structures \mathcal{P} and $\tilde{\mathcal{P}}$ on two holomorphic bundles \mathcal{V} respectively $\tilde{\mathcal{V}}$ over Σ are called *isomorphic* if the singular divisors $D = \tilde{D}$ coincide, the parabolic weights are the same $\alpha_j = \tilde{\alpha}_j$ for all j , and if there is a holomorphic bundle isomorphism $\Phi: \mathcal{V} \rightarrow \tilde{\mathcal{V}}$ mapping the parabolic lines onto each other, i.e., $\Phi(\ell_k) = \tilde{\ell}_j$ for all j .

Definition 20. A parabolic structure \mathcal{P} on the holomorphic bundle \mathcal{V} is called (semi-)stable if and only if all holomorphic subbundles have negative (non-positive) parabolic degree. A parabolic structure is called polystable if it is either stable or the underlying holomorphic bundle \mathcal{V} is the direct sum of two holomorphic line subbundles of parabolic degree 0.

The notion of a parabolic structure was introduced to generalize the Narasimhan-Seshadri theorem [37] to the case of punctured Riemann surfaces [35].

Definition 21. A (trace-free) logarithmic connection on the holomorphic bundle $\mathcal{V} \rightarrow \Sigma$ with singular divisor $D = p_1 + \dots + p_n$ is a connection ∇ on $\mathcal{V}|_{\Sigma^\circ}$ with $\bar{\partial}^\nabla = \bar{\partial}_V|_{\Sigma^\circ}$, such that with respect to any holomorphic $\text{SL}(2, \mathbb{C})$ frame F of \mathcal{V} on an open subset $U \subset \Sigma$, the connection 1-form $\alpha = \alpha_F$ of ∇ with respect to F is a meromorphic $\mathfrak{sl}(2, \mathbb{C})$ -valued 1-form with at most first order poles at $U \cap \{p_1, \dots, p_n\}$ and holomorphic elsewhere.

Note that a logarithmic connection on a Riemann surface is automatically flat, as its connection 1-form is meromorphic. A holomorphic frame F of \mathcal{V} restricted to $U \subset \Sigma$ can be regarded as a holomorphic isomorphism from the trivial holomorphic rank 2 bundle to \mathcal{V} restricted to U . Thus, the *residue*

$$\text{res}_{p_j} \nabla := F \circ \text{res}_{p_j} \alpha_F \circ F^{-1} \in \mathfrak{sl}(V_p)$$

of a logarithmic connection ∇ on a holomorphic bundle \mathcal{V} at a singular point p_j is well-defined, i.e., independent of the choice of the holomorphic frame.

Assume that the two eigenvalues $\pm\alpha_j$ of $\text{res}_{p_j} \nabla$ do not differ by an integer. Then, the (local) monodromy of ∇ along a simple closed curve surrounding p_j lies in the conjugacy class of

$$(24) \quad \exp(2\pi i \text{res}_{p_j} \nabla) \cong \begin{pmatrix} e^{2\pi i \alpha_j} & 0 \\ 0 & e^{-2\pi i \alpha_j} \end{pmatrix}.$$

In particular, if the eigenvalues $\pm\alpha_j$ are real with $\alpha_j \in (0, \frac{1}{2})$, then the local monodromy lies in the conjugacy class of unitary matrices. In the following, we always assume that this is the case, i.e., we impose the condition

$$\det(\text{res}_{p_j} \nabla) \in (-\frac{1}{4}, 0)$$

on the trace-free residues. Then a logarithmic connection determines a parabolic structure with underlying holomorphic bundle \mathcal{V} and singular divisor $D = p_1 + \dots + p_n$ such that the parabolic weight $0 < \alpha_j < \frac{1}{2}$ at p_j is the positive eigenvalue of $\text{res}_{p_j} \nabla$ and the quasiparabolic line ℓ_j is the corresponding eigenline. We will refer to the positive eigenvalues of the residues $\text{res}_{p_j} \nabla$ as the parabolic weights of ∇ . If the monodromy representation is irreducible and unitary (up to overall conjugation), then the parabolic structure is stable. The converse statement also holds by the Mehta-Seshadri theorem [35]: for every stable parabolic structure \mathcal{P} there exists a (unique) logarithmic connection ∇ with unitary monodromy (up to conjugation) inducing the parabolic structure \mathcal{P} .

This correspondence extends to reducible unitary logarithmic connections with strictly polystable parabolic structures: the induced parabolic structure of a unitary connection with abelian monodromy is the direct sum of two holomorphic line subbundles of parabolic degree 0. And by abelian Hodge theory and the residue theorem, the converse is true as well.

3.1.1. *Parabolic structures on the 4-punctured sphere.* In this paper, we mainly study the case of rank 2 trace-free parabolic structures over $\Sigma = \mathbb{CP}^1$ with 4 singular points $p_1, \dots, p_4 \in \mathbb{CP}^1$ such that the parabolic weights α_j satisfy

$$(25) \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = t \in (0, \tfrac{1}{2}).$$

By the Birkhoff-Grothendieck theorem, every holomorphic rank 2 vector bundle over \mathbb{CP}^1 with trivial determinant is of the form $\mathcal{V} = \mathcal{O}(k) \oplus \mathcal{O}(-k)$ for some unique $k \in \mathbb{N}^{\geq 0}$.

Lemma 22. *For $k \geq 2$, the holomorphic bundle $\mathcal{V} = \mathcal{O}(k) \oplus \mathcal{O}(-k) \rightarrow \mathbb{CP}^1$ does not admit a logarithmic connection on the 4-punctured sphere with parabolic weights (25). In particular, there is no stable or semi-stable parabolic structure with these weights.*

Proof. Assume there is such a connection ∇ on $\mathcal{V} = \mathcal{O}(k) \oplus \mathcal{O}(-k)$. Writing ∇ with respect to this splitting gives

$$\nabla = \begin{pmatrix} D & \beta_+ \\ \beta_- & D^* \end{pmatrix}.$$

Then, the upper-left entry D is a logarithmic connection on $\mathcal{O}(k)$ (the definition is analogous to the $\mathfrak{sl}(2, \mathbb{C})$ case) with dual connection D^* , and $\beta_+ \in H^0(\mathbb{CP}^1, K\mathcal{O}(2k+4))$ and $\beta_- \in H^0(\mathbb{CP}^1, K\mathcal{O}(-2k+4))$, where the $\mathcal{O}(4)$ factor comes from the 4 singularities. Since $K = K_{\mathbb{CP}^1} \cong \mathcal{O}(-2)$ we have by degree count that $\beta_- \equiv 0$. This is a contradiction, since in this case D would be a meromorphic connection on $L = \mathcal{O}(k)$ with residues $\pm t$ at the 4 singular points, thus the residue formula gives $0 = \text{pdeg}(L) \geq k - 4t > 0$.

The same computation gives that the holomorphic bundle \mathcal{V} does not admit a stable or semi-stable parabolic structure, since the parabolic degree of $\mathcal{O}(k)$ is always positive. \square

The two remaining cases $k = 1$ and $k = 0$ behave quite differently.

Lemma 23. *Let $t \in (0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{1}{2})$. Then there is, up to isomorphism, exactly one parabolic structure \mathcal{P} on $\mathcal{V} = \mathcal{O}(1) \oplus \mathcal{O}(-1)$ over \mathbb{CP}^1 with 4 singular points and parabolic weights (25) which admits a logarithmic connection. The parabolic structure \mathcal{P} is stable if and only if $t > \frac{1}{4}$. Furthermore, up to isomorphism, the space of logarithmic connections on $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ is a complex affine line.*

Proof. We refer to [23, Proposition 18] for the proof. \square

If the bundle $\mathcal{V} \cong \mathcal{O} \oplus \mathcal{O}$ is trivial, we can compare the different quasiparabolic lines over the different singular points with respect to a fixed background \mathbb{C}^2 . Since we will be focusing on $t \sim 0$ later, we only discuss stability for $t < \frac{1}{4}$ here.

Lemma 24. *Let $t \in (0, \frac{1}{4})$. A parabolic structure on $\mathcal{O} \oplus \mathcal{O}$ over \mathbb{CP}^1 with 4 singular points and parabolic weights (25) admits a logarithmic connection if and only if every parabolic line ℓ_j , $j = 1, \dots, 4$, coincides with at most one other line ℓ_k for $k = 1, \dots, 4$ and $k \neq j$. The parabolic structure \mathcal{P} is stable if and only if the parabolic lines ℓ_j , $j = 1, \dots, 4$, are pairwise distinct, and \mathcal{P} is the direct sum of two holomorphic line subbundles of parabolic degree 0 if and only if we have two pairs of (distinct) parabolic lines.*

Proof. Let ∇ be a logarithmic connection on $\mathcal{V} = \mathcal{O} \oplus \mathcal{O}$. Let L be some holomorphic line subbundle of degree 0, i.e., it is isomorphic to \mathcal{O} . Take a complementary subbundle L^* , which is isomorphic to \mathcal{O} as well, and decompose

$$\nabla = \begin{pmatrix} D & \beta_+ \\ \beta_- & D^* \end{pmatrix}$$

with respect to $\mathcal{V} = L \oplus L^*$, where D is a logarithmic connection on L , and

$$\beta_- \in H^0(\mathbb{CP}^1, K\mathcal{O}(4-n)),$$

$n \in \{0, \dots, 4\}$ being the number of points p_j for which $L_{p_j} = \ell_j$. For $n > 2$ this implies $\beta_- = 0$ which then gives a contradiction to the residue theorem for the line bundle connection D . Hence, we obtain $n < 3$. The converse direction can be shown easily by giving explicit formulas using Fuchsian systems, or it follows directly by the Mehta-Seshadri theorem once we know that they are stable.

To show stability, note first that there is no holomorphic sub line bundle of $\mathcal{O} \oplus \mathcal{O}$ with positive degree. Moreover, the parabolic degree of a line subbundle $\mathcal{O}(k) \subset \mathcal{O} \oplus \mathcal{O}$ with $k < 0$ is always negative since $t < \frac{1}{4}$. Thus it remains to consider line subbundles L that are isomorphic to \mathcal{O} , which means L is constant. If all parabolic lines ℓ_j are different, then $n \leq 1$ and we have $\text{pdeg}(L) \leq -2t < 0$, from which stability follows.

The case $n = 2$ follows from similar arguments together with the residue theorem. \square

Proposition 25. *The moduli space \mathcal{M}_{Bun} of polystable parabolic structures \mathcal{P} over \mathbb{CP}^1 with 4 singular points p_1, \dots, p_4 and with parabolic weights given by $t \in (0, \frac{1}{4})$ can be identified with \mathbb{CP}^1 with three \mathbb{Z}_2 -orbifold points $0, 1, \infty$.*

Proof. Since every polystable parabolic bundle admits a (unitary) logarithmic connection by the Mehta-Seshadri theorem the underlying holomorphic bundle is $\mathcal{O} \oplus \mathcal{O}$ by Lemma 22 and Lemma 23. Hence, holomorphic gauge transformations of the underlying holomorphic bundle \mathcal{V} are given by conjugation with constant $\text{SL}(2, \mathbb{C})$ matrices. If \mathcal{P} is stable, all parabolic lines are pairwise distinct. Fixing the gauge freedom is then equivalent to normalizing the parabolic lines, i.e.,

$$\ell_1 = 0, \quad \ell_2 = 1, \quad \text{and} \quad \ell_3 = \infty.$$

Then,

$$\ell_4 = w \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$$

uniquely determines a stable parabolic structure (and for each $w \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ there is a stable parabolic structure by Lemma 24). We call w the parabolic modulus of the parabolic structure \mathcal{P} . By identifying

$$(26) \quad \begin{aligned} w = 0 &\iff \ell_1 = \ell_4 \text{ and } \ell_2 = \ell_3 \\ w = 1 &\iff \ell_2 = \ell_4 \text{ and } \ell_1 = \ell_3 \\ w = \infty &\iff \ell_3 = \ell_4 \text{ and } \ell_1 = \ell_2 \end{aligned}$$

we can fill in the missing three points of \mathbb{CP}^1 . We first claim that it is not possible to find a holomorphic family $w \mapsto \mathcal{P}(w)$ of polystable parabolic structures which represents the modulus w in any open neighbourhood of $w = 0, 1$, or $w = \infty$. Consider the case $w = 0$ and parametrize (in terms of a centered local holomorphic coordinate ζ)

$$(27) \quad \begin{aligned} \ell_1(\zeta) &= \mathbb{C}(f_1(\zeta)\zeta, 1)^T & \ell_2(\zeta) &= \mathbb{C}(1 + f_2(\zeta)\zeta, 1)^T \\ \ell_3(\zeta) &= \mathbb{C}(1 + f_3(\zeta)\zeta, 1)^T & \ell_4(\zeta) &= \mathbb{C}(f_4(\zeta)\zeta, 1)^T. \end{aligned}$$

Assume that the lines only meet in first order in ζ , which translates to the condition $f_1(0) \neq f_3(0)$ and $f_2(0) \neq f_4(0)$. Consider (for small $\zeta \neq 0$) the Moebius transformations

$$x \in \mathbb{CP}^1 \mapsto \frac{\zeta(x - \zeta f_1(\zeta))(f_2(\zeta) - f_3(\zeta))}{(1 - x - \zeta f_3(\zeta))(-1 + \zeta(f_1(\zeta) - f_2(\zeta)))} \in \mathbb{CP}^1$$

which satisfy

$$(28) \quad \begin{aligned} \ell_1(\zeta) &\mapsto 0, \quad \ell_2(\zeta) \mapsto 1, \quad \ell_3(\zeta) \mapsto \infty \\ w(\zeta) &= \ell_4(\zeta) \mapsto -\zeta^2 \frac{(f_2(\zeta) - f_3(\zeta))(f_1(\zeta) - f_4(\zeta))}{(-1 - \zeta f_1(\zeta) - \zeta f_2(\zeta))(1 + \zeta f_3(\zeta) - \zeta f_4(\zeta))}. \end{aligned}$$

Hence, $w(\zeta)$ has at $\zeta = 0$ a simple branch point as claimed. The other two cases $w = 1$ and $w = \infty$ follow analogously.

It remains to exhibit the \mathbb{Z}_2 -orbifold structure on the moduli space of polystable parabolic structures with parabolic weights t at all 4 singular points. As we have seen above, the moduli space \mathcal{M}_{Bun} is

given by the set of quadruples of parabolic lines $\ell_j \subset \mathbb{C}$, $j = 1, \dots, 4$, subject to the condition that the 4 lines are pairwise distinct or that they are pairwise the same as in (26), up to the action of $\mathrm{SL}(2, \mathbb{C})$. Consider the map

$$(29) \quad \mathbb{CP}^1 \rightarrow \mathcal{M}_{\mathrm{Bun}} \cong \mathbb{CP}^1; [u, v] \mapsto (\ell_1 = [u, v], \ell_2 = [v, -u], \ell_3 = [-u, v], \ell_4 = [v, u]) / \sim$$

where \sim denotes the equivalence relation induced by the $\mathrm{SL}(2, \mathbb{C})$ -action. This map is well-defined, as the 4 lines are either pairwise distinct ($[u, v] \notin \{[\pm 1, 1], [\pm i, 1], [0, 1], [1, 0]\}$), or $\ell_1 = \ell_2$ and $\ell_3 = \ell_4$ (if $[u, v] = [\pm i, 1]$), or $\ell_1 = \ell_3$ and $\ell_2 = \ell_4$ (if $[u, v] \in \{[0, 1], [1, 0]\}$), or $\ell_1 = \ell_4$ and $\ell_2 = \ell_3$ (if $[u, v] = [\pm 1, 1]$). Consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on \mathbb{CP}^1 generated by $[u, v] \mapsto [-u, v]$ and $[u, v] \mapsto [v, u]$. Acting with $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ respectively with $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, we see that the map (29) is invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action. Furthermore, this map is rational of degree 4, and the only fix points (of the 3 non-trivial elements of order 2) are $[\pm 1, 1]$, $[\pm i, 1]$, or $[0, 1]$ and $[1, 0]$, respectively. This proves the proposition. \square

As we have seen in Proposition 25 the \mathbb{CP}^1 with three orbifold singularities $0, 1, \infty$ is given by the quotient of \mathbb{CP}^1 by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action generated by $\zeta \mapsto -\zeta$ and $\zeta \mapsto \frac{1}{\zeta}$. We will encounter this $\mathbb{Z}_2 \times \mathbb{Z}_2$ action again in our symmetric setup, see Proposition 32 below.

3.1.2. Strongly parabolic Higgs fields. Let ∇ and $\tilde{\nabla}$ be two logarithmic connections with the same induced parabolic structure \mathcal{P} on the holomorphic bundle \mathcal{V} . The difference $\tilde{\nabla} - \nabla$ is a $\mathfrak{sl}(\mathcal{V})$ -valued meromorphic 1-form. Since the eigenvalues of the residues of $\tilde{\nabla}$ and ∇ are the same, and the eigenlines ℓ_j for the positive eigenvalues coincide, the residues

$$R_j := \mathrm{res}_{p_j}(\tilde{\nabla} - \nabla) = \mathrm{res}_{p_j}(\tilde{\nabla}) - \mathrm{res}_{p_j}(\nabla)$$

must be nilpotent with $\ell_j \subset \ker R_j$. This motivates the following definition:

Definition 26. Let \mathcal{P} be a parabolic structure on the holomorphic bundle \mathcal{V} with singular divisor $D = p_1 + \dots + p_n$. A strongly parabolic Higgs field is a meromorphic 1-form

$$\Psi \in H^0(\Sigma, K_\Sigma \mathfrak{sl}(\mathcal{V}, \mathbb{C}) \otimes \mathcal{O}_\Sigma(D))$$

with $(\mathrm{res}_{p_j} \Psi) \ell_j = 0$ for all j . A strongly parabolic Higgs pair (\mathcal{P}, Ψ) is called stable if every Ψ -invariant holomorphic line subbundle of \mathcal{V} has negative degree, and strictly polystable if \mathcal{V} is the direct sum of two Ψ -invariant line bundles of parabolic degree 0.

We restrict ourselves in the following to the case of the 4-punctured sphere with parabolic weight $t \in (0, \frac{1}{4})$ at each puncture. Note that, similarly to the proof of Lemma 22, it can be shown that there is no stable parabolic Higgs pair on $\mathcal{O}(k) \oplus \mathcal{O}(-k)$ with $k \geq 2$ over \mathbb{CP}^1 and only four singular points (together with our restrictions for the range of the weights). As before we treat the cases of $\mathcal{V} = \mathcal{O}(-1) \oplus \mathcal{O}(1)$ and $\mathcal{V} = \mathcal{O} \oplus \mathcal{O}$ separately.

Lemma 27. Let (\mathcal{P}, Ψ) be a stable strongly parabolic Higgs pair with underlying holomorphic bundle $\mathcal{V} = \mathcal{O}(-1) \oplus \mathcal{O}(1)$. Then, up to isomorphisms of \mathcal{P} , all four parabolic lines are contained in $\mathcal{O}(-1)$, and the Higgs field Ψ takes the form

$$\Psi = \begin{pmatrix} 0 & \alpha \\ \gamma & 0 \end{pmatrix}$$

where $\gamma \in H^0(\mathbb{CP}^1, \mathrm{Hom}(\mathcal{O}(-1), K_{\mathbb{CP}^1} \mathcal{O}(1))) \cong \mathbb{C}$ is a constant and α is a non-zero meromorphic section of $\mathrm{Hom}(\mathcal{O}(1), K_{\mathbb{CP}^1} \mathcal{O}(-1))$ with 4 simple poles at p_1, \dots, p_4 which is unique up to a scale that can be fixed by an isomorphism of \mathcal{P} .

Proof. We can assume without loss of generality that ℓ_1, ℓ_2 and ℓ_3 are contained in $\mathcal{O}(-1) \subset \mathcal{V}$, i.e., $\mathcal{O}(-1)|_{p_j} = \ell_j$, for $j = 1, 2, 3$. Write

$$\Psi = \begin{pmatrix} \beta & \alpha \\ \gamma & -\beta \end{pmatrix}$$

with respect to the decomposition $\mathcal{V} = \mathcal{O}(-1) \oplus \mathcal{O}(1)$. Then, β is a meromorphic 1-form on \mathbb{CP}^1 with at most a simple pole at p_4 , thus β must vanish. If $\alpha = 0$ then $\mathcal{O}(1)$ would be an invariant subbundle

of positive parabolic degree. Hence, $\alpha \neq 0$ by stability. Since $\text{Hom}(\mathcal{O}(1), K_{\mathbb{C}P^1}\mathcal{O}(-1)) \cong \mathcal{O}(-4)$ is of degree -4 and $\alpha \neq 0$ can have at most simple poles, α must have a simple pole at each p_j . Moreover, this implies α is unique up to scale, which can be fixed by a constant diagonal gauge. Furthermore, we obtain that γ cannot have a pole at p_1, \dots, p_4 , as the residues of Ψ must be nilpotent, thus γ must be a constant which also implies that $\mathcal{O}(-1)|_{p_4} = \ell_4$. \square

Lemma 28. *Let (\mathcal{P}, Ψ) be a stable Higgs pair with strongly parabolic Higgs field Ψ and underlying holomorphic structure $\mathcal{O} \oplus \mathcal{O}$. Then, \mathcal{P} is polystable. Conversely, for every polystable \mathcal{P} on $\mathcal{O} \oplus \mathcal{O}$ there is a 1-dimensional vector space (unique up to parabolic isomorphisms) of strongly parabolic Higgs fields Ψ such that the Higgs pair (\mathcal{P}, Ψ) is stable if $\Psi \neq 0$.*

Proof. We first show that the underlying parabolic structure \mathcal{P} for a stable strongly parabolic Higgs pair on $\mathcal{O} \oplus \mathcal{O}$, is polystable. Assume \mathcal{P} is not polystable, then there is a unique degree zero line bundle L with non-negative parabolic degree. Thus, L must contain at least two parabolic lines, i.e., $L|_{p_j} = \ell_j$ for two different $j \in \{1, \dots, 4\}$. A strongly parabolic Higgs field $\Psi \in H^0(K_{\mathbb{C}P^1}\mathfrak{sl}(2, \mathbb{C}))$ is given by

$$\Psi = \sum_{j=1}^4 R_j \frac{dz}{z - p_j},$$

with $R_j \in \mathfrak{sl}(2, \mathbb{C})$ and $\sum_{j=1}^4 R_j = 0$. Thus if L contains three or four parabolic lines, in which case L has positive parabolic degree, it must be invariant under Ψ in contradiction to the Higgs pair being stable. If L contains exactly two parabolic lines, then it is of parabolic degree zero, and we can find a complementary constant line bundle \tilde{L} , which contains either one or two of the remaining parabolic lines. Since the quasiparabolic lines lie in the kernel of R_j , the matrices R_j with respect to the decomposition $\mathcal{O} \oplus \mathcal{O} = L \oplus \tilde{L}$ are either upper or lower triangular with vanishing diagonals whenever L or \tilde{L} contains the quasiparabolic line ℓ_j . Thus by residue theorem, as the sum of all residues R_j must be zero, we obtain that \tilde{L} must contain two parabolic lines and thus has parabolic degree 0. But then the parabolic bundle is the direct sum of two line bundles of parabolic degree zero and hence \mathcal{P} must be polystable, which gives a contradiction.

Next, we show that dimension of the space of (stable) strongly parabolic Higgs fields for a given stable parabolic structure is at most 1-dimensional (up to isomorphism). If \mathcal{P} is stable, i.e., the four quasiparabolic eigenlines are distinct, a short computation shows that there exists no non-zero strongly parabolic Higgs field of \mathcal{P} with vanishing residue at one of the singular points by the residue theorem. This shows that the space of strongly parabolic Higgs fields is at most 1-dimensional, as the linear combination of two (non-zero) strongly parabolic Higgs fields such that the residue R_j at one singular point vanishes must be identically zero.

The space of stable strongly parabolic Higgs fields is at least 1-dimensional, up to conjugation, since every stable parabolic structure \mathcal{P} with underlying holomorphic bundle $\mathcal{O} \oplus \mathcal{O}$ and parabolic weights $t \in (0, \frac{1}{4})$ is determined by the quasiparabolic lines

$$(30) \quad \ell_1 = (u, v)^T \mathbb{C}, \quad \ell_2 = (v, -u)^T \mathbb{C}, \quad \ell_3 = (u, -v)^T \mathbb{C}, \quad \ell_4 = (v, u)^T \mathbb{C}$$

for suitable $u, v \in \mathbb{C}^2 \setminus \{0\}$. Then, it can be easily verified that

$$(31) \quad \Psi = \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix} \frac{dz}{z - p_1} + \begin{pmatrix} -uv & -v^2 \\ u^2 & uv \end{pmatrix} \frac{dz}{z - p_2} + \begin{pmatrix} uv & u^2 \\ -v^2 & -uv \end{pmatrix} \frac{dz}{z - p_3} + \begin{pmatrix} -uv & v^2 \\ -u^2 & uv \end{pmatrix} \frac{dz}{z - p_4}$$

is a non-zero strongly parabolic Higgs field for \mathcal{P} . Since \mathcal{P} is already stable, the Higgs pair (\mathcal{P}, Ψ) is stable as well.

If \mathcal{P} is the direct sum of two line bundles of parabolic degree 0, then a stable strongly parabolic Higgs field must be off-diagonal (with respect to the decomposition of the rank two bundle into the two line bundles), with the off-diagonal entries being two non-zero meromorphic 1-forms with simple poles at the two of the four singular points, where the quasiparabolic lines coincide with the respective holomorphic line bundle. This gives a two-dimensional space (with two complex lines removed) of

possible strongly parabolic Higgs fields. Because for non-vanishing off-diagonals the determinant of the Higgs field is non-vanishing with simple poles, these Higgs fields are stable. By a diagonal gauge, we can fix the ratio of the meromorphic 1-forms, and we obtain a complex line (without the origin, where the Higgs field vanishes) of stable parabolic Higgs fields. \square

Lemma 29. *Let $(u, v) \in \mathbb{C}^2 \setminus \{0\}$ and consider the strongly parabolic Higgs field Ψ as in (31). Then the underlying parabolic structures are always polystable and moreover, they are stable except for*

$$uv = 0, \quad u^2 = v^2 \quad \text{or} \quad u^2 = -v^2.$$

Proof. Since $4t < 1$ every holomorphic subbundle of the holomorphically trivial rank 2 bundle with negative degree also has negative parabolic degree. Let L be a holomorphic line subbundle of degree 0, i.e., a constant line in \mathbb{C}^2 . Thus the parabolic degree of L can be maximized by choosing $L = \ell_j$ to be one of the quasiparabolic lines. These lines ℓ_1, \dots, ℓ_4 are given by (30). If all four lines are distinct, then any such L has negative parabolic degree $-2t$ and the parabolic structure is stable. From the explicit formulas in (30) it follows that at most two of the lines ℓ_1, \dots, ℓ_4 can coincide. And two lines coincide if and only if $uv = 0$, $u^2 = v^2$ or $u^2 = -v^2$. In this case we have that there are two complementary lines L and \tilde{L} with parabolic degree zero and \mathcal{P} is polystable but not stable. \square

The parabolic Higgs pair, consisting of the parabolic structure \mathcal{P} with trivial underlying holomorphic structure and the strongly parabolic Higgs field Ψ as in (31), is uniquely determined by the nilpotent $\mathfrak{sl}(2, \mathbb{C})$ -matrix

$$(32) \quad R_1 = \text{res}_{p_1} \Psi = \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix}.$$

In fact, we have

$$(33) \quad \Psi = R_1 \frac{dz}{z - p_1} + R_2 \frac{dz}{z - p_2} + R_3 \frac{dz}{z - p_3} + R_4 \frac{dz}{z - p_4}$$

with

$$(34) \quad R_2 = (CD)^{-1} R_1 (CD), \quad R_3 = D^{-1} R_1 D, \quad R_4 = C^{-1} R_1 C$$

for anti-commuting

$$(35) \quad C = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Note that the conjugation of R_1 by C , D or CD leads to a conjugation of the corresponding Higgs field Ψ , which is not true in general for arbitrary $g \in \text{SL}(2, \mathbb{C})$, as g does neither commute nor anti-commute with C and D . In fact, a direct computation gives:

Lemma 30. *Let $0 \neq R_1 \in \mathfrak{sl}(2, \mathbb{C})$ be nilpotent and $g \in \text{SL}(2, \mathbb{C})$. Then the parabolic Higgs fields Ψ and $\tilde{\Psi}$ corresponding to R_1 and $\tilde{R}_1 = g^{-1} R_1 g$ via (33) and (34) are gauge equivalent if and only if*

$$g \in \langle C, D \rangle,$$

where $\langle C, D \rangle$ denote the gauge group generated by conjugation with C and D . Therefore, the group of (non-trivial) transformations preserving the gauge orbit is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Remark 31. *Note that a non-zero stable strongly parabolic Higgs field on $\mathcal{O} \oplus \mathcal{O}$ uniquely determines the quasiparabolic lines. Hence if the parabolic weights are given, the Higgs field induces a parabolic structure. We call a parabolic Higgs field Ψ of the form (31) symmetric.*

As a corollary of the above observations, we have the following classification.

Proposition 32. *For $t \in (0, \frac{1}{4})$ fixed, the moduli space of polystable strongly parabolic Higgs fields over \mathbb{CP}^1 with 4 singular points contains an open dense subset \mathcal{U} which is isomorphic to the holomorphic cotangent bundle $T^*\mathbb{CP}^1$ modulo the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action induced by conjugation by C and D .*

Proof. We define \mathcal{U} to be the (open) subset of the moduli space of polystable strongly parabolic Higgs pairs over $\mathbb{C}P^1$ with 4 singular points on $\mathcal{V} = \mathcal{O} \oplus \mathcal{O}$.

Whenever the Higgs field is non-zero, the parabolic structure is determined by the symmetric Higgs field Ψ . This Higgs field in turn is determined by the non-zero nilpotent residue $R_1 \in \mathfrak{sl}(2, \mathbb{C})$. The matrix R_1 is determined unique up to sign by

$$(u, v) \in \mathbb{C}^2 \setminus \{0\} \mapsto \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix}.$$

Recall that the kernel of R_1 determines the induced quasiparabolic lines via (34) or (30). Moreover, by Lemma 30, the gauge class of a strongly parabolic Higgs field (or the induced parabolic structure) determines R_1 uniquely up to conjugation by the gauge group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by C and D .

When $(u, v) \rightarrow 0$ the induced parabolic structure is determined by the ratio $[u : v]$ in the limit. Thus the moduli space of strongly parabolic structures \mathcal{U} on $\mathcal{O} \oplus \mathcal{O}$ is given by the blow up of $\mathbb{C}^2/\mathbb{Z}_2$ at $(u, v) = 0$ up to taking the quotient by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action. The blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ can then be naturally identified with the cotangent bundle of $\mathbb{C}P^1$. \square

The complement of \mathcal{U} is the complex line of parabolic Higgs fields with underlying holomorphic bundle $\mathcal{V} = \mathcal{O}(1) \oplus \mathcal{O}(-1)$. It is possible to explicitly glue this line to \mathcal{U} and turn the moduli space into a complex orbifold, see for example [36]. We do not give more details of this, since our loop group methods only work on \mathcal{U} as of now.

3.2. Non-abelian Hodge correspondence in the strongly parabolic case. In this section we recall Simpson's non-abelian Hodge correspondence [42] in the case of rank two strongly parabolic Higgs bundles and non-trivial weights, see also [18] or [31].

The space of hermitian metrics with determinant one is naturally diffeomorphic to the hyperbolic 3-space. Thus we can measure the distance between two metrics as the distance of the two corresponding points in the hyperbolic space using the metric d . Let \mathcal{P} be a parabolic structure on a holomorphic rank two bundle \mathcal{V} with trivial determinant bundle, z be a local holomorphic coordinate centered at a singular point $p_j \in \Sigma$, and let $\ell_j \subset \mathcal{V}_{p_j}$ be the parabolic line and $\alpha_j \in (0, \frac{1}{2})$ be the corresponding parabolic weight. Choose a local holomorphic frame (s_1, s_2) of \mathcal{V} on $U_j \ni p_j$ such that

$$s_1 \wedge s_2 = 1 \in H^0(U_j, \Lambda^2 \mathcal{V}) = H^0(U_j, \mathcal{O})$$

and

$$s_1(p_j) \in \ell_j.$$

Consider the hermitian metric h_j given by

$$h_j = \begin{pmatrix} (z\bar{z})^{\alpha_j} & 0 \\ 0 & (z\bar{z})^{-\alpha_j} \end{pmatrix}$$

with respect to the holomorphic frame (s_1, s_2) . We call h_j a model metric with respect to the parabolic structure at p_j . A hermitian metric h on \mathcal{V} over $\Sigma \setminus \{p_1, \dots, p_n\}$ is called *tame* at p_j with respect to the parabolic structure if and only if

$$d(h, h_j) \text{ is bounded on } U_j \setminus \{p_j\}.$$

Since any two model metrics have finite distance to each other the notion of tameness is independent of the choice of holomorphic frame with the above properties.

Definition 33. Let \mathcal{P} be a parabolic structure on a holomorphic vector bundle \mathcal{V} over Σ with singular divisor $p_1 + \dots + p_n$. A hermitian metric on a holomorphic bundle \mathcal{V} over $\Sigma \setminus \{p_1, \dots, p_n\}$ is called *tame with respect to \mathcal{P}* if it is tame at all of its singular points p_j .

Given a tame hermitian metric h and a holomorphic structure $\bar{\partial}$ we denote its Chern connection on $\Sigma \setminus \{p_1, \dots, p_n\}$ by $D^h = D^{\bar{\partial}, h}$. Similarly, for a strongly parabolic Higgs field Ψ we denote its hermitian conjugate by $\Psi^* = \Psi^{*, h} \in \Gamma(\Sigma \setminus \{p_1, \dots, p_n\}, \overline{K}\mathfrak{sl}(\mathcal{V}))$.

FIGURE 1. Two views on an equivariant minimal surface in \mathbb{H}^3 obtained from a strongly parabolic nilpotent Higgs field on \mathbb{CP}^1 with four singular points and parabolic weight $t = \alpha_j = \frac{1}{6}$. Image by Nick Schmitt.

Definition 34. Let (\mathcal{P}, Ψ) be a polystable strongly parabolic Higgs pair. A metric h is called tame harmonic with respect to the Higgs pair if the metric is tame with respect to \mathcal{P} and if the connection

$$\nabla := D^h + \Psi + \Psi^{*,h}$$

is flat.

A tame harmonic metric gives a solution of Hitchin's self-duality equations over $\Sigma \setminus \{p_1, \dots, p_n\}$ with appropriate growth conditions at the punctures. The notation of a (tame) harmonic metric is motivated by the fact that there is a direct link to harmonic maps into hyperbolic 3-space: let $U \subset \Sigma \setminus \{p_1, \dots, p_n\}$ be open and $(s_1, s_2): U \rightarrow \mathrm{SL}(2, \mathbb{C})$ be a parallel frame with respect to the flat connection ∇ . Then, the map

$$(h(s_i, s_j))_{i,j}: U \rightarrow \{A \in \mathrm{SL}(2, \mathbb{C}) \mid \bar{A}^T = A \text{ and } A > 0\} \cong \mathbb{H}^3$$

is a harmonic map to hyperbolic 3-space \mathbb{H}^3 . Globally, this gives an equivariant harmonic map (with respect to the monodromy representation of the flat connection ∇) from the universal covering of $\Sigma \setminus \{p_1, \dots, p_n\}$ into the hyperbolic 3-space. We can now state the non-abelian Hodge correspondence for the parabolic case (pNAH).

Theorem 6. [44] *For every polystable strongly parabolic Higgs pairs with singular divisor $D = p_1 + \dots + p_n$ and parabolic weights $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in (0, \frac{1}{2})$ there is a tame harmonic metric h . The tame harmonic metric is unique if the Higgs pair is stable. This induces a bijection between the space of polystable strongly parabolic Higgs pairs with singular divisor D and parabolic weights α modulo isomorphisms and the space of totally reducible flat logarithmic connections with singularities at p_j of conjugacy class (24) modulo gauge transformations by associating*

$$(\mathcal{V}, \mathcal{P}, \Psi) \mapsto \nabla := D^h + \Psi + \Psi^{*,h}.$$

The parabolic structure induced by the logarithmic connection $\nabla = D^h + \Psi + \Psi^{*,h}$ is not isomorphic to the parabolic structure underlying the stable strongly parabolic Higgs field in general, but the parabolic weights are the same. Just as in the compact case, we obtain from a tame harmonic metric h for a strongly parabolic Higgs pair $(\mathcal{V}, \mathcal{P}, \Psi)$ an associated \mathbb{C}^* family of flat connections

$$(36) \quad \lambda \in \mathbb{C}^* \mapsto \nabla^\lambda := D^h + \lambda^{-1}\Psi + \lambda\Psi^{*,h}.$$

It follows from Simpson's construction [44] (see also [45]) that for all $\lambda \in \mathbb{C}^*$, the connections ∇^λ extends naturally to logarithmic connections on Σ with the same parabolic weights as the initial parabolic structure \mathcal{P} . In particular, the parabolic weights of ∇^λ are independent of $\lambda \in \mathbb{C}^*$.

3.2.1. Rational weights. In the case of rational parabolic weights

$$\alpha_1 = \frac{l_1}{k_1}, \dots, \alpha_n = \frac{l_n}{k_n} \in \mathbb{Q} \cap (0, \frac{1}{2}),$$

Theorem 6 is directly linked to the non-abelian Hodge correspondence on compact Riemann surfaces through coverings of Σ , see [38, Section 3 and Section 5] for details. The underlying geometric idea is that the equivariant harmonic map to hyperbolic 3-space obtained from the tame harmonic metric associated to a polystable strongly parabolic Higgs pair is actually the equivariant harmonic map associated to a polystable Higgs pair on some compact surface, see Figure 1. We will explain the details here only in the case of Σ being the 4-punctured sphere and parabolic weights $t = \alpha_j = \frac{l}{k}$, $l, k \in \mathbb{N}$, $2l < k$, but these constructions also work in the general case of rational weights, see [38].

Consider the k -fold covering of \mathbb{CP}^1 defined by the equation

$$(37) \quad \Sigma_k: y^k = \frac{(z - p_1)(z - p_3)}{(z - p_2)(z - p_4)},$$

which is totally branched over p_1, \dots, p_4 , and let $\pi = z: \Sigma_k \rightarrow \mathbb{CP}^1$ denote the covering map. Note that Σ_k is a compact Riemann surface of genus $k-1$ and there is an action of the finite abelian group \mathbb{Z}_k on Σ_k such that $\Sigma_k/\mathbb{Z}_k = \mathbb{CP}^1$. Let $\mathcal{V} \rightarrow \mathbb{CP}^1$ be the underlying holomorphic bundle of the parabolic structure \mathcal{P} . By abuse of notation we also denote its pull-back bundle by $\mathcal{V} = \pi^*\mathcal{V} \rightarrow \Sigma_k$. We construct a new holomorphic bundle $\hat{\mathcal{V}} \rightarrow \Sigma_k$ as follows. Let $x: U_j \subset \Sigma_k \rightarrow \mathbb{C}$ be a local holomorphic coordinate centered at $\pi^{-1}(p_j)$ satisfying $\sigma^*x = e^{\frac{2\pi i}{k}}x$ for a generator σ of the \mathbb{Z}_k -action. Let $g_j: \mathcal{V}|_{U_j} \rightarrow \mathbb{C}^2$ be a holomorphic $\mathrm{SL}(2, \mathbb{C})$ trivialisation of \mathcal{V} over U such that $g_j(\ell_j) = \mathbb{C}(1, 0)$, where ℓ_j is the parabolic line at $\pi^{-1}(p_j)$. Then, $\hat{\mathcal{V}}$ is the holomorphic bundle which is \mathcal{V} over $\Sigma_k \setminus \pi^{-1}\{p_1, \dots, p_4\}$ and over U_j the $\mathrm{SL}(2, \mathbb{C})$ trivialisation $\hat{g}_j: \hat{\mathcal{V}}|_{U_j} \rightarrow \mathbb{C}^2$ is defined by

$$(38) \quad (g_j \circ \hat{g}_j^{-1})|_{U_j \setminus \pi^{-1}(p_j)} = \begin{pmatrix} x^{-l} & 0 \\ 0 & x^l \end{pmatrix}.$$

Note that $\hat{\mathcal{V}}$ has trivial determinant bundle. Moreover, the pull-back of the Higgs field $\pi^*\Psi$ is given by

$$g_j \circ \pi^*\Psi \circ g_j^{-1} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \frac{dx}{x} + A_1 x^{k-1} dx + \text{higher order terms in } x$$

for some $a \in \mathbb{C}$ and $A_1, \dots \in \mathfrak{sl}(2, \mathbb{C})$. Thus by conjugating with $g_j \circ \hat{g}_j^{-1}$ on $U_j \setminus \pi^{-1}(p_j)$ the Higgs field extends smoothly to p_j and gives rise to $\hat{\Psi} \in H^0(\Sigma_k, K\mathfrak{sl}(\hat{\mathcal{V}}))$. By construction $\hat{\mathcal{V}}$ has a natural non-trivial \mathbb{Z}_k -action extending

$$\hat{\sigma}: U_j \times \mathbb{C}^2 \rightarrow U_j \times \mathbb{C}^2; \left(p, \begin{pmatrix} a \\ b \end{pmatrix}\right) \mapsto (\sigma(p), \begin{pmatrix} e^{\frac{2\pi i l}{k}} a \\ e^{-\frac{2\pi i l}{k}} b \end{pmatrix})$$

in the trivialisation \hat{g}_j , and $\hat{\Psi}$ is equivariant:

$$\sigma^*\hat{\Psi} = \hat{\sigma}^{-1}\hat{\Psi}\hat{\sigma}.$$

Furthermore, let ∇ be a logarithmic connection with induced parabolic structure \mathcal{P} . Recall that this implies that

$$\mathrm{res}_{p_j} \nabla|_{\ell_j} = \frac{l}{k} \mathrm{Id}_{\ell_j}.$$

A short computation similar to the case for a strongly parabolic Higgs field then shows that $\pi^*\nabla$ gives rise to an equivariant global holomorphic connection $\hat{\nabla}$ on $\hat{\mathcal{V}} \rightarrow \Sigma_k$

$$\sigma^*\hat{\nabla} = \hat{\nabla}.\hat{\sigma}.$$

We will refer to both operations $(\mathcal{V}, \mathcal{P}, \Psi) \mapsto (\hat{\mathcal{V}}, \hat{\Psi})$ and $\nabla \mapsto \hat{\nabla}$ as *twisted lifts*. Note that gauge equivalent strongly parabolic Higgs pairs (or respectively logarithmic connections) have gauge equivalent twisted lifts. In other words, let $\mathcal{M}(\Sigma_k)$ and $\mathcal{M}(\mathbb{CP}^1)$ denote the moduli spaces of (parabolic) Higgs bundles or flat (logarithmic) connections on Σ_k and the 4-punctured \mathbb{CP}^1 , respectively, then

$$(39) \quad \hat{\cdot}: \mathcal{M}(\mathbb{CP}^1) \rightarrow \mathcal{M}(\Sigma_k)$$

is a well-defined and smooth map between these moduli spaces. Every polystable strongly parabolic Higgs pair (\mathcal{P}, Ψ) gives rise to a tame harmonic metric h , see Theorem 6. For rational weights this can be deduced directly from the fact that a stable and \mathbb{Z}_k -symmetric Higgs pair $(\hat{\mathcal{V}}, \hat{\Psi})$ on Σ_k gives rise to a harmonic metric \hat{h} which is, by uniqueness of the solution, equivariant with respect to the \mathbb{Z}_k -action. Then twisting leads to an invariant harmonic metric h on π^*V descending to the quotient 4-punctured sphere. Using (38), it can be shown that the metric h on the quotient is tame.

In particular, we obtain a commutative non-abelian Hodge diagram

$$(40) \quad \begin{array}{ccc} \{(\mathcal{P}, \Psi) \mid \text{polystable}\} / \sim & \xrightarrow{\text{twisted lift}} & \{(\hat{\mathcal{V}}, \Psi) \mid \text{polystable}\} / \sim \\ \downarrow \text{pNAH} & & \downarrow \text{NAH} \\ \{\nabla \mid \text{logarithmic and totally reducible}\} / \sim & \xrightarrow{\text{twisted lift}} & \{\hat{\nabla} \mid \text{totally reducible}\} / \sim \end{array}$$

where \sim denotes modulo isomorphism/gauge equivalence leading to

Theorem 7. *Let $(\mathcal{V}, \mathcal{P}, \Psi)$ be a polystable strongly parabolic Higgs pair with associated family of flat logarithmic connections $\nabla^\lambda = D^h + \lambda^{-1}\Psi + \lambda\Psi^*$. Then, the associated family $D^{\hat{h}} + \lambda^{-1}\hat{\Psi} + \lambda\hat{\Psi}^*$ of the twisted lift $(\hat{\mathcal{V}}, \hat{\Psi})$ is (gauge equivalent) to the twisted lift $\hat{\nabla}^\lambda$ of the family ∇^λ .*

3.3. The parabolic Deligne-Hitchin moduli space. Analogously to the case of compact Riemann surfaces, Deligne-Hitchin moduli spaces can be defined in the strongly parabolic setup, see [3] and [45] for more details. We restrict ourselves again to the $\text{SL}(2, \mathbb{C})$ case with fixed local conjugacy classes.

Definition 35. *Let Σ be a compact Riemann surface and $p_1, \dots, p_n \in \Sigma$ be pairwise distinct points. A parabolic $\text{SL}(2, \mathbb{C})$ λ -connection with singular divisor $D = p_1 + \dots + p_n$ is a quadruple $(\lambda, \mathcal{V}, \mathcal{P}, \mathcal{D})$ consisting of a holomorphic $\text{SL}(2, \mathbb{C})$ bundle \mathcal{V} over Σ with a trace-free parabolic structure \mathcal{P} and holomorphic trace-free λ -connection \mathcal{D} on $\mathcal{V} \mid_{\Sigma \setminus \{p_1, \dots, p_n\}}$ such that with respect to any local holomorphic frame of \mathcal{V} around p_j , the λ -connection 1-form of \mathcal{D} is meromorphic with first order pole at p_j only, and such that the quasiparabolic line ℓ_j at p_j is an eigenline of the residue $\text{res}_{p_j} \mathcal{D}$ with eigenvalue $(\lambda\alpha_j)$, where α_j is the parabolic weight at p_j .*

For $\lambda \neq 0$, $\frac{1}{\lambda}\mathcal{D} + \bar{\partial}^\mathcal{V}$ is a logarithmic connection with parabolic structure \mathcal{P} . For $\lambda = 0$, \mathcal{D} is a strongly parabolic Higgs field for the parabolic structure \mathcal{P} .

A parabolic λ -connection is stable (semi-stable) if every \mathcal{D} -invariant holomorphic line subbundle has negative (non-positive) parabolic degree, and unstable otherwise. As before, we also have the notion of polystability and for $\lambda \neq 0$, there are no unstable parabolic λ -connections.

An isomorphism between two parabolic λ -connections $(\lambda, \mathcal{V}_1, \mathcal{P}_1, \mathcal{D}_1)$ and $(\lambda, \mathcal{V}_2, \mathcal{P}_2, \mathcal{D}_2)$ is given by a $\text{SL}(2, \mathbb{C})$ gauge transformation g between the two parabolic structures $(\mathcal{V}_1, \mathcal{P}_1)$ and $(\mathcal{V}_2, \mathcal{P}_2)$ such that $\mathcal{D}_2 \circ g = g \circ \mathcal{D}_1$. The only automorphisms of a stable parabolic λ -connection are $\pm \text{Id}$.

In order to construct a moduli space, we fix the Riemann surface Σ , the divisor D and the parabolic weights, but neither the holomorphic bundle nor the quasiparabolic lines. Then, the parabolic Hodge moduli space is the space of all polystable λ -connections with $\lambda \in \mathbb{C}$ modulo isomorphism. It is denoted by $\mathcal{M}_{\text{Hod}}^{D, \alpha_1, \dots, \alpha_n}(\Sigma)$, or by $\mathcal{M}_{\text{Hod}}^{\text{par}}(\Sigma)$ for short. It is a smooth complex manifold at its stable locus, and it fibers over \mathbb{C} .

Let ∇ be a logarithmic $\text{SL}(2, \mathbb{C})$ connection on Σ with singular divisor D and induced parabolic structure \mathcal{P} . Let $\bar{\Sigma}$ be the complex conjugate Riemann surface, with divisor $\bar{D} = p_1 + \dots + p_n$. Then ∇ is a flat connection on $\bar{\Sigma} \setminus \{p_1, \dots, p_n\}$. By Deligne extension, there is a logarithmic $\text{SL}(2, \mathbb{C})$ connection $\tilde{\nabla}$ on $\bar{\Sigma}$ with the same parabolic weights as ∇ , which is gauge equivalent to ∇ on the smooth surface $\Sigma \setminus \{p_1, \dots, p_n\} = \bar{\Sigma} \setminus \{p_1, \dots, p_n\}$. More explicitly, choose a local holomorphic coordinate around $p_j \in \Sigma$ and a trivialization of \mathcal{V} such that the connection ∇ is given by

$$\nabla = d + \begin{pmatrix} \alpha_j & 0 \\ 0 & -\alpha_j \end{pmatrix} \frac{dz}{z},$$

and choose a gauge in this trivialization given by $g_j = \text{diag}((z\bar{z})^{-\alpha_j}, (z\bar{z})^{\alpha_j})$, then the logarithmic connection $\tilde{\nabla}$ on $\bar{\Sigma}$ is given by

$$\tilde{\nabla} = \nabla \cdot g_j = d + \begin{pmatrix} -\alpha_j & 0 \\ 0 & \alpha_j \end{pmatrix} \frac{d\bar{z}}{\bar{z}}$$

around $\bar{p}_j \in \bar{\Sigma}$ in the same trivialization. It is easy to check that this gives a well-defined map from the moduli space of logarithmic connections $\mathcal{M}_{dR}^{D, \alpha_1, \dots, \alpha_n}(\Sigma)$ on Σ to the moduli space of logarithmic connections $\mathcal{M}_{dR}^{\bar{D}, \alpha_1, \dots, \alpha_n}(\bar{\Sigma})$ on $\bar{\Sigma}$. Furthermore, this map can be extended to $\lambda \in \mathbb{C}^*$ to identify the Hodge moduli spaces of Σ and $\bar{\Sigma}$ restricted to \mathbb{C}^*

$$\mathcal{G}: \mathcal{M}_{Hod}^{par}(\Sigma) |_{\mathbb{C}^*} = \mathbb{C}^* \times \mathcal{M}_{dR}^{D, \alpha_1, \dots, \alpha_n}(\Sigma) \longrightarrow \mathbb{C}^* \times \mathcal{M}_{dR}^{\bar{D}, \alpha_1, \dots, \alpha_n}(\bar{\Sigma}) = \mathcal{M}_{Hod}^{par}(\bar{\Sigma}) |_{\mathbb{C}^*}.$$

As in the smooth case, \mathcal{G} maps the stable (and smooth) locus to the stable (and smooth) locus. In this way, we obtain the parabolic Deligne-Hitchin moduli space

$$\mathcal{M}_{DH}^{par}(\Sigma, D, \alpha_1, \dots, \alpha_n) = \mathcal{M}_{Hod}^{par}(\Sigma) \cup_{\mathcal{G}} \mathcal{M}_{Hod}^{par}(\bar{\Sigma}) \longrightarrow \mathbb{C}P^1.$$

If $\alpha_1 = \dots = \alpha_n = t$ we will denote the space by $\mathcal{M}_{DH}^{par}(\Sigma, D, \alpha_1, \dots, \alpha_n) = \mathcal{M}_{DH}^{par}(\Sigma, D, t)$. The parabolic Deligne-Hitchin moduli space is naturally equipped with a holomorphic \mathbb{C}^* action covering

$$\mu \in \mathbb{C}^* \mapsto (\lambda \in \mathbb{C}P^1 \mapsto \mu\lambda \in \mathbb{C}P^1).$$

As before, we denote the automorphism obtained by multiplying with $\mu = -1$ by N . Furthermore, there exists a natural real structure C . In order to define C we recall that the underlying complex rank 2 vector bundle has a fixed C^∞ trivialization $V = \Sigma \times \mathbb{C}^2$ with complex conjugation $c: V \rightarrow \bar{V} \cong V$. For a linear differential operator A let $\bar{A} := c^{-1} \circ A \circ c$. Thus if A is complex linear, \bar{A} is also complex linear but in general not gauge equivalent to A . If $\bar{\partial}$ is a holomorphic structure on $V \rightarrow \Sigma$, then $\bar{\bar{\partial}}$ is a holomorphic structure on $V \rightarrow \bar{\Sigma}$.

Lemma 36. *If $(\lambda, \bar{\partial}, \mathcal{P}, \mathcal{D})$ is a parabolic λ -connection on Σ with singular divisor $D = p_1 + \dots + p_n$, then*

$$(\bar{\lambda}, \bar{\bar{\partial}}, \bar{\mathcal{P}}, \bar{\mathcal{D}})$$

is a parabolic λ -connection on $\bar{\Sigma}$, where $\bar{\mathcal{P}}$ is given by the quasiparabolic lines $\bar{\ell}_j = c(\ell_j)$ and the parabolic weights $\bar{\alpha}_j = \alpha_j$.

If two parabolic λ -connections are isomorphic via a gauge g then their complex conjugate λ -connections on $\bar{\Sigma}$ are isomorphic via the gauge $c^{-1} \circ g \circ c$. In particular, there exists an anti-holomorphic involution C of \mathcal{M}_{DH}^{par} covering $\lambda \mapsto \bar{\lambda}^{-1}$ and commuting with N .

Example 37. *Let $(\lambda, \bar{\partial}, \mathcal{P}, \mathcal{D})$ be a parabolic λ -connection on Σ with $\lambda \in S^1$ such that $\bar{\partial} + \frac{1}{\lambda}\mathcal{D}$ is unitary. Then, $[\lambda, \bar{\partial}, \mathcal{P}, \mathcal{D}] \in \mathcal{M}_{DH}^{par}$ is a fixed point of C .*

For a given polystable strongly parabolic Higgs pair $p = (\mathcal{P}, \Psi)$ together with its tame harmonic metric h , the associated family of flat connections (36) gives rise to a holomorphic section s_p of \mathcal{M}_{DH}^{par} called twistor line, see [45, Theorem 2.7] for details. With respect to the real structure $\mathcal{T} = CN = NC$ this section is real, i.e., $\mathcal{T}(s_p(-\bar{\lambda}^{-1})) = s_p(\lambda)$ for all $\lambda \in \mathbb{C}^*$.

We now discuss the necessary and sufficient conditions of a real holomorphic section s of \mathcal{M}_{DH}^{par} to be a twistor line. This is motivated by (and analogous to) the compact case described in Section 2.

Lemma 38. *Let $\epsilon > 0$. Let $\lambda \in \mathbb{D}_{1+\epsilon}^* \mapsto \nabla^\lambda$ be a holomorphic family of irreducible logarithmic $SL(2, \mathbb{C})$ connections with first order pole at $\lambda = 0$ and $\Psi = \text{res}_{\lambda=0} \nabla^\lambda \in \Gamma(K_{\Sigma} \mathfrak{sl}(\mathcal{V}(\lambda)))$ on a family of holomorphic bundles $\mathcal{V}(\lambda) \rightarrow \Sigma$ with λ -independent parabolic divisor $p_1 + \dots + p_n$ and parabolic weights $\alpha_j \in (0, \frac{1}{2})$.*

Assume that there exists $g = g(\lambda): \Sigma \setminus \{p_1, \dots, p_n\} \rightarrow \Lambda SL(2, \mathbb{C})$ such that

$$(41) \quad \overline{\nabla^{-\bar{\lambda}^{-1}}} = \nabla^\lambda \cdot g(\lambda).$$

Then ∇^λ gives rise to a real holomorphic section s of the parabolic Deligne-Hitchin moduli space.

Proof. Note that $\lambda \mapsto \nabla^\lambda$ gives rise to a local section of \mathcal{M}_{DH}^{par} over $\mathbb{D}_{1+\epsilon}^*$ via

$$\mathbb{D}_{1+\epsilon}^* \times \mathcal{M}_{dR}^{D, \alpha_1, \dots, \alpha_n} \subset \mathcal{M}_{Hod}^{par}(\Sigma) |_{\mathbb{C}^*} \subset \mathcal{M}_{Hod}^{par}(\Sigma) \subset \mathcal{M}_{DH}^{par}(\Sigma).$$

We first extend this section to $\lambda = 0$. Consider the holomorphic structure $\bar{\partial}_{\mathcal{V}}$. Then, Ψ is meromorphic with at most first order poles at p_1, \dots, p_n . By assumption the eigenvalues $\pm\alpha_j$ of $\text{res}_{p_j} \nabla^\lambda$ are independent of λ . This implies that $\text{res}_{p_j} \Psi$ is nilpotent for all $j = 1, \dots, n$. We define the quasiparabolic line at p_j as follows: if $\text{res}_{p_j}(\Psi) \neq 0$, then $\ell_j := \ker(\text{res}_{p_j}(\Psi))$, else we define the line to be $\ell_j := \text{Eig}((\text{res}_{p_j} \nabla) |_{\lambda=0, \alpha_j})$. This gives rise to a parabolic structure \mathcal{P} on \mathcal{V} with Ψ being a compatible strongly parabolic Higgs field. To extend the section to $\lambda = 0$, we need to ensure that the strongly parabolic Higgs pair (\mathcal{P}, Ψ) is polystable.

Remark 39. *In this paper, all $\mathbb{D}_{1+\varepsilon}^*$ -families of logarithmic connections considered will induce stable strongly parabolic Higgs pairs at $\lambda = 0$. Thus there is no further gauge necessary to obtain a proper section of the Deligne-Hitchin moduli space.*

Assume the parabolic Higgs pair (\mathcal{P}, Ψ) is unstable, then there exists an invariant holomorphic line bundle L with respect to $\bar{\partial}_{\mathcal{V}}$ (on Σ) of positive parabolic degree. Consider a complementary complex line bundle \tilde{L} , which is not holomorphic with respect to $\bar{\partial}_{\mathcal{V}}$ in general. Define the holomorphic family of gauge transformations by $\lambda \in \mathbb{C}^* \mapsto \tilde{g}(\lambda) = \text{diag}(1, \lambda)$ with respect to the C^∞ decomposition $V = L \oplus \tilde{L}$. Then, it can be shown analogously to [22, Section 1.4] that the family of logarithmic connections

$$(42) \quad \tilde{\nabla}^\lambda := \nabla^\lambda \tilde{g}(\lambda) = \lambda^{-1} \tilde{\Psi} + \tilde{\nabla} + \text{higher order terms in } \lambda$$

is gauge equivalent to ∇^λ for $\lambda \neq 0$ and gives rise to a stable parabolic structure at $\lambda = 0$ with strongly parabolic Higgs field $\tilde{\Psi}$.

If the parabolic structure (\mathcal{P}, Ψ) is semi-stable but not polystable, we can replace the Higgs pair at $\lambda = 0$ by a polystable pair in the same gauge orbit. (This case does not lead to twistor lines, but it does neither occur for the real holomorphic sections constructed by in this paper. In fact, either the parabolic Higgs field is non-vanishing and induces a stable parabolic Higgs pair by Lemma 28, or the Higgs field vanishes and the section of the parabolic Deligne-Hitchin moduli space is induced by a unitary logarithmic connection by Theorem 9 below.)

To extend the section to \mathbb{CP}^1 recall that the underlying C^∞ trivialization of the holomorphic bundles $\mathcal{V}(\lambda)$ is of fixed type, so that the complex conjugation c is well-defined. Then, by Lemma 36 and (41), s extends holomorphically to $\{\lambda \in \mathbb{C} \cup \{\infty\} \mid |\lambda| > \frac{1}{1+\varepsilon}\}$, i.e., s gives rise to a global holomorphic section of $\mathcal{M}_{DH}^{par}(\Sigma) \rightarrow \mathbb{CP}^1$, which is real by (41). \square

We call the transformation $\nabla^\lambda \mapsto \tilde{\nabla}^\lambda$ in (42) the *twist* of the family ∇^λ . The twist transformation can also be applied if the Higgs pair is stable. In fact, assume that ∇^λ is the associated family of a tame harmonic metric for some nilpotent strongly parabolic Higgs field, which can be interpreted as an equivariant conformal harmonic map f , i.e., a minimal surface, into the hyperbolic 3-space. Then, the twisted family $\tilde{\nabla}^\lambda$ corresponds to the associated family of the equivariant harmonic conformal Gauss map of f into the deSitter 3-space, see [22].

In the following, we will always assume, without loss of generality, that we start with a family of logarithmic connections ∇^λ inducing a stable (or polystable) strongly parabolic Higgs pair, without applying the twist transformation first. In particular, if $\nabla^\lambda = \lambda^{-1} \Psi + \nabla + \lambda \Psi_1 + \dots$ such that the determinant of the strongly parabolic Higgs field Ψ has only simple zeros or poles, the corresponding Higgs pair must be stable, as it cannot have a holomorphic Ψ -invariant line bundle.

Proposition 40. *Let s be the real holomorphic section of \mathcal{M}_{DH}^{par} given by a family of irreducible logarithmic connections ∇^λ as in Lemma 38 together with the gauge g satisfying (41). Then, necessary conditions for s to be a twistor line are*

- g_x lies in the big cell for all $x \in \Sigma \setminus \{p_1, \dots, p_n\}$, i.e., $g_x = g_x^+ g_x^-$ for some $g_x^+ \in \Lambda_+ \text{SL}(2, \mathbb{C})$ and $g_x^- \in \Lambda_- \text{SL}(2, \mathbb{C})$,
- $g(\lambda) \overline{g(-\bar{\lambda}^{-1})} = -\text{Id}$.

Proof. If the family of logarithmic connections ∇^λ induces a twistor line, then ∇^λ is gauge equivalent to the associated family of flat connections of a self-duality solution (36) by a positive gauge \tilde{g} , i.e., $\tilde{g}_x \in \Lambda_+ \mathrm{SL}(2, \mathbb{C})$ for all $x \in \Sigma \setminus \{p_1, \dots, p_n\}$. Since by Example 15 associated families are negative sections and the gauge is given by (12), we obtain by (41) and by irreducibility of ∇^λ that $\tilde{g}^{-1}g \in \Lambda_- \mathrm{SL}(2, \mathbb{C})$. The second condition can be derived as in the compact case, for details see [22, Section 1.3]. \square

The next proposition shows that being a twistor line is an open condition.

Proposition 41. *Let s be a twistor line of $\mathcal{M}_{DH}^{par}(\Sigma, D, \alpha_1, \dots, \alpha_n)$ with rational parabolic weights $\alpha_j \in (0, \frac{1}{4}) \cap \mathbb{Q}$. Then there exists an open neighborhood U around s in the space of the real holomorphic sections of $\mathcal{M}_{DH}^{par}(\Sigma, D, \alpha_1, \dots, \alpha_n)$ such that every $\tilde{s} \in U$ is a twistor line.*

Proof. Consider the covering surface $\hat{\Sigma} \rightarrow \Sigma$ which branches over p_1, \dots, p_n of appropriate order. Over $\hat{\Sigma}$ we obtain, after applying the twisted lift construction, a holomorphic family of smooth connections

$$\hat{\nabla}^\lambda = \lambda^{-1} \hat{\Psi} + \hat{\nabla} + \text{higher order terms in } \lambda,$$

see Theorem 7, such that the induced Higgs pair at $\lambda = 0$ is stable. Since s is a twistor line of $\mathcal{M}_{DH}^{par}(\Sigma, D, \alpha_1, \dots, \alpha_n)$, the family $\hat{\nabla}^\lambda$ on $\hat{\Sigma}$ gives rise to a twistor line of $\mathcal{M}_{DH}(\hat{\Sigma})$ by Theorem 7 and the gauge g with

$$\hat{\nabla}^\lambda . g = \overline{\hat{\nabla}^{-\bar{\lambda}^{-1}}}$$

possesses a global corresponding loop group factorization $g = g^+ g^-$ on $\hat{\Sigma}$.

Let \tilde{s} be another real holomorphic section of $\mathcal{M}_{DH}^{par}(\Sigma, D, \alpha_1, \dots, \alpha_n)$ lying in an appropriate open neighborhood of s . Then its twisted lift $\hat{\tilde{s}}$ is a holomorphic section of $\mathcal{M}_{DH}(\hat{\Sigma})$ which can be represented by a family $\hat{\tilde{\nabla}}^\lambda$ of flat and smooth connections on $\hat{\Sigma}$. The family $\hat{\tilde{\nabla}}^\lambda$ can be chosen in a way such that for all $\lambda \in K$, where K is a compact subset of \mathbb{C}^* containing the unit circle, $\hat{\tilde{\nabla}}^\lambda$ lies in an appropriate open neighborhood of $\hat{\nabla}^\lambda$ in the space of flat connections. Consequently, the family of $\mathrm{SL}(2, \mathbb{C})$ gauge transformations \tilde{g} satisfying

$$\hat{\tilde{\nabla}}^\lambda . \tilde{g}(\lambda) = \overline{\hat{\tilde{\nabla}}^{-\bar{\lambda}^{-1}}}$$

lies in an open neighborhood of g for all $\lambda \in \{\lambda \in \mathbb{C} \mid \lambda \in K, -\bar{\lambda}^{-1} \in K\}$. By Theorem 5, we obtain that \tilde{g} also lies in the big cell, and using the construction in Section 2.6 we obtain $\hat{\tilde{s}}$ is a twistor line of $\mathcal{M}_{DH}(\hat{\Sigma})$. Then Theorem 7 gives that \tilde{s} is also a twistor line of $\mathcal{M}_{DH}^{par}(\Sigma, D, \alpha_1, \dots, \alpha_n)$ as claimed. \square

Remark 42. *Proposition 41 also holds for non-rational parabolic weights. More generally, the same conclusion holds if we allow the (positive) parabolic weights as well as the conformal structure of Σ and the singular points p_1, \dots, p_n to vary. This follows from the results of [43] combined with [31]. In fact, as a generalization of Theorem 4, the space of twistor lines is open and closed in the space of real holomorphic sections.*

3.4. The symplectic form on the space of logarithmic connections. The symplectic form on the moduli space of flat connections over compact surfaces [1, 16] has been generalized to moduli spaces of flat connections with prescribed local conjugacy classes over punctured Riemann surfaces, see for example [2] or [4]. We provide a short self-contained account to the construction of this symplectic form, and describe its relationship to the symplectic form on a appropriate compact covering surface in the case of rational parabolic weights.

Let Σ be a compact Riemann surface and $p_1, \dots, p_n \in \Sigma$ be pairwise distinct points. Fix $\mathrm{SL}(2, \mathbb{C})$ conjugacy classes

$$\mathcal{C}_1, \dots, \mathcal{C}_n$$

of diagonal matrices $C_1, \dots, C_n \in \mathfrak{sl}(2, \mathbb{C})$ at the punctures such that the eigenvalues of each C_j are contained in $(0, \frac{1}{2})$. Let z_j be a centered holomorphic coordinate at p_j for $j = 1, \dots, n$. For C_1, \dots, C_n fixed let \mathcal{A} denote the infinite dimensional space of all flat $\mathrm{SL}(2, \mathbb{C})$ connections ∇ on $\Sigma^0 := \Sigma \setminus \{p_1, \dots, p_n\}$ which are of the form

$$(43) \quad \nabla = A_j \frac{dz_j}{z_j} + \text{smooth connection},$$

where z_j is a centered coordinate around p_j and $A_j \in \mathbb{C}_j$.

Lemma 43. *Let $X \in \Omega^1(\Sigma^0, \mathfrak{sl}(2, \mathbb{C}))$ be a tangent vector to $\nabla \in \mathcal{A}$. Then, there exists smooth $\xi \in \Gamma(\Sigma, \mathfrak{sl}(2, \mathbb{C}))$ and $\hat{X} \in \Omega^1(\Sigma, \mathfrak{sl}(2, \mathbb{C}))$ such that*

$$X = d^\nabla \xi + \hat{X} \quad \text{and} \quad d^\nabla \hat{X} = 0$$

on the punctured surface Σ^0 .

Proof. Since X is a tangent vector at ∇ to the infinite dimensional space of flat connections, we have in particular $d^\nabla X = 0$. Moreover, X preserves the form (43), and therefore we can write

$$X = [A_j, \xi_j] \frac{dz_j}{z_j} + \tilde{X}$$

around p_j for appropriate $\xi_j \in \mathfrak{sl}(2, \mathbb{C})$ and a smooth \tilde{X} on Σ . Let $\xi \in \Gamma(\Sigma, \mathfrak{sl}(2, \mathbb{C}))$ be a section with $\xi(p_j) = \xi_j$. Then the splitting

$$X = d^\nabla \xi + (X - d^\nabla \xi)$$

is of the required form. \square

Lemma 44. *Let $\nabla \in \mathcal{A}$ with A_j as in (43) and consider two tangent vectors*

$$X = d^\nabla \xi + \hat{X}, \quad Y = d^\nabla \mu + \hat{Y} \in T_\nabla \mathcal{A}$$

for $\xi, \mu \in \Gamma(\Sigma, \mathfrak{sl}(2, \mathbb{C}))$ and $\hat{X}, \hat{Y} \in \Omega^1(\Sigma, \mathfrak{sl}(2, \mathbb{C}))$ as in Lemma 43. Then

$$\frac{1}{8\pi} \int_{\Sigma^0} \mathrm{tr}(X \wedge Y) = \frac{1}{8\pi} \int_{\Sigma} \mathrm{tr}(\hat{X} \wedge \hat{Y}) - i \sum_{j=1}^n \frac{1}{8\mathrm{tr}(A_j^2)} \mathrm{tr}(A_j [\mathrm{Res}_{p_j}(X), \mathrm{Res}_{p_j}(Y)]).$$

Proof. Consider the punctured Riemann surface Σ^0 and centered holomorphic coordinates z_j at the punctures p_j . For $t > 0$ small let

$$\gamma_t^j: S^1 \longrightarrow \Sigma^0; \quad e^{2\pi i \varphi} \longmapsto (z_j)^{-1}(te^{2\pi i \varphi}).$$

When splitting X, Y according to

$$X = d^\nabla \xi + \hat{X}, \quad Y = d^\nabla \mu + \hat{Y}$$

we obtain

$$\int_{\Sigma^0} \mathrm{tr}(X \wedge Y) = \int_{\Sigma^0} \mathrm{tr}(\hat{X} \wedge \hat{Y}) + \mathrm{tr}(d^\nabla \xi \wedge \hat{Y}) + \mathrm{tr}(\hat{X} \wedge d^\nabla \mu) + \mathrm{tr}(d^\nabla \xi \wedge d^\nabla \mu).$$

For the second term in the above expression we have

$$(44) \quad \begin{aligned} \int_{\Sigma^0} \mathrm{tr}(d^\nabla \xi \wedge \hat{Y}) &= \int_{\Sigma^0} d \mathrm{tr}(\xi \hat{Y}) - \int_{\Sigma^0} \mathrm{tr}(\xi(d^\nabla \hat{Y})) \\ &= - \lim_{t \rightarrow 0} \sum_j \int_{\gamma_t^j} \mathrm{tr}(\xi \hat{Y}) = 0 \end{aligned}$$

since $d^\nabla \hat{Y} = 0$ and \hat{Y}, ξ are both smooth on Σ . Analogously, we have

$$\int_{\Sigma^0} \mathrm{tr}(\hat{X} \wedge d^\nabla \mu) = 0.$$

For the last term we compute

$$\begin{aligned}
 \int_{\Sigma^0} \text{tr}(d^\nabla \xi \wedge d^\nabla \mu) &= \int_{\Sigma^0} d \text{tr}(\xi d^\nabla \mu) = - \lim_{t \rightarrow 0} \sum_j \int_{\gamma_t^j} \text{tr}(\xi d^\nabla \mu) \\
 &= - \lim_{t \rightarrow 0} \sum_j \int_{\gamma_t^j} \text{tr}(\xi [A_j, \mu] \frac{dz_j}{z_j}) - \lim_{t \rightarrow 0} \sum_j \int_{\gamma_t^j} \text{tr}(\text{something smooth}) \\
 &= -2\pi i \sum_j \text{tr}(\xi(p_j)[A_j, \mu(p_j)]) = 2\pi i \sum_j \text{tr}(A_j[\xi(p_j), \mu(p_j)]) \\
 &= -2\pi i \sum_j \frac{1}{2 \text{tr}(A_j^2)} \text{tr}(A_j[[A_j, \xi(p_j)], [A_j, \mu(p_j)]]) \\
 &= -2\pi i \sum_j \frac{1}{2 \text{tr}(A_j^2)} \text{tr}(A_j[\text{Res}_{p_j}(X), \text{Res}_{p_j}(Y)])
 \end{aligned}
 \tag{45}$$

as claimed. \square

Define a holomorphic complex bilinear and skew-symmetric form on the (infinite dimensional) tangent space $T_\nabla \mathcal{A}$ by

$$\begin{aligned}
 \mathcal{O}: T_\nabla \mathcal{A} \times T_\nabla \mathcal{A} &\rightarrow \mathbb{C} \\
 (X, Y) &\mapsto \frac{1}{8\pi} \int_{\Sigma^0} \text{tr}(X \wedge Y) + i \sum_j \frac{1}{8 \text{tr}((\text{Res}_{p_j}(\nabla))^2)} \text{tr}(\text{Res}_{p_j}(\nabla)[\text{Res}_{p_j}(X), \text{Res}_{p_j}(Y)]).
 \end{aligned}
 \tag{46}$$

From Lemma 44 we then obtain:

Corollary 45. *For $d^\nabla \xi, Y \in T_\nabla \mathcal{A}$ we have*

$$\mathcal{O}(d^\nabla \xi, Y) = 0.$$

Therefore, the bilinear form \mathcal{O} descends to a well defined holomorphic 2-form on the quotient of \mathcal{A} by gauge transformations.

It can be shown that \mathcal{O} is indeed a symplectic form.

Remark 46. *If the infinitesimal deformation of the flat connection preserves the holomorphic structure, i.e., the tangent vector fields X, Y are both meromorphic, then the first summand of \mathcal{O} in (46) vanishes and only the residue terms remain. For $A \in \mathcal{C}_j$, the complex skew bilinear form*

$$X = [A, \xi], Y = [A, \mu] \in T_A \mathcal{C}_j \mapsto \text{tr}(A[\xi, \mu]) = -\frac{1}{2 \text{tr}(A^2)} \text{tr}(A[X, Y])$$

is well-defined and known as the Kirillov symplectic form on the adjoint orbit \mathcal{C}_j .

3.4.1. Rational weights. Assume that the parabolic weights are all rational. Then, the symplectic form \mathcal{O} and the Goldman symplectic form coincide up to scaling and taking the twisted lift. We give the proof in the case of the 4-punctured sphere, with rational weights $\alpha_1 = \dots = \alpha_4 = \frac{l}{k}$. As before, consider the k -fold covering $\Sigma_k \rightarrow \mathbb{CP}^1$ given by (37) which totally branches over the singular points.

Proposition 47. *Let $\mathcal{M}(\mathbb{CP}^1)$ be the space of logarithmic connections on \mathbb{CP}^1 with singular points p_1, \dots, p_4 and rational weights $\alpha_1 = \dots = \alpha_4 = \frac{l}{k}$ equipped with its symplectic form \mathcal{O} . Let $\pi: \Sigma_k \rightarrow \mathbb{CP}^1$ be the k -fold covering given by (37) which totally branches over the singular points, and let $\mathcal{M}(\Sigma_k)$ be the moduli space of flat connections on Σ_k with Goldman symplectic form Ω . Then*

$$32\pi k \mathcal{O} = \Omega.$$

Proof. Because of Corollary 45 \mathcal{O} is well-defined on the moduli space $\mathcal{M}(\mathbb{CP}^1)$, and for tangent vectors X, Y , we can choose representatives that vanish in an open set $U \subset \mathbb{CP}^1$ containing all the singular points. Then, only the first summand in (46) contributes to $\mathcal{O}(X, Y)$. On the other hand, the pull-back

of π^*X and π^*Y to Σ_k vanish on $\pi^{-1}(U)$, and the tangent vectors \hat{X} and \hat{Y} of $\mathcal{M}(\Sigma_k)$ are represented by π^*X and π^*Y in the trivialization of the holomorphic bundle $\hat{\mathcal{V}} \cong \mathcal{V}$ over $\Sigma_k \setminus \pi^{-1}\{p_1, \dots, p_4\}$. Since $\pi: \Sigma_k \rightarrow \mathbb{CP}^1$ is of degree k , and due to the different scalings in (17) and (46), the result follows. \square

Remark 48. *Proposition 47 directly generalizes to λ -connections. As a consequence, we are able to compute the twisted holomorphic symplectic form (4) on the open dense subset of $\mathcal{M}_{DH}(\mathbb{CP}^1, p_1 + \dots + p_4, t)$ consisting of Fuchsian λ -connections.*

4. INITIAL CONDITIONS AT $t = 0$.

In this section we write down λ -dependent connection 1-forms on the 4-puncture sphere $\mathbb{CP}^1 \setminus \{p_1, \dots, p_4\}$ depending on a parabolic weight $\alpha = t \in [0, \frac{1}{4})$ and a strongly parabolic Higgs field. We then explicitly compute the initial conditions at $t = 0$ which we will deform via an implicit function theorem argument in Section 5 to obtain equivariant tame harmonic maps from $\mathbb{CP}^1 \setminus \{p_1, \dots, p_4\}$ into \mathbb{H}^3 for $t > 0$.

4.1. The potential. To construct real holomorphic sections explicitly, we restrict to the case where the underlying holomorphic structure is trivial for all $\lambda \in \mathbb{D}_a = \{\lambda \mid |\lambda|^2 < a^2\}$ for some $a > 1$. This restriction is motivated by the fact that an open and dense subset of the moduli space of strongly parabolic Higgs field (respectively logarithmic connections) has trivial underlying holomorphic bundle, see Proposition 32 for strongly parabolic Higgs fields (and Lemma 23 for logarithmic connections). In general (e.g. for large Higgs field with trivial underlying holomorphic structure), the associated family of flat connections does not have trivial underlying holomorphic bundle for all λ in the punctured unit disc. On the other hand, for fixed weight t and fixed compact subset K of the λ -spectral plane, we will show that there is a constant C such that for any strongly parabolic Higgs field with trivial underlying holomorphic structure and norm less than C , the associated logarithmic connection ∇^λ is Fuchsian for all $\lambda \in K \setminus \{0\}$. This will follow by continuity of the solution depending on the parabolic Higgs data (with trivial underlying holomorphic bundle), and the fact that for vanishing Higgs field the associated family is constant (and Fuchsian).

A *potential* is given by a holomorphic and complex linear loop algebra-valued 1-form

$$\eta \in \Omega^{1,0}(\mathbb{CP}^1 \setminus \{p_1, \dots, p_4\}, \Lambda \mathfrak{sl}(2, \mathbb{C}))$$

with

$$(\lambda\eta) \in \Omega^{1,0}(\mathbb{CP}^1 \setminus \{p_1, \dots, p_4\}, \Lambda_+ \mathfrak{sl}(2, \mathbb{C})),$$

where $\Lambda \mathfrak{sl}(2, \mathbb{C})$ and $\Lambda_+ \mathfrak{sl}(2, \mathbb{C})$ are as in Section 2.5 and $(\lambda\eta)$ denotes pointwise multiplication. Motivated by Lemma 38 and its proof we call the residue at $\lambda = 0$

$$\Psi = \eta_{-1} := \text{Res}_{\lambda=0}(\eta)$$

the parabolic Higgs field of the potential η . On the 4-punctured sphere we further specialize to η being a Fuchsian system for every $\lambda \in \mathbb{D}_a \setminus \{0\}$, i.e., we consider potentials of the form

$$\eta = \sum_{j=1}^4 A_j \frac{dz}{z - p_j} \quad \text{satisfying} \quad \sum_{j=0}^4 A_j = 0$$

with z being the homogenous coordinate of \mathbb{CP}^1 , $(\lambda A_j) \in \Lambda_+ \mathfrak{sl}(2, \mathbb{C})$. To be more explicit, the coefficients of A_j , as functions of λ , are in the functional space $\mathcal{W}_a^{\geq -1}$ where for $l \in \mathbb{Z}$

$$\mathcal{W}_a^{\geq l} := \left\{ h = \sum_{k=l}^{\infty} h_k \lambda^k \text{ such that } \sum_{k=l}^{\infty} |h_k| a^{|k|} < \infty \right\}.$$

For $l \geq 0$ we obtain the space of absolutely convergent power series in the disk of radius $a > 1$. We denote the subspace of convergent power series with vanishing constant term h_0 by \mathcal{W}_a^+ . Whenever

the dependence on a particular a is not important, we will omit this index. More generally, \mathcal{W}_a denotes the space of convergent Laurent series on the annulus $\mathbb{A}_a := \{\frac{1}{a^2} < |\lambda|^2 < a^2\}$. Every element $h \in \mathcal{W} = \mathcal{W}_a$ can be decomposed into its positive part $h^+ \in \mathcal{W}^+$, its constant part $h^0 := h_0$ and its negative part $h^- = h - h^+ - h^0$.

Remark 49. *Even though the choice of $a > 1$ is irrelevant for the application of the implicit function theorem for $t \sim 0$, we expect that for larger t and general strongly parabolic Higgs field (with trivial underlying holomorphic structure), the associated family of flat connection is not Fuchsian for all λ satisfying $|\lambda|^2 \leq a^2$. Instead, we expect unstable logarithmic connections (where the power series expansion in λ will necessarily have poles) moving into the unit disc within the t -deformation. This is in contrast to the case of the equations describing (equivariant) harmonic maps into the 3-sphere, which differ from the self-duality equation only by a sign, where unstable points cannot cross the unit circle, see [23].*

To fix further notations let \mathbb{C}^+ denote the upper-right quadrant of the plane

$$\mathbb{C}^+ = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0, \operatorname{Im}(z) > 0\}.$$

Let $p \in \mathbb{C}^+$ and consider the 4-punctured sphere

$$\Sigma = \Sigma_p = \mathbb{C}P^1 \setminus \{p_1, p_2, p_3, p_4\}$$

with

$$(47) \quad p_1 = p, \quad p_2 = -1/p, \quad p_3 = -p \quad \text{and} \quad p_4 = 1/p.$$

Up to Möbius transformations, every 4-punctured sphere is of this form. By definition Σ is invariant under the holomorphic involutions $\delta(z) = -z$ and $\tau(z) = 1/z$. Consider the Pauli matrices

$$\mathbf{m}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathbf{m}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{m}_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and the holomorphic 1-forms on Σ

$$(48) \quad \begin{aligned} \omega_1 &= \frac{dz}{z-p_1} - \frac{dz}{z-p_2} + \frac{dz}{z-p_3} - \frac{dz}{z-p_4} \\ \omega_2 &= \frac{dz}{z-p_1} - \frac{dz}{z-p_2} - \frac{dz}{z-p_3} + \frac{dz}{z-p_4} \\ \omega_3 &= \frac{dz}{z-p_1} + \frac{dz}{z-p_2} - \frac{dz}{z-p_3} - \frac{dz}{z-p_4}. \end{aligned}$$

Then the ω_i have the symmetries

$$(49) \quad \begin{cases} \delta^* \omega_1 = \omega_1, & \delta^* \omega_2 = -\omega_2, & \delta^* \omega_3 = -\omega_3 \\ \tau^* \omega_1 = -\omega_1, & \tau^* \omega_2 = \omega_2, & \tau^* \omega_3 = -\omega_3. \end{cases}$$

Ansatz 50. *In the following we will consider potentials of the form*

$$(50) \quad \eta_t = t \sum_{j=1}^3 x_j(t) \mathbf{m}_j \omega_j$$

where $t \sim 0$ is a real parameter and $x_1(t), x_2(t), x_3(t) \in \mathcal{W}^{\geq -1}$ are parameters depending on t .

We aim to determine η_t in dependence of t through the implicit function theorem by imposing the reality condition (41). The maps $t \mapsto x_j(t)$, $j = 1, 2, 3$, are always assumed to be smooth in some neighbourhood $(-\epsilon, \epsilon)$ of $t = 0$. We denote the parameter vector by

$$x(t) = (x_1(t), x_2(t), x_3(t)) \in (\mathcal{W}^{\geq -1})^3.$$

Remark 51. For $t \sim 0$, the parameter $x(t)$ is not uniquely determined by the reality condition (41) as we have a non-trivial moduli space of solutions, which is parametrized by Higgs data. Hence, we have to incorporate the dependency on the parabolic Higgs pair when applying the implicit function theorem. We omit this dependency for now, and refer to Section 4.2 below and to Theorem 8 below for precise statements.

With this ansatz η_t is a Fuchsian system for every $\lambda \in \mathbb{D}_a \setminus \{0\}$ and the residues at a puncture is given by a $\Lambda\mathfrak{sl}(2, \mathbb{C})$ element

$$\text{Res}_{z=p_j} \eta_t = tA_j$$

with

$$(51) \quad \begin{aligned} A_1 &= \begin{pmatrix} x_1(t) & x_2(t) + ix_3(t) \\ x_2(t) - ix_3(t) & -x_1(t) \end{pmatrix} & A_2 &= \begin{pmatrix} -x_1(t) & -x_2(t) + ix_3(t) \\ -x_2(t) - ix_3(t) & x_1(t) \end{pmatrix} \\ A_3 &= \begin{pmatrix} x_1(t) & -x_2(t) - ix_3(t) \\ -x_2(t) + ix_3(t) & -x_1(t) \end{pmatrix} & A_4 &= \begin{pmatrix} -x_1(t) & x_2(t) - ix_3(t) \\ x_2(t) + ix_3(t) & x_1(t) \end{pmatrix} \end{aligned}$$

satisfying

$$(52) \quad \det(A_j) = -x_1(t)^2 - x_2(t)^2 - x_3(t)^2$$

for all $j = 1, \dots, 4$. Moreover, $\Psi = \text{res}_{\lambda=0} \eta_t$ takes the form (33) with $R_j = \text{res}_{\lambda=0} A_j$.

This ansatz is chosen such that the potential has the following symmetries

$$(53) \quad \begin{aligned} \delta^* \eta_t &= D^{-1} \eta_t D \quad \text{with} \quad D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ \tau^* \eta_t &= C^{-1} \eta_t C \quad \text{with} \quad C = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \end{aligned}$$

We will show that these symmetries can be imposed without loss of generality if we start with a symmetric Higgs field and $t \sim 0$ is small.

Remark 52. With this ansatz we only study harmonic maps for which the monodromy becomes trivial for $t \rightarrow 0$ and consequently the corresponding Higgs fields $\Psi(t) \rightarrow 0$. In fact, evaluating Ansatz (50) at $t = 0$ gives the trivial connection with trivial monodromy due to the factor t in Ansatz (50).

4.2. The choice of initial data at $t = 0$, and parabolic Higgs fields. In this section, we discuss the initial data at $t = 0$ for the implicit function theorem. These initial data are naturally parametrized by the coordinates (u, v) from (31), which determine the Higgs pair.

We denote the initial value of the parameters with an underscore $\underline{x}_j := x_j(0)$, and accordingly \underline{A}_j . These are chosen such that $\underline{x}_1, \underline{x}_2, \underline{x}_3$ at $t = 0$ satisfy

$$(54) \quad \begin{cases} \det(\underline{A}_1) = -1 \\ \underline{A}_1 = \underline{A}_1^*, \end{cases}$$

where $()^*$ -operator for matrices is defined as

$$M^*(\lambda) = \overline{M(-1/\bar{\lambda})}^T.$$

The first equation in (54) implies that the local monodromies of the associated Fuchsian systems $d + \eta_t$ lie in the conjugacy class of $\text{diag}(\exp(2\pi it), \exp(-2\pi it))$ for every $\lambda \in \mathbb{D}_a$. The second equation in (54) is an infinitesimal version of \mathcal{T} -reality at $t = 0$. The equations (54) on \underline{A}_1 are equivalent to

$$\begin{cases} \underline{x}_1^2 + \underline{x}_2^2 + \underline{x}_3^2 = 1 \\ \underline{x}_j = \underline{x}_j^*, \quad \forall j, \end{cases}$$

where the induced $()^*$ -operator for functions is defined to be

$$(55) \quad f^*(\lambda) = \overline{f(-1/\bar{\lambda})}.$$

Since potentials have at most a first order pole at $\lambda = 0$ the eligible \underline{x}_j must be a degree 1 Laurent polynomial

$$(56) \quad \underline{x}_j = \underline{x}_{j,-1}\lambda^{-1} + \underline{x}_{j,0} + \underline{x}_{j,1}\lambda$$

with

$$(57) \quad \begin{cases} \underline{x}_{j,0} \in \mathbb{R} \\ \underline{x}_{j,1} = -\overline{\underline{x}_{j,-1}}. \end{cases}$$

Then $\underline{x}_1^2 + \underline{x}_2^2 + \underline{x}_3^2 = 1$ is equivalent to

$$(58) \quad \begin{cases} \sum_{j=1}^3 \underline{x}_{j,-1}^2 = 0 \\ \sum_{j=1}^3 \underline{x}_{j,-1} \underline{x}_{j,0} = 0 \\ \sum_{j=1}^3 \underline{x}_{j,0}^2 - 2|\underline{x}_{j,-1}|^2 = 1. \end{cases}$$

To solve the first equation of (58), consider the standard parametrization of the quadric $\{x^2 + y^2 + z^2 = 0\}$ in \mathbb{C}^3 given by

$$(59) \quad \underline{x}_{1,-1} = uv, \quad \underline{x}_{2,-1} = \frac{1}{2}(v^2 - u^2) \quad \text{and} \quad \underline{x}_{3,-1} = \frac{i}{2}(u^2 + v^2)$$

with $(u, v) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Then the second equation of (58) gives

$$(60) \quad uv \underline{x}_{1,0} + \frac{1}{2}(v^2 - u^2) \underline{x}_{2,0} + \frac{i}{2}(u^2 + v^2) \underline{x}_{3,0} = 0$$

and its (real) solutions are

$$(61) \quad \underline{x}_{1,0} = \rho(|u|^2 - |v|^2), \quad \underline{x}_{2,0} = 2\rho \operatorname{Re}(u\bar{v}) \quad \text{and} \quad \underline{x}_{3,0} = 2\rho \operatorname{Im}(u\bar{v}), \quad \text{for some } \rho \in \mathbb{R}.$$

Finally, the third equation of (58) becomes

$$(62) \quad \rho^2(|u|^2 + |v|^2)^2 - (|u|^2 + |v|^2)^2 = 1$$

which determines ρ up to sign. The geometric meaning of these equations and its relationship to the classical Weierstrass representation of minimal surfaces in Euclidean 3-space is discussed in Section 6.1. We will show in Section 5.4 below that the correct sign of ρ to obtain twistor lines is

$$(63) \quad \rho = \sqrt{1 + (|u|^2 + |v|^2)^{-2}} > 0.$$

But we obtain \mathcal{T} -real holomorphic sections of $\mathcal{M}_{DH}(\mathbb{C}P^1, p_1 + \dots + p_4, t)$ for $t \sim 0$ for both choices of sign in (62).

We have thus fixed the initial conditions of η_t depending on a pair $(u, v) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ which parametrizes the space of all eligible non-vanishing λ -residues of the η_t . This gives a 8-fold covering of the open subset \mathcal{U} of the parabolic Higgs bundle moduli space specified by a trivial underlying holomorphic bundle. In fact, (u, v) uniquely determines the nilpotent residue R_1 of the Higgs field at p_1 up to sign, and by Lemma 30, R_1 determines the gauge class of the Higgs pair uniquely up to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action generated by conjugation with C and D .

Convention 53. *In the following, we slightly abuse notation and neglect the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action when referring to completion of the nilpotent orbit $\{R_1 \in \mathfrak{sl}(2, \mathbb{C}) \mid \det(R_1) = 0\}$ as the moduli space of parabolic Higgs fields. This 4-fold covering can actually be identified with an open dense subset of the space of parabolic Higgs fields on the 1-punctured torus with parabolic weight $\frac{1}{2} - 2t$.*

We will also ignore the complex line of stable parabolic Higgs fields with underlying holomorphic structure $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ at $t = 0$, since the $t \rightarrow 0$ limit of the corresponding representations is not the trivial representation.

Remark 54. When considering Ansatz (50) for $t > 0$ we have to rephrase the second condition in (54) which encodes the reality condition at $t = 0$. Instead to ensure (41) we have to require that η_t^* and η_t lie in the same gauge class. By Lemma 38 this corresponds to constructing families of flat connections that descend to real holomorphic sections of $\mathcal{M}_{DH}(\mathbb{CP}^1, p_1 + \dots + p_4, t)$ for prescribed strongly parabolic Higgs field at $\lambda = 0$ by varying parabolic weight t .

Using the above (u, v) -parametrization the parabolic Higgs field $\Psi = \text{Res}_{\lambda=0} \eta_t$ is given by (31) and has determinant

$$(64) \quad \det(\Psi) = \frac{-4t^2(u^4 - (p^2 + p^{-2})u^2v^2 + v^4)}{z^4 - (p^2 + p^{-2})z^2 + 1},$$

where the singular points p_1, \dots, p_4 are determined by $p \in \mathbb{C}^+$ via (47). The 0-eigenlines of the nilpotent residues of Ψ viewed as points in \mathbb{CP}^1 have homogenous coordinates u/v , $-v/u$, $-u/v$ and v/u , respectively. The cross-ratio of these four eigenlines, which together with the parabolic weight t determines the parabolic structure on the trivial holomorphic bundle, is computed to be

$$(65) \quad \left(\frac{u}{v}, \frac{-v}{u}; \frac{-u}{v}, \frac{v}{u} \right) = \frac{-4u^2v^2}{(u^2 - v^2)^2},$$

where the cross-ratio is defined to be

$$(z_1, z_2; z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}.$$

The cross-ratio (65) is already uniquely determined by the ratio u/v , up to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry generated by $u/v \mapsto -u/v$, $u/v \mapsto v/u$ in accordance with Proposition 32 (since scaling u and v simultaneously just scales the Higgs field).

5. CONSTRUCTING REAL HOLOMORPHIC SECTIONS

In order for the potential $d + \eta_t$ to be the lift of a real holomorphic section s we impose the condition that

$$(66) \quad d + \eta_t(\lambda) \text{ is gauge equivalent to } d + \overline{\eta_t(-\bar{\lambda}^{-1})}$$

for all $\lambda \in \mathbb{A}_a = \{\lambda \mid a^{-2} < |\lambda|^2 < a^2\}$ as in (41). To make this condition more explicit, we identify the space of gauge equivalence classes of flat connections with the space of representations of $\pi_1(\Sigma, z_0)$ modulo conjugation via the monodromy representation. Let $p \in \mathbb{C}^+$ and $\Sigma = \Sigma_p$ be the 4-punctured sphere and fix the base point $z_0 = 0$. Choose the generators $\gamma_1, \gamma_2, \gamma_3$ and γ_4 of the fundamental group $\pi_1(\Sigma, 0)$ as in [26], i.e. let γ_1 be the composition of the real half-line from 0 to $+\infty$ with the imaginary half-line from $+i\infty$ to 0, and more generally γ_k is the product of the half-line from 0 to $i^{k-1}\infty$ with the half line from $i^k\infty$ to 0, so γ_k encloses p_k and $\gamma_1\gamma_2\gamma_3\gamma_4 = 1$. Let Φ_t be the fundamental solution of the Cauchy Problem

$$(67) \quad d_\Sigma \Phi_t = \Phi_t \eta_t \quad \text{with initial condition} \quad \Phi_t(z = 0) = \text{Id}$$

and let $M_k(t) = \mathcal{M}(\Phi_t, \gamma_k)$ be the monodromy of Φ_t along γ_k . This gives the *left* monodromy representation of the connection $d + \eta_t$, i.e.,

$$M_1(t)M_2(t)M_3(t)M_4(t) = \text{Id}.$$

The identification of the left and the right monodromy representation is done by taking the inverse, which is in accordance with the fact that for a solution Φ of $d\Phi = \Phi\eta$ its inverse solves $d\Phi^{-1} = -\eta\Phi^{-1}$. In particular, two connections are gauge equivalent if and only if their left monodromies are conjugate to each other. In order to directly use results and explicit computations from our previous work (e.g. [25, 26]), we will work with the left monodromy in the following.

Fricke coordinates. The moduli space of representations can be parametrized using the so-called Fricke coordinates. Define

$$s_k := \text{tr}(M_k); \quad s_{kl} = \text{tr}(M_k M_l).$$

Since $M_k \in \text{SL}(2, \mathbb{C})$, the trace s_k determines the eigenvalues of M_k , i.e., the conjugacy class of the local monodromy M_k . For the symmetric case considered in this paper we restrict to

$$s_1 = s_2 = s_3 = s_4 = 2 \cos(2\pi t)$$

for $t \in (0, \frac{1}{4})$ and the following classical result by Fricke-Voigt holds.

Proposition 55. *Consider a $\text{SL}(2, \mathbb{C})$ -representation on the 4-punctured sphere Σ . Let*

$$s = s_1 = \dots = s_4 \in (0, 2)$$

and let $U = s_{12}$, $V = s_{23}$, $W = s_{13}$. Then the following algebraic equation holds

$$(68) \quad U^2 + V^2 + W^2 + UVW - 2s^2(U + V + W) + 4(s^2 - 1) + s^4 = 0.$$

When satisfying (68) the parameters s and U, V, W together determine a monodromy representation $\rho: \pi_1(\Sigma) \rightarrow \text{SL}(2, \mathbb{C})$ from the first fundamental group of Σ into $\text{SL}(2, \mathbb{C})$. By imposing the symmetries (53), this representation is unique up to conjugation.

Proof. For the first (classical) part of the proposition see for example [17]. Whenever the representations are irreducible, [17] moreover shows that they are uniquely determined by their global traces U, V, W up to conjugation, even without symmetry assumptions. By [17, Lemma 5] a representation is reducible if and only if

$$s_{ij} \in \{2, -2 + s^2\} = \{2, 2 \cos(4\pi t)\}$$

for all i, j . For $t \in (0, \frac{1}{4})$ (68) then implies that

$$(U, V, W) \in \{(2, 2, 2 \cos(4\pi t)), (2, 2 \cos(4\pi t), 2), (2 \cos(4\pi t), 2, 2)\}.$$

It can be easily checked that for each of the 3 possibilities there is a Fuchsian potential, unique up to conjugation with C , D , and CD , satisfying the symmetry assumptions (53) and inducing the corresponding totally reducible representation. \square

Using the above proposition the reality condition (66) on the potentials η_t is equivalent to

$$s_{jk} = s_{jk}^* \quad \text{for } (j, k) \in \{(1, 2), (1, 3), (2, 3)\},$$

with $(\cdot)^*$ as defined in (55). The goal of this section is thus to solve the following Monodromy Problem

$$(69) \quad \begin{cases} s_{jk} = s_{jk}^* & \text{for } (j, k) \in \{(1, 2), (1, 3), (2, 3)\} \\ \sum_{j=1}^3 x_j(t)^2 = 1 \end{cases}$$

using a similar implicit function theorem argument as in [25, 26]. By Lemma 38 and Proposition 55 we then obtain real holomorphic sections of $\mathcal{M}_{DH}(\mathbb{CP}^1, p_1 + \dots + p_4, t)$. It then remains to show that the so-constructed real holomorphic sections are negative and in the connected component of twistor lines. This is done in Theorem 10 below.

5.1. Setup. As in [26], consider for a fixed potential η the extended frame Φ satisfying $d\Phi = \Phi\eta$ and $\Phi(z=0) = \text{Id}$. Let $\mathcal{P} = \Phi(z=1)$ and $\mathcal{Q} = \Phi(z=i)$, where we omitted the index t . Then the traces s_{jk} are given by squares of holomorphic functions in terms of the entries of $\mathcal{P} = (\mathcal{P}_{ij})$ and $\mathcal{Q} = (\mathcal{Q}_{ij})$ as follows:

Proposition 56. *With the notation above we have*

$$(70) \quad s_{12} = 2 - 4\mathfrak{p}^2$$

$$(71) \quad s_{23} = 2 - 4\mathfrak{q}^2$$

$$(72) \quad s_{13} = 2 - 4\mathfrak{r}^2$$

with

$$\mathfrak{p} = \mathcal{P}_{11}\mathcal{P}_{21} - \mathcal{P}_{12}\mathcal{P}_{22}, \quad \mathfrak{q} = i(\mathcal{Q}_{11}\mathcal{Q}_{21} + \mathcal{Q}_{12}\mathcal{Q}_{22})$$

and

$$\begin{aligned} \mathfrak{r} = & \frac{i}{2}(\mathcal{P}_{22}\mathcal{Q}_{11} + \mathcal{P}_{12}\mathcal{Q}_{21})^2 + \frac{i}{2}(\mathcal{P}_{22}\mathcal{Q}_{12} + \mathcal{P}_{12}\mathcal{Q}_{22})^2 \\ & - \frac{i}{2}(\mathcal{P}_{21}\mathcal{Q}_{11} + \mathcal{P}_{11}\mathcal{Q}_{21})^2 - \frac{i}{2}(\mathcal{P}_{21}\mathcal{Q}_{12} + \mathcal{P}_{11}\mathcal{Q}_{22})^2. \end{aligned}$$

Proof. The equations (70) and (71) have been proven in [26, Proposition 16]. It remains to show (72). Using the symmetries δ and $\tau \circ \delta$ which fix $z = 1$ and $z = i$ respectively we have as in [26, Proposition 16]

$$(73) \quad \begin{aligned} \Phi(+\infty) &= \mathcal{P}C\mathcal{P}^{-1}C^{-1}, & \Phi(+i\infty) &= \mathcal{Q}DC\mathcal{Q}^{-1}C^{-1}D^{-1}, \\ M_1 &= \Phi(+\infty)\Phi(+i\infty)^{-1}, & M_3 &= DM_1D^{-1}. \end{aligned}$$

This gives with $C^2 = D^2 = -\text{Id}$ and $CD = -DC$ that

$$M_1M_3 = -(\mathcal{P}C\mathcal{P}^{-1}C^{-1}DC\mathcal{Q}C^{-1}D^{-1}\mathcal{Q}^{-1}D)^2 = -(\mathcal{P}C\mathcal{P}^{-1}D\mathcal{Q}CD\mathcal{Q}^{-1}D)^2.$$

For $A \in SL(2, \mathbb{C})$ we have

$$\text{tr}(-A^2) = 2 - \text{tr}(A)^2,$$

hence (72) holds with

$$\mathfrak{r} = \frac{1}{2}\text{tr}(\mathcal{P}C\mathcal{P}^{-1}D\mathcal{Q}DC\mathcal{Q}^{-1}D)$$

which coincides with the formula given in Proposition 56 after a tedious computation. \square

Proposition 57. *At $t = 0$, we have*

$$\mathfrak{p}(0) = \mathfrak{q}(0) = \mathfrak{r}(0) = 0$$

with derivatives

$$\mathfrak{p}'(0) = 2\pi x_3(0), \quad \mathfrak{q}'(0) = 2\pi x_2(0) \quad \text{and} \quad \mathfrak{r}'(0) = 2\pi x_1(0).$$

Proof. At $t = 0$ we have $\eta_t = 0$ thus all monodromies are trivial and $\mathcal{P} = \mathcal{Q} = \text{Id}$ from which the first point follows. For the assertion on the derivatives define as in [26]

$$(74) \quad \Omega_j(z) = \int_0^z \omega_j(z)$$

for $j = 1, 2, 3$, where the integral is computed on the segment from 0 to z . Then

$$\mathcal{P}'(0) = \sum_{j=1}^3 x_j(0)\Omega_j(1)\mathfrak{m}_j, \quad \text{and} \quad \mathcal{Q}'(0) = \sum_{j=1}^3 x_j(0)\Omega_j(i)\mathfrak{m}_j,$$

from which we can compute

$$\begin{aligned} \mathfrak{p}'(0) &= \mathcal{P}'_{21}(0) - \mathcal{P}'_{12}(0) = -2ix_3(0)\Omega_3(1) \\ \mathfrak{q}'(0) &= i(\mathcal{Q}'_{21}(0) + \mathcal{Q}'_{12}(0)) = 2ix_2(0)\Omega_2(i) \\ \mathfrak{r}'(0) &= i(\mathcal{P}'_{22}(0) + \mathcal{Q}'_{11}(0) - \mathcal{P}'_{11}(0) - \mathcal{Q}'_{22}(0)) = -2ix_1(0)(\Omega_1(1) - \Omega_1(i)). \end{aligned}$$

By the Residue Theorem, we have for $j = 1, 2, 3$

$$2\pi i = \int_{\gamma_1} \omega_j = \int_0^1 \omega_j - \int_0^1 \tau^* \omega_j - \int_0^i \omega_j + \int_0^i (\tau\delta)^* \omega_j.$$

Using the symmetries (49), this gives

$$(75) \quad \Omega_1(1) - \Omega_1(i) = \pi i, \quad \Omega_2(i) = -\pi i \quad \text{and} \quad \Omega_3(1) = \pi i$$

proving Proposition 57. \square

In view of Proposition 56, the monodromy problem (69) can be reformulated to be

$$(76) \quad \begin{cases} \mathfrak{p} = \mathfrak{p}^* \\ \mathfrak{q} = \mathfrak{q}^* \\ \mathfrak{r} = \mathfrak{r}^* \\ \sum_{j=1}^3 x_j(t)^2 = 1 \end{cases}$$

and the following proposition implies that it suffices to solve for two of the three traces only.

Proposition 58. *Assume that $x_1(t), x_2(t), x_3(t): (-\varepsilon, \varepsilon) \rightarrow \mathcal{W}^{\geq 1}$ are analytic in t , with $x_j(0) = \underline{x}_j$, satisfying the following equations for all t*

$$(77) \quad \begin{cases} \mathfrak{p} = \mathfrak{p}^* \\ \mathfrak{q} = \mathfrak{q}^* \\ \sum_{j=1}^3 x_j(t)^2 = 1. \end{cases}$$

Then also $\mathfrak{r} = \mathfrak{r}^$ for all t . Analogous statements hold by cyclicly permutating $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$.*

Proof. Let

$$U = s_{12}, \quad V = s_{23}, \quad W = s_{13} \quad \text{and} \quad s = \text{tr}(M_k)$$

as in Proposition 55 satisfying the quadratic equation (68)

$$Q := U^2 + V^2 + W^2 + UVW - 2s^2(U + V + W) + 4(s^2 - 1) + s^4 = 0.$$

By substitution of $U = 2 - 4\mathfrak{p}^2$, $V = 2 - 4\mathfrak{q}^2$ and $W = 2 - 4\mathfrak{r}^2$, Q factors as

$$Q = Q_1 Q_2 \quad \text{with} \quad Q_j = s^2 + 4(\mathfrak{p}^2 + \mathfrak{q}^2 + \mathfrak{r}^2 - 1) + 8(-1)^j \mathfrak{p}\mathfrak{q}\mathfrak{r}.$$

Since $Q = 0$ for all t , and Q_1, Q_2 are analytic functions of t , one of them must be identically zero. (We will see in Remark 72 below that $j = 2$.) The discriminant of Q_j , considered as a polynomial in the variable \mathfrak{r} is give by

$$\Delta = 64(1 - \mathfrak{p}^2)(1 - \mathfrak{q}^2) - 16s^2$$

and is independent of $j = 1, 2$. Since \mathcal{P}, \mathcal{Q} are well-defined analytic functions in t (with values in \mathcal{W}_a), \mathfrak{r} is a well-defined analytic function in t as well by Proposition 56. Therefore, Δ admits a well-defined square root δ such that

$$\mathfrak{r} = (-1)^{j+1} \mathfrak{p}\mathfrak{q} + \frac{\delta}{8}$$

for all t . From the hypotheses of the proposition, we have $\Delta = \Delta^*$ hence $\delta^* = \varepsilon\delta$, where the sign $\varepsilon = \pm 1$ does not depend on t because \mathfrak{r}^* is a well-defined analytic function in t . Hence

$$\mathfrak{r}^* = (-1)^{j+1} \mathfrak{p}\mathfrak{q} + \varepsilon \frac{\delta}{8}.$$

The sign ε can be determined using the first order derivatives at $t = 0$. We have

$$\mathfrak{r}' = 2\pi \underline{x}_1 = \frac{\delta'}{8} \quad \text{and} \quad \mathfrak{r}'^* = 2\pi \underline{x}_1^* = 2\pi \underline{x}_1 = \varepsilon \frac{\delta'}{8}.$$

Since $\underline{x}_1 \neq 0$, we obtain $\varepsilon = 1$. Hence $\mathfrak{r} = \mathfrak{r}^*$ for all t . \square

5.2. Solving the Monodromy Problem. As remarked before, see Remark 51, the potentials $d + \eta_t$ are not uniquely determined by the reality conditions (69) or (76), but also depend on the Higgs data. By Proposition 32, we identify the moduli space of strongly parabolic Higgs fields on $V = \mathcal{O} \oplus \mathcal{O}$, up to taking the quotient by $\mathbb{Z}_2 \times \mathbb{Z}_2$ action, with the completion of the nilpotent orbit in $\mathfrak{sl}(2, \mathbb{C})$, which is in turn given by the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ at the origin. First, consider the regular case of $(u, v) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and consider for $j = 1, 2, 3$ the quadratic polynomials

$$P_j(\lambda) = \lambda \underline{x}_j(\lambda) = \underline{x}_{j,1} \lambda^2 + \underline{x}_{j,0} \lambda - \overline{\underline{x}_{j,1}},$$

where the \underline{x}_j are defined in (56) satisfying (57), (59), (61) and (62), and denote the discriminant of P_j by

$$\Delta_j = \underline{x}_{j,0}^2 + 4|\underline{x}_{j,1}|^2 \in \mathbb{R}.$$

The following arguments work for both signs of ρ in (62), compare with Lemma 63 below. When statements do depend on the choice of sign, e.g., in Theorem 9, we state them for the correct choice (63) only.

Proposition 59.

- (1) *The polynomial P_j has a complex root μ_j with $|\mu_j| < 1$ if and only if $\underline{x}_{j,0} \neq 0$. In this case $\mu_j \in \mathbb{D}_1$ depends real-analytically on $(u, v) \neq (0, 0)$.*
- (2) *With the same notation, $P_k(\mu_j)$ and $P_\ell(\mu_j)$ are \mathbb{R} -independent complex numbers, if $\{j, k, \ell\} = \{1, 2, 3\}$.*

Proof. If $\underline{x}_{j,1} = 0$, the first point is trivial, because in this case $P_j = \underline{x}_{j,0} \lambda$ and $\underline{x}_{j,0} \neq 0$ since $(u, v) \neq (0, 0)$. Hence assume in the following $\underline{x}_{j,1} \neq 0$. If μ is a root of P_j , then $\mu \neq 0$. Moreover, $\underline{x}_j = \underline{x}_j^*$ gives that $-\frac{1}{\mu} \neq \mu$ is the other root of P_j . Thus

$$|\mu| = 1 \Leftrightarrow \mu - \frac{1}{\mu} = 0 \Leftrightarrow \underline{x}_{j,0} = 0.$$

For $\underline{x}_{j,0} \neq 0$ the root μ_j with $|\mu_j| < 1$ is given by

$$\mu_j = \frac{-\underline{x}_{j,0} + \text{sign}(\underline{x}_{j,0})\sqrt{\Delta_j}}{2\underline{x}_{j,1}},$$

which we can rewrite as

$$\mu_j = \frac{2\overline{\underline{x}_{j,1}}}{\underline{x}_{j,0}} f\left(\frac{4|\underline{x}_{j,1}|^2}{\underline{x}_{j,0}^2}\right) \quad \text{with} \quad f(z) = \frac{\sqrt{1+z}-1}{z}.$$

The function f extends holomorphically to $z = 0$, therefore μ_j depends analytically on (u, v) . To prove the second point, assume for simplicity of notation that $j = 1$. Suppose by contradiction that $P_2(\mu_1)$ and $P_3(\mu_1)$ are linearly dependent over \mathbb{R} . First assume that $\mu_1 \neq 0$. Then the complex numbers $\underline{x}_2(\mu_1), \underline{x}_3(\mu_1)$ are linearly dependent over \mathbb{R} . Moreover, $\underline{x}_1(\mu_1) = 0$ so

$$\underline{x}_2(\mu_1)^2 + \underline{x}_3(\mu_1)^2 = 1$$

and this implies that $\underline{x}_2(\mu_1)$ and $\underline{x}_3(\mu_1)$ are real. Since all $\underline{x}_{k,0}$ are real, we obtain

$$\underline{x}_{k,1}\mu_1 - \overline{\underline{x}_{k,1}}\mu_1^{-1} \in \mathbb{R} \quad \text{for } k = 1, 2, 3.$$

Then

$$\sum_{k=1}^3 (\underline{x}_{k,1}\mu_1 - \overline{\underline{x}_{k,1}}\mu_1^{-1})^2 \geq 0.$$

Expanding the squares and using $\sum_{k=1}^3 \underline{x}_{k,1}^2 = 0$ we obtain

$$\sum_{k=1}^3 |\underline{x}_{k,1}|^2 \leq 0$$

which implies $u = v = 0$ which is a contradiction.

If $\mu_1 = 0$, we have $P_2(0) = \underline{x}_{2,-1}$ and $P_3(0) = \underline{x}_{3,-1}$. From $P_1(0) = \underline{x}_{1,-1} = 0$ we obtain

$$\underline{x}_{2,-1}^2 + \underline{x}_{3,-1}^2 = 0$$

and since $\underline{x}_{2,-1}$ and $\underline{x}_{3,-1}$ are linearly dependent over \mathbb{R} , $\underline{x}_{2,-1} = \underline{x}_{3,-1} = 0$ which again results in $u = v = 0$ leading to a contradiction. \square

Theorem 8. *Let $(u, v) \neq (0, 0)$ be fixed. This determines $\underline{x}_{j,-1}$ for $j = 1, 2, 3$. Then, there are $\epsilon_0 > 0$ and $a > 1$ such that there exists unique values of the parameters $x(t) = (x_1(t), x_2(t), x_3(t)) \in (\mathcal{W}_a^{\geq -1})^3$ in a neighborhood of \underline{x} for all $t \in (-\epsilon_0, \epsilon_0)$ depending real analytically on (t, p, u, v) solving (69) with $x(0) = \underline{x}$ and prescribed $x_{j,-1}(t) = \underline{x}_{j,-1}$. Moreover, ϵ_0 and $a > 1$ are uniform with respect to (p, u, v) on compact subsets of $\mathbb{C}^+ \times \mathbb{C}^2 \setminus \{(0, 0)\}$.*

Proof. Fix $(u, v) \neq (0, 0)$. By Proposition 59 and (61), at least one of the polynomials P_j has a root μ_j inside the unit λ -disc. By symmetry of the roles played by the parameters $\mathbf{p}, \mathbf{q}, \mathbf{r}$, we may assume without loss of generality that $j = 1$. By Proposition 58, it suffices to solve Problem (77). We fix $a > 1$ such that $a|\mu_1| < 1$ and consider the corresponding function space $\mathcal{W}_a^{\geq 0}$. We introduce a parameter $y = (y_1, y_2, y_3)$ in a neighborhood of 0 in $(\mathcal{W}_a^{\geq 0})^3$ and set

$$(78) \quad x_k(t) = \underline{x}_k + y_k, \quad k = 1, 2, 3.$$

Note that the negative part of the potential is fixed to its initial value $x_{k,-1} = \underline{x}_{k,-1}$. Since \mathbf{p} and \mathbf{q} are analytic functions of (t, y) which vanish at $t = 0$, the functions

$$\hat{\mathbf{p}}(t, y) := \frac{1}{t}\mathbf{p}(t, y) \quad \text{and} \quad \hat{\mathbf{q}}(t, y) := \frac{1}{t}\mathbf{q}(t, y)$$

extend analytically at $t = 0$ and by Proposition 57 we have at $t = 0$

$$(79) \quad \hat{\mathbf{p}}(0, y) = 2\pi(\underline{x}_3 + y_3) \quad \text{and} \quad \hat{\mathbf{q}}(0, y) = 2\pi(\underline{x}_2 + y_2).$$

We define

$$(80) \quad \begin{aligned} \mathcal{F}(t, y) &:= \hat{\mathbf{p}}(t, y) - \hat{\mathbf{p}}(t, y)^* \\ \mathcal{G}(t, y) &:= \hat{\mathbf{q}}(t, y) - \hat{\mathbf{q}}(t, y)^* \\ \mathcal{K}(y) &:= \sum_{k=1}^3 x_k(t)^2 = \sum_{k=1}^3 (\underline{x}_k + y_k)^2. \end{aligned}$$

Then solving Problem(77) is equivalent to solving the equations $\mathcal{F} = \mathcal{G} = 0$ and $\mathcal{K} = 1$. By our choice of the central value (56) and due to (79), these equations hold at $(t, y) = (0, 0)$. By definition we have $\mathcal{F}^* = -\mathcal{F}$ so $\mathcal{F} = 0$ is equivalent to $\mathcal{F}^+ = 0$ and $\text{Im}(\mathcal{F}^0) = 0$, and the analogous statement holds for \mathcal{G} as well. By Proposition 57 and substituting (78) the partial derivatives with respect to $y = (y_1, y_2, y_3)$ are

$$(81) \quad \begin{aligned} d\mathcal{F}(0, 0)^+ &= 2\pi dy_3^+ \\ \text{Im}(d\mathcal{F}(0, 0)^0) &= 2\pi \text{Im}(dy_{3,0}) \\ d\mathcal{G}(0, 0)^+ &= 2\pi dy_2^+ \\ \text{Im}(d\mathcal{G}(0, 0)^0) &= 2\pi \text{Im}(dy_{2,0}). \end{aligned}$$

Hence, the partial derivative of

$$(\mathcal{F}^+, \mathcal{G}^+, \text{Im}(\mathcal{F}^0), \text{Im}(\mathcal{G}^0))$$

with respect to

$$(y_2^+, y_3^+, \text{Im}(y_{2,0}), \text{Im}(y_{3,0}))$$

is an automorphism of $(\mathcal{W}_a^+)^2 \times \mathbb{R}^2$. The implicit function theorem therefore uniquely determines $(y_2^+, y_3^+, \text{Im}(y_{2,0}), \text{Im}(y_{3,0})) \in (\mathcal{W}_a^+)^2 \times \mathbb{R}^2$ as analytic functions of t and the remaining parameters y_1 , $\text{Re}(y_{2,0})$ and $\text{Re}(y_{3,0})$. Furthermore, the partial derivative of $(y_2^+, y_3^+, \text{Im}(y_{2,0}), \text{Im}(y_{3,0}))$ at $(t, y) = (0, 0)$ with respect to these remaining parameters is zero by (81).

It remains to solve the equation $\mathcal{K} = 1$. We write the Euclidean division of the polynomial P_k by $(\lambda - \mu_1)$ as

$$P_k(\lambda) = (\lambda - \mu_1)Q_k + P_k(\mu_1)$$

with $Q_k \in \mathbb{C}[\lambda]$. Note that the Q_k are real analytic in (u, v) by Proposition 59. Observe that since

$$\sum_{k=1}^3 \underline{x}_{k,-1}^2 = 0,$$

\mathcal{K} has no λ^{-2} term so $\lambda\mathcal{K} \in \mathcal{W}^{\geq 0}$. We write the division of $\lambda\mathcal{K}$ by $(\lambda - \mu_1)$ as

$$\lambda\mathcal{K} = (\lambda - \mu_1)\mathcal{S} + \mathcal{R}$$

where $\mathcal{R} \in \mathbb{C}$ and $\mathcal{S} \in \mathcal{W}^{\geq 0}$. Note that since $|\mu_1| < 1$, \mathcal{R} and \mathcal{S} are analytic functions of all parameters by [26, Proposition 5]. We have, since $P_1(\mu_1) = 0$ and $dy_k = \text{Re}(dy_{k,0})$ for $k = 2, 3$

$$d(\lambda\mathcal{K})(0,0) = \sum_{k=1}^3 2P_k(\lambda) dy_k = 2(\lambda - \mu_1)Q_1 dy_1 + \sum_{k=2}^3 2((\lambda - \mu_1)Q_k + P_k(\mu_1))\text{Re}(dy_{k,0})$$

so by uniqueness of the division

$$\begin{aligned} d\mathcal{R}(0,0) &= 2P_2(\mu_1)\text{Re}(dy_{2,0}) + 2P_3(\mu_1)\text{Re}(dy_{3,0}) \\ d\mathcal{S}(0,0) &= 2Q_1 dy_1 + 2Q_2\text{Re}(dy_{2,0}) + 2Q_3\text{Re}(dy_{3,0}). \end{aligned}$$

If $\underline{x}_{1,1} \neq 0$, we have $\mu_1 \neq 0$ and the other root of P_1 is $-1/\overline{\mu_1}$ so

$$Q_1 = \underline{x}_{1,1} \left(\lambda + \frac{1}{\mu_1} \right)$$

is invertible in $\mathcal{W}_a^{\geq 0}$ because $\frac{1}{|\mu_1|} > a$. (Note that $\lambda - c$ is invertible in $\mathcal{W}_a^{\geq 0}$ if and only if $|c| > a$.)

If $\underline{x}_{1,1} = 0$, we have

$$P_1 = \underline{x}_{1,0}\lambda$$

so $Q_1 = \underline{x}_{1,0} \in \mathbb{C}^*$ is invertible in $\mathcal{W}_a^{\geq 0}$ as well. By Proposition 59, $P_2(\mu_1)$ and $P_3(\mu_1)$ are \mathbb{R} -independent complex numbers, and the partial derivative of $(\mathcal{S}, \mathcal{R})$ with respect to $(y_1, \text{Re}(y_{2,0}), \text{Re}(y_{3,0}))$ at $(t, y) = (0, 0)$ is an isomorphism from $\mathcal{W}_a^{\geq 0} \times \mathbb{R}^2$ to $\mathcal{W}_a^{\geq 0} \times \mathbb{C}$. Thus, the implicit function theorem uniquely determines $(y_1, \text{Re}(y_{2,0}))$ and $\text{Re}(y_{3,0})$ as analytic functions of t in a neighborhood of $t = 0$. \square

5.3. The limit $(u, v) \rightarrow (0, 0)$. In this section, we highlight the dependency of the solutions $x(t) = x(t, p, u, v)$ provided by Theorem 5 on the parameters p, u, v when necessary, and we drop the dependency of all parameters when convenient.

By Proposition 32 the Higgs bundle moduli space is given by the blow up of $\mathbb{C}^2/\mathbb{Z}_2$ at the origin. Rather than expecting the solution $x(t, p, u, v)$ to extend continuously to $(u, v) = (0, 0)$, the limit $(u, v) \rightarrow (0, 0)$ should therefore depend on the direction in the blow-up. We write

$$u = r\tilde{u} \quad \text{and} \quad v = r\tilde{v} \quad \text{with} \quad |\tilde{u}|^2 + |\tilde{v}|^2 = 1.$$

Let $0 < |r| \leq \frac{1}{2}$ so that $(u, v) \neq (0, 0)$. With a slight abuse of notation, we write $x = x(t, p, r, \tilde{u}, \tilde{v})$. Recall that $(u, v) \rightarrow (-u, -v)$ does not change the initial value \underline{x} , so the map $x(t, p, r, \tilde{u}, \tilde{v})$ is even with respect to r .

Our goal is to prove

Theorem 9.

- (1) *There exists $\epsilon_2 > 0$ such that for $|t| < \epsilon_2$, the function $x(t, p, r, \tilde{u}, \tilde{v})$ extends analytically to $r = 0$. Moreover, ϵ_2 is uniform with respect to p in compact subsets of \mathbb{C}^+ and (\tilde{u}, \tilde{v}) in \mathbb{S}^3 .*

(2) At $r = 0$, $x(t, p, 0, \tilde{u}, \tilde{v})$ does not depend on λ and solves the following problem:

$$(82) \quad \begin{cases} \exists U \in SL(2, \mathbb{C}), \quad \forall k, \quad UM_k U^{-1} \in SU(2) \\ x_1^2 + x_2^2 + x_3^2 = 1. \end{cases}$$

(3) At $(t, r) = (0, 0)$, we have for $\rho > 0$

$$(83) \quad x_1 = (|\tilde{u}|^2 - |\tilde{v}|^2), \quad x_2 = 2 \operatorname{Re}(\tilde{u} \tilde{v}) \quad \text{and} \quad x_3 = 2 \operatorname{Im}(\tilde{u} \tilde{v}).$$

Remark 60. For $\rho < 0$, we have to replace in (3) the limit (x_1, x_2, x_3) by $-(x_1, x_2, x_3)$.

Proof of Theorem 9. Rewrite the central value \underline{x} in terms of $(r, \tilde{u}, \tilde{v})$ as

$$\underline{x}_j = r^2 \tilde{\underline{x}}_{j,-1} \lambda^{-1} + \tilde{\rho}(r) \tilde{\underline{x}}_{j,0} + r^2 \tilde{\underline{x}}_{j,1} \lambda$$

with

$$\begin{aligned} \tilde{\underline{x}}_{1,-1} &= \tilde{u} \tilde{v}, \quad \tilde{\underline{x}}_{2,-1} = \frac{1}{2}(\tilde{v}^2 - \tilde{u}^2), \quad \tilde{\underline{x}}_{3,-1} = \frac{i}{2}(\tilde{u}^2 + \tilde{v}^2), \\ \tilde{\underline{x}}_{1,0} &= |\tilde{u}|^2 - |\tilde{v}|^2, \quad \tilde{\underline{x}}_{2,0} = 2 \operatorname{Re}(\tilde{u} \tilde{v}), \quad \tilde{\underline{x}}_{3,0} = 2 \operatorname{Im}(\tilde{u} \tilde{v}), \\ \tilde{\underline{x}}_{j,1} &= -\overline{\tilde{\underline{x}}_{j,-1}} \end{aligned}$$

and

$$\tilde{\rho}(r) := r^2 \rho(r) = \sqrt{1 + r^4} = 1 + O(r^4).$$

Observe that

$$(84) \quad \sum_{k=1}^3 \tilde{\underline{x}}_{k,-1}^2 = 0, \quad \sum_{k=1}^3 \tilde{\underline{x}}_{k,0}^2 = 1, \quad \sum_{k=1}^3 \tilde{\underline{x}}_{k,-1} \tilde{\underline{x}}_{k,0} = 0$$

and

$$(85) \quad \sum_{k=1}^3 |\tilde{\underline{x}}_{k,-1}|^2 = \frac{1}{2}.$$

For given $(\tilde{u}, \tilde{v}) \in \mathbb{S}^3$, fix $j \in \{1, 2, 3\}$ such that $\tilde{\underline{x}}_{j,0} \neq 0$. (This is possible by (84)). We take as ansatz that the parameter y_j is of the following form

$$(86) \quad y_j = y_{j,0} + r^2 \tilde{y}_j^+ \quad \text{with} \quad \tilde{y}_j^+ \in \mathcal{W}_a^+.$$

We solve the equations $\mathcal{F} = \mathcal{G} = 0$ using the implicit function theorem as in Section 5.2. This uniquely determines the parameters y_k^+ and $\operatorname{Im}(y_{k,0})$ with $k \neq j$ as functions of (t, r) and the remaining parameters $y_{j,0}$, \tilde{y}_j^+ and $\operatorname{Re}(y_{k,0})$ with $k \neq j$. Moreover, at $r = 0$, $y_j^+ = 0$ and $y_k^+ = 0$, $k \neq j$, solve the equation $\mathcal{F}^+ = \mathcal{G}^+ = 0$. Since y_k^+ is an even function of r (due to uniqueness), this means that we can also write for $k \neq j$

$$y_k = y_{k,0} + r^2 \tilde{y}_k^+ \quad \text{with} \quad \tilde{y}_k^+ \in \mathcal{W}^+.$$

We decompose $\mathcal{K} = x_1^2 + x_2^2 + x_3^2$ as

$$\mathcal{K} = \mathcal{K}_{-1} \lambda^{-1} + \mathcal{K}_0 + \mathcal{K}^+ \quad \text{with} \quad \mathcal{K}^+ \in \mathcal{W}^+.$$

When $r = 0$, \mathcal{K} does not depend on λ , i.e., $\mathcal{K}_{-1} = 0$ and $\mathcal{K}^+ = 0$. Thus since \mathcal{K}_{-1} and \mathcal{K}^+ are even functions of r , this means that $\tilde{\mathcal{K}}_{-1} = r^{-2} \mathcal{K}_{-1}$ and $\tilde{\mathcal{K}}^+ = r^{-2} \mathcal{K}^+$ extend analytically at $r = 0$. More explicitly,

$$\begin{aligned} \mathcal{K} &= \sum_{k=1}^3 \left(r^2 \tilde{\underline{x}}_{k,-1} \lambda^{-1} + \tilde{\rho} \tilde{\underline{x}}_{k,0} + r^2 \tilde{\underline{x}}_{k,1} \lambda + y_{k,0} + r^2 \tilde{y}_k^+ \right)^2 \\ &= \sum_{k=1}^3 \left(\tilde{\underline{x}}_{k,0} + y_{k,0} \right)^2 + 2r^2 \left(\tilde{\underline{x}}_{k,0} + y_{k,0} \right) \left(\tilde{\underline{x}}_{k,-1} \lambda^{-1} + \tilde{\underline{x}}_{k,1} \lambda + \tilde{y}_k^+ \right) + O(r^4) \end{aligned}$$

from which we obtain at $r = 0$

$$\begin{aligned}\tilde{\mathcal{K}}_{-1} \big|_{r=0} &= 2 \sum_{k=1}^3 (\tilde{x}_{k,0} + y_{k,0}) \tilde{x}_{k,-1} \\ \mathcal{K}_0 \big|_{r=0} &= \sum_{k=1}^3 (\tilde{x}_{k,0} + y_{k,0})^2 \\ \tilde{\mathcal{K}}^+ \big|_{r=0} &= 2 \sum_{k=1}^3 (\tilde{x}_{k,0} + y_{k,0}) (\tilde{x}_{k,1} \lambda + \tilde{y}_k^+).\end{aligned}$$

In particular, at the central value $y = 0$, we have by Equation (84) that $\mathcal{K}_0 = 1$, $\tilde{\mathcal{K}}_{-1} = 0$ and $\tilde{\mathcal{K}}^+ = 0$ and the differentials with respect to y at $(t, r, y) = (0, 0, 0)$ are

$$\begin{aligned}d\tilde{\mathcal{K}}_{-1} &= 2 \sum_{k=1}^3 \tilde{x}_{k,-1} dy_{k,0} \\ d\mathcal{K}_0 &= 2 \sum_{k=1}^3 \tilde{x}_{k,0} dy_{k,0} \\ d\tilde{\mathcal{K}}^+ &= 2 \sum_{k=1}^3 \tilde{x}_{k,0} d\tilde{y}_k^+ + \lambda \tilde{x}_{k,1} dy_{k,0}.\end{aligned}$$

Keep in mind that \tilde{y}_k^+ and $\text{Im}(y_k^0)$ for $k \neq j$ have already been determined and their differential with respect to the remaining parameters $y_{j,0}$, \tilde{y}_j^+ and $\text{Re}(y_{k,0})$ with $k \neq j$, and are zero at $(t, r) = (0, 0)$. Consider the polynomials $\tilde{P}_k(\lambda) = \tilde{x}_{k,0}\lambda + \tilde{x}_{k,-1}$. Recall that $\tilde{x}_{j,0} \neq 0$, and let $\tilde{\mu}_j$ be the root of P_j . Then

$$d\tilde{\mathcal{K}}_{-1} + \tilde{\mu}_j d\mathcal{K}_0 = 2 \sum_{k \neq j} \tilde{P}_k(\tilde{\mu}_j) \text{Re}(dy_{k,0}).$$

By Claim 61 below, this is an isomorphism from \mathbb{R}^2 to \mathbb{C} . Hence the partial derivative of $(\tilde{\mathcal{K}}_{-1}, \mathcal{K}_0, \tilde{\mathcal{K}}^+)$ with respect to $((\text{Re}(y_{k,0}))_{k \neq j}, y_j^0, \tilde{y}_j^+)$ is an isomorphism from $\mathbb{R}^2 \times \mathbb{C} \times \mathcal{W}^+$ to $\mathbb{C}^2 \times \mathcal{W}^+$. The implicit function theorem then gives that there exists $\epsilon_1 > 0$ such that for $|t| < \epsilon_1$ and $|r| < \epsilon_1$, there exists unique values of the parameters which solve Problem (69). When $\epsilon_1/2 \leq |r| \leq 1/2$, Theorem 8 gives us a uniform $\epsilon_0 > 0$ such that for $|t| < \epsilon_0$, there exists unique values of the parameters which solve Problem (69). They certainly satisfy the ansatz (86) since $|r| \geq \epsilon_1/2$. Take $\epsilon_2 = \min(\epsilon_1, \epsilon_0)$. In the overlap $\epsilon_1/2 \leq |r| < \epsilon_1$, the solutions agree for $|t| < \epsilon_2$ by uniqueness of the implicit function theorem.

So far, we have proven point (1), and point (3) follows by construction. Since for $r = 0$, the connections are independent of λ , the reality condition (69) implies that the traces s_{jk} are real. Thus, the monodromy representation is either conjugated to a real representation or to a unitary representation. Using (3), it immediately follows that the traces for all $(j, k) \in \{(1, 2), (1, 3), (2, 3)\}$ $s_{jk} \in [-2, 2]$ for small t , which implies unitarity. \square

Claim 61. *For $k \neq j$ the two complex numbers $\tilde{P}_k(\tilde{\mu}_j)$ are linearly independent over \mathbb{R} .*

Proof. Assume for the simplicity of notation that $j = 1$ and that $\tilde{P}_2(\tilde{\mu}_1)$ and $\tilde{P}_3(\tilde{\mu}_1)$ are linearly dependent. If $\mu_1 \neq 0$, let

$$\alpha_k = \tilde{\mu}_1^{-1} \tilde{P}_k(\tilde{\mu}_1) = \tilde{x}_{k,0} + \tilde{x}_{k,-1} \tilde{\mu}_1^{-1}.$$

Using Equation (84) we have

$$\sum_{k=1}^3 \alpha_k^2 = 1.$$

Since $\alpha_1 = 0$ and α_2, α_3 are linearly dependent, all α_k must be real numbers. Since $\tilde{x}_{k,0}$ are real, each $\tilde{x}_{k,-1}\tilde{\mu}_1^{-1}$ must be real. Then using Equation (84) again, we have

$$\sum_{k=1}^3 (\tilde{x}_{k,-1}\tilde{\mu}_1^{-1})^2 = 0$$

implying that $\tilde{x}_{k,-1} = 0$ for all k contradicting Equation (85). Thus, $\tilde{P}_2(\tilde{\mu}_1)$ and $\tilde{P}_3(\tilde{\mu}_1)$ must be linearly independent, if $\mu_1 \neq 0$.

If $\mu_1 = 0$, then $\tilde{x}_{1,-1} = 0$. Then, if $\tilde{x}_{2,-1}, \tilde{x}_{3,-1}$ are linearly dependent, we would obtain

$$\sum_{k=1}^3 (\tilde{x}_{k,-1})^2 = 0,$$

which again gives $\tilde{x}_{k,-1} = 0$ for all k contradicting Equation (85). \square

Remark 62. *The constructed real holomorphic sections are uniquely determined by the residue $A_1(t, \lambda)$ at $z = p$ of the potential η_t (when fixing the sign in (62)). The deformation of $A = A_1$ in the parameter t can in fact be expressed by a Lax pair type equation*

$$A' = [A, X],$$

for some $X \in \Lambda^+ \mathfrak{sl}(2, \mathbb{C})$. Moreover, X is unique up to adding $g \cdot A$, where $g: \lambda \mapsto g(\lambda) \in \mathbb{C}$ is holomorphic around $\lambda = 0$.

To see this recall

$$A = \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{pmatrix}$$

satisfies $\det A = -1$ and thus $A^2 = \text{Id}$. Therefore, we obtain $A'A + AA' = 0$ which gives

$$x'_1 x_1 + x'_2 x_2 + x'_3 x_3 = 0.$$

Let

$$A' = \begin{pmatrix} x'_1 & x'_2 + ix'_3 \\ x'_2 - ix'_3 & -x'_1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}.$$

Then we can choose X to be given by

$$\begin{aligned} \alpha &= -\frac{1}{2}(x_2 - ix_3)(x'_2 + ix'_3) + x_1 B \\ \beta &= \frac{1}{2}x_1(x'_2 + ix'_3) + (x_2 + ix_3)B \\ \gamma &= -\frac{1}{2}x_1(x'_2 - ix'_3) + (x_2 - ix_3)(B + a). \end{aligned} \tag{87}$$

with $B = -\frac{1}{2}(x'_{2,0} + ix'_{3,0})\frac{x_{1,-1}}{x_{2,-1} + ix_{3,-1}}$ chosen to remove the negative powers of λ .

5.4. Twistor lines. It remains to determine whether the real holomorphic sections constructed in Theorem 8 and Theorem 9 are actually twistor lines. This finally determines the sign in (63). We start with the following observation:

Lemma 63. *Denote by $x^\pm(t) \in (\mathcal{W}_a^{\geq -1})^3$ the solutions of the monodromy provided by Theorem 8 for the two possible choices (61) of ρ in their initial data (56). We then have*

$$x^-(-t)(-\lambda) = x^+(t)(\lambda)$$

for all $\lambda \in \mathbb{D}_a^*$.

Proof. This directly follows from the uniqueness part in the implicit function theorem, compare with the proof of [26, Proposition 24]. \square

Remark 64. *This Lemma shows how to relate the two different choices of the sign for ρ in the initial data by changing the sign for the deformation parameter. Since we will show that only one of the sign choices leads to actual twistor lines, we will restrict to solutions for $t > 0$ in the following.*

Lemma 65. *For small $t > 0$, let $x_{u,v}^-(t) \in (\mathcal{W}_a^{\geq -1})^3$ be the solution of the monodromy problem provided by Theorem 8 with initial data (56) and $\rho < 0$ in (61), where $(u, v) \in \mathbb{C}^2 \setminus \{0\}$ parametrizes the Higgs field via (59). Let $s_{u,v}^-(t)$ be the corresponding real holomorphic section of the parabolic Deligne-Hitchin moduli space. Then the limit*

$$\lim_{r \rightarrow 0} s_{ru,rv}^- = s_{[u,v]}$$

is not a real holomorphic section, and $s_{ru,rv}^-$ are not twistor lines.

Proof. By Theorem 9, the families of flat connections extend to $r = 0$. This is true for both possible signs of ρ , and it can be directly computed that the corresponding limit for negative ρ differs from (83) by an overall sign. Therefore, in the limit $(t, r) \rightarrow 0$, the quasiparabolic lines (the eigenlines with respect to the *positive* eigenvalue of the residue A_j determined by x_1, x_2, x_3) at any fixed $\lambda_0 \in \mathbb{C}^*$ are given by

$$(88) \quad \ell_1 = (-\bar{v}, \bar{u})^T \mathbb{C}, \quad \ell_2 = (\bar{u}, \bar{v})^T \mathbb{C}, \quad \ell_3 = (\bar{v}, \bar{u})^T \mathbb{C}, \quad \ell_4 = (-\bar{u}, \bar{v})^T \mathbb{C}.$$

On the other hand, the quasiparabolic lines of $s_{ru,rv}^-$ for $r \neq 0$ at $\lambda = 0$ are given by (30). Hence, the parabolic structure at $\lambda = 0$ differs from (88) for $(u, v) \neq 0$. Therefore, for $(u, v) \in \mathbb{C}^2 \setminus \{0\}$, the $r \rightarrow 0$ limit cannot exist.

But this contradicts the fact that the space of twistor lines is open and closed: if $s_{ru,rv}^-$ would be twistor lines for all $r \neq 0$, the limit would exist and would be a twistor line as well. Thus $s_{ru,rv}^-$ cannot be a twistor line for all (u, v) . \square

Lemma 65 shows that taking the limit $r \rightarrow 0$ is delicate in general. But using additional symmetries, we have

Lemma 66. *Consider the 4-punctured sphere given by $p = \exp(\frac{\pi i}{4})$, and let $(\tilde{u}, \tilde{v}) = \frac{1}{\sqrt{2}}(1, \exp(\frac{\pi i}{4}))$. Consider the solution provided by Theorem 9 for $\rho > 0$. Then, we have at $r = 0$ that (83) holds for all $t > 0$ small enough. In particular, the corresponding real holomorphic sections parametrised by r and t have a limit for $r \rightarrow 0$ which is a twistor line.*

Proof. The aim is to show that for this special choice of parameters in Theorem 9 and (83) holds for all $t \in (0, \frac{1}{4})$, where the implicit function theorem applies, rather than only at $(r, t) = 0$.

First, we show that the logarithmic connection given by (83) with singular divisor determined by p and quasiparabolic lines given by (\tilde{u}, \tilde{v}) has unitary monodromy for all $t \in (0, \frac{1}{4})$. Due the choice of p , there is an additional symmetry $z \mapsto iz$ which maps the set of singular points into itself. Moreover, the special choice of (\tilde{u}, \tilde{v}) gives that the logarithmic connection is equivariant with respect to this order four symmetry, and therefore descend to a logarithmic $\mathrm{SL}(2, \mathbb{C})$ connection $\tilde{\nabla}$ on the 3-punctured sphere. On the 3-punctured sphere, any monodromy representation is uniquely determined (up to conjugation) by the parabolic weights, and $\tilde{\nabla}$ must be unitary for $t \in (0, \frac{1}{4})$, see [8, Lemma 2.2] for details. Moreover, the parabolic structure for the pull-back of $\tilde{\nabla}$ to the 4-punctured sphere is uniquely determined and thus given by (\tilde{u}, \tilde{v}) .

Next, consider for $(\tilde{u}, \tilde{v}) = \frac{1}{\sqrt{2}}(1, \exp(\frac{\pi i}{4}))$ and $r \neq 0$ the solution $(x_1, x_2, x_3) = x(t, p, r, \tilde{u}, \tilde{v})$ provided by Theorem 9. Then we have

$$(x_1, x_2, x_3)(-\lambda) = -(x_1, x_3, x_2)(\lambda)$$

for all $\lambda \in \mathbb{C}^*$ by the uniqueness part of the implicit function theorem, together with the symmetry $z \mapsto iz$. As a consequence, the $r \rightarrow 0$ limit has the corresponding symmetry as well. But by Theorem 9, point (2), the limit is given by a unitary logarithmic connection ∇ independent of λ , symmetric with respect to $z \mapsto iz$. This gives that ∇ descends to the 3-punctured sphere and thus has unitary monodromy for $t \in (0, \frac{1}{4})$. Therefore ∇ is the pull-back of $\tilde{\nabla}$, and thus (83) holds for all small $t \in (0, \frac{1}{4})$. \square

Theorem 10. *The real sections constructed in Theorem 8 and Theorem 9 (for $t > 0$ and $\rho > 0$) are twistor lines.*

Proof. For $p = \exp(\frac{\pi i}{4})$ and small rational t , this follows from Lemma 66 and Proposition 41. The general case then follows by continuous dependency of real holomorphic sections from the parameters $(p, t, r, \tilde{u}, \tilde{v})$ (Theorem 9) and (p, t, u, v) (Theorem 8), respectively, together with Remark 42 for non-rational weights. \square

6. THE HYPER-KÄHLER STRUCTURE AND THE NON-ABELIAN HODGE CORRESPONDENCE

In this section we explicitly describe the (rescaled) metric and the non-abelian Hodge correspondence at the limit $t = 0$. We then compute the first order derivatives with respect to t at $t = 0$.

6.1. The non-abelian Hodge correspondence at $t = 0$. The parabolic non-abelian Hodge correspondence on the rank 2 hermitian bundle \mathcal{V} is a diffeomorphism that associates to each stable strongly parabolic Higgs pair $(\bar{\partial}_{\mathcal{V}}, \Phi)$ the logarithmic connection $\nabla^{\lambda=1}$ of the associated family of flat connections. In the case of Fuchsian potentials on a 4-punctured sphere with parabolic weight $t \in (0, \frac{1}{4})$, the underlying holomorphic structure is trivial and the Higgs pair is given by the strongly parabolic Higgs field Ψ only. Due to the symmetry assumptions, Ψ is fully determined by its residue at $p \in \mathbb{C}^+$ and since $\sum_{j=1}^3 x_{j,-1}^2 = 0$, this residue is nilpotent. On the other hand, the connection 1-form for the flat $\mathrm{SL}(2, \mathbb{C})$ -connection $\nabla^{\lambda=1}$ is determined by $A_1(\lambda = 1)$ satisfying $\det A_1 = -1$ and $\mathrm{tr} A_1 = 0$. Hence A_1 lies in the $\mathrm{SL}(2, \mathbb{C})$ adjoint orbit of $(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})$.

Consider the complex 3-dimensional vector space $\mathfrak{sl}(2, \mathbb{C})$ with its complex bilinear inner product

$$\langle \xi, \eta \rangle = -\frac{1}{2} \mathrm{tr}(\xi \eta).$$

Then its associated quadratic form is the determinant \det . Decompose $\mathfrak{sl}(2, \mathbb{C})$ into real subspaces

$$\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$$

consisting of the subspace of skew-hermitian ($A = -\bar{A}^T$) and the subspace of hermitian symmetric ($A = \bar{A}^T$) trace-free matrices. Note that $\langle \cdot, \cdot \rangle$ is positive-definite on the 3-dimensional real subspace $\mathfrak{su}(2)$ and negative definite on $i\mathfrak{su}(2)$.

Lemma 67. *There is a diffeomorphism between the $\mathrm{SL}(2, \mathbb{C})$ orbit through $(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})$ without the hermitian symmetric matrices and the nilpotent $\mathrm{SL}(2, \mathbb{C})$ orbit in $\mathfrak{sl}(2, \mathbb{C})$.*

Proof. Let Ψ be an element in the nilpotent orbit. Then, $\Phi = \Psi - \bar{\Psi}^T$ is skew-hermitian and there is a unique skew-hermitian $N \in \mathfrak{su}(2)$ of length 1 such that

$$\langle N, \Psi - \bar{\Psi}^T \rangle = 0, \quad \langle N, i\Psi + i\bar{\Psi}^T \rangle = 0$$

and

$$N \times (\Psi - \bar{\Psi}^T) := \frac{1}{2} [N, \Psi - \bar{\Psi}^T] = i\Psi + i\bar{\Psi}^T.$$

Moreover,

$$(89) \quad A_{\Psi} := -\sqrt{1 + \langle \Phi, \Phi \rangle} iN + \Phi \in \mathfrak{sl}(2, \mathbb{C})$$

has determinant -1 and the map

$$\Psi \longmapsto A_{\Psi}$$

is smooth. Note that

$$A_1(\lambda) = \lambda^{-1} \Psi - \sqrt{1 + \langle \Phi, \Phi \rangle} iN - \lambda \bar{\Psi}^T$$

is of the form (54). Plugging in the formulas for the Higgs field Ψ in terms of the (u, v) coordinates, $-\sqrt{1 + \langle \Phi, \Phi \rangle} iN$ is the constant term of the initial values (62) with ρ being positive.

For the converse direction, let $A \in \mathfrak{sl}(2, \mathbb{C})$ be of determinant -1 , such that A is not hermitian symmetric, i.e.,

$$\Phi := \frac{1}{2}(A - \bar{A}^T) \neq 0.$$

Since $\det(A) = -1$ and $\langle \cdot, \cdot \rangle$ is positive definite on $\mathfrak{su}(2)$, also

$$\xi := \frac{1}{2}(A + \bar{A}^T)$$

does not vanish. Define

$$N := \frac{i}{\sqrt{-\langle \xi, \xi \rangle}} \xi \in \mathfrak{su}(2).$$

Note that $\langle N, \Phi \rangle = 0$ since $\det(A)$ is real. Then,

$$\Psi_A := \frac{1}{2}(\Phi - iN \times \Phi)$$

is nilpotent, and using (89)

$$A_{\Psi_A} = A.$$

Moreover, the map $A \mapsto \Psi_A$ is also smooth. \square

Recall that we identify the nilpotent orbit with $\mathbb{C}^2 \setminus (0, 0)/\mathbb{Z}_2$ via the $2 : 1$ map

$$\mathbb{C}^2 \setminus \{(0, 0)\} \longrightarrow \{\Psi \in \mathfrak{sl}(2, \mathbb{C}) \mid \Psi \neq 0; \det(\Psi) = 0\}, \quad (u, v) \longmapsto \Psi := \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix}.$$

Note that

$$\Psi \begin{pmatrix} u \\ v \end{pmatrix} = 0.$$

In the following, we consider the *rescaled* space of Higgs bundles, i.e., we study Higgs fields of the form $t\Psi$ on parabolic structures with trivial underlying holomorphic bundle and parabolic weight $t > 0$ small. On the deRham moduli space side, we consider logarithmic connections of the form $d + t\eta$ which converge to the trivial connection for $t \rightarrow 0$. We obtain

Theorem 11. *The diffeomorphism in Lemma 67 extends to a diffeomorphism of the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ at $(0, 0)$ which is $T^*\mathbb{CP}^1$, to the full adjoint $\mathrm{SL}(2, \mathbb{C})$ orbit through $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (including hermitian symmetric matrices). This map is the limit of the non-abelian Hodge correspondence for $t \rightarrow 0$ for rescaled strongly parabolic Higgs fields with trivial underlying holomorphic bundle.*

Proof. For $(u, v) \rightarrow 0$ take $u = r\tilde{u}$ and $v = r\tilde{v}$ with $|\tilde{u}|^2 = |\tilde{v}|^2 = 1$ and $r \in \mathbb{R}_{>0}$ and consider $r \rightarrow 0$. Let $\Psi(u, v) = r^2\Psi(\tilde{u}, \tilde{v})$ be the associated nilpotent matrix. Then the map $\Psi \mapsto A_\Psi$ extends to $r = 0$ with

$$\lim_{r \rightarrow 0} A_{\Psi(u, v)} = -i\tilde{N}$$

with $\tilde{N} \in \mathfrak{su}(2, \mathbb{C})$ of length 1 satisfying

$$\langle \tilde{N}, \tilde{\Psi} - \overline{\tilde{\Psi}}^T \rangle = 0 \quad \text{and} \quad \langle \tilde{N}, i\tilde{\Psi} + i\overline{\tilde{\Psi}}^T \rangle = 0.$$

Conversely, all hermitian symmetric matrices of determinant -1 can be realized as a limit. It follows from Theorem 8 and Theorem 9 that this map is the non-abelian Hodge correspondence for $t \rightarrow 0$. \square

6.2. The rescaled metric at $t = 0$. By Theorem 11 the rescaled Higgs bundle moduli space at $t = 0$ is the completion of the nilpotent $\mathrm{SL}(2, \mathbb{C})$ orbit, identified with the blow-up of $\mathbb{C}^2 \setminus \{(0, 0)\}/\mathbb{Z}_2$ at the origin, which is mapped by the limit of the non-abelian Hodge correspondence at $t = 0$ to the $\mathrm{SL}(2, \mathbb{C})$ orbit through $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The latter can be interpreted as the rescaled limit of the Betti moduli space at the identity, compare with the vanishing of Q_j in the proof of Proposition 58. Next, we show the rescaled limit hyper-Kähler metric on the rescaled Higgs bundle moduli space at $t = 0$ is the Eguchi-Hanson metric. The scaling factor $\frac{1}{t}$ is chosen so that the central sphere $\mathbb{C}P^1$ in \mathcal{M}_{Higgs} , which is the moduli space of semi-stable parabolic bundles (i.e., Higgs pairs with vanishing Higgs fields), has constant volume independent of the weight t . This is in accordance with the scaling factor k in Proposition 47 when $t = \frac{l}{k}$ is rational.

Lemma 68. *Let $t \sim 0$ and consider the space of solutions $\mathcal{M}(t)$ provided by Theorem 8 and Theorem 9, parametrized by $(u, v) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. In terms of the parameters $x_1(t), x_2(t), x_3(t)$ in (51), the rescaled twisted holomorphic symplectic form is given by*

$$\varpi = \frac{dx_2(t) \wedge dx_3(t)}{x_1(t)}.$$

Proof. By Proposition 18 the twisted holomorphic symplectic form is given by the Goldman symplectic form, which can be computed by (46). Moreover, since we consider Fuchsian systems, the integral on Σ^0 in (46) vanishes and we only need to compute the boundary terms. By construction all 4 residues $B_j = tA_j$ give the same contribution. A direct computation shows that

$$(90) \quad \frac{1}{8\mathrm{tr}(B_1^2)} \mathrm{tr}[(B_1[dB_1, dB_1])] = -\frac{t}{4}i(x_1dx_2 \wedge dx_3 - x_2dx_1 \wedge dx_3 + x_3dx_1 \wedge dx_2).$$

On the other hand,

$$0 = x_1dx_1 + x_2dx_2 + x_3dx_3$$

which combines with (90) (together with the rescaling by $\frac{1}{t}$) to

$$\varpi = \frac{4i}{t} \frac{1}{8\mathrm{tr}(B_1^2)} \mathrm{tr}[(B_1[dB_1 \wedge dB_1])] = \frac{dx_2 \wedge dx_3}{x_1}$$

proving the lemma. \square

For an explicit formula of the rescaled twisted symplectic form in Lemma 68 at $t = 0$ let $A = A_1(\lambda)$ be the residue of the rescaled potential $\frac{1}{t}\eta_{t=0}$ (see (50)) at $t = 0$. Recall that

$$A = \lambda^{-1} \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix} + \rho \begin{pmatrix} |u|^2 - |v|^2 & 2u\bar{v} \\ 2\bar{u}v & |v|^2 - |u|^2 \end{pmatrix} + \lambda \begin{pmatrix} -\bar{u}v & -\bar{v}^2 \\ \bar{u}^2 & \bar{u}\bar{v} \end{pmatrix},$$

i.e., we compute the symplectic form using the Higgs field coordinates (u, v) , where $r^2 = |u|^2 + |v|^2$ and $\rho = \sqrt{1 + r^{-4}}$. A direct computation then gives at $t = 0$

$$(91) \quad \begin{aligned} \varpi = 2i \left(-\frac{r^6 + |v|^2}{\rho r^6} du \wedge d\bar{u} - \lambda^{-1} du \wedge dv + \frac{\bar{u}v}{\rho r^6} du \wedge d\bar{v} \right. \\ \left. - \frac{u\bar{v}}{\rho r^6} d\bar{u} \wedge dv - \lambda d\bar{u} \wedge d\bar{v} - \frac{r^6 + |u|^2}{\rho r^6} dv \wedge d\bar{v} \right). \end{aligned}$$

On the other hand,

$$\varpi = \lambda^{-1}(\omega_J + i\omega_K) - 2\omega_I - \lambda(\omega_J - i\omega_K)$$

from which we obtain

$$\begin{aligned} \omega_I &= \frac{i}{\rho r^6} ((r^6 + |v|^2) du \wedge d\bar{u} - \bar{u}v du \wedge d\bar{v} + u\bar{v} d\bar{u} \wedge dv + (r^6 + |u|^2) dv \wedge d\bar{v}) \\ \omega_J &= i(-du \wedge dv + d\bar{u} \wedge d\bar{v}) \\ \omega_K &= -(du \wedge dv + d\bar{u} \wedge d\bar{v}). \end{aligned}$$

Proposition 69. *The rescaled hyper-Kähler metric of strongly parabolic Higgs bundles on the 4-punctured sphere at $t = 0$ is the Eguchi-Hanson space modulo a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action.*

Proof. Since (u, v) corresponds to Higgs bundle coordinates, we use the complex structure I to compute $g = \omega_I(\cdot, I\cdot)$. Consider the tangent space basis

$$\mathcal{B} = \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial \bar{u}}, \frac{\partial}{\partial v}, \frac{\partial}{\partial \bar{v}} \right)$$

Then the complex structure I can be represented by the diagonal matrix $\text{diag}(i, -i, i, -i)$. and we obtain

$$g = \sqrt{1 + r^{-4}} \left[du \otimes d\bar{u} + d\bar{u} \otimes du + dv \otimes d\bar{v} + d\bar{v} \otimes dv - \frac{1}{r^2(1 + r^4)} \left(u\bar{u}(du \otimes d\bar{u} + d\bar{u} \otimes du) + \bar{u}v(dv \otimes d\bar{v} + d\bar{v} \otimes dv) + u\bar{v}(dv \otimes d\bar{u} + d\bar{u} \otimes dv) + v\bar{v}(dv \otimes d\bar{v} + d\bar{v} \otimes dv) \right) \right]$$

which by [34, Equation 2] identifies with the Eguchi-Hanson metric with $n = 2$ and $a = 1$. \square

Obviously, the Eguchi-Hanson metric is independent of the conformal type of the 4-punctured sphere. On the other hand, it is known that the conformal type of the underlying 4-punctured sphere can be recovered from the Hitchin metric of the parabolic Higgs bundle moduli space, see [12] and [18]. We will see the non-trivial dependence of the metric on the conformal type in the first order approximation of the metric in t .

6.3. First order approximations. The advantage of our setup is that we obtain, analogously to [26], an iterative way of computing the power series expansion of the twistor lines leading to an approach towards explicitly computing the non-abelian Hodge correspondence and all involved geometric quantities for the case at hand. In this section, we compute the first order derivatives of the parameters (x_1, x_2, x_3) and derive in particular first order derivatives of the relative twisted holomorphic symplectic form. This yields derivatives of the non-abelian Hodge correspondence as well as derivatives of the hyper-Kähler metric.

6.3.1. First order derivatives of the parameters. Define for $1 \leq j, k \leq 3$ and ω_k, Ω_j given in (48) and (74), respectively,

$$\Omega_{jk}(z) = \int_0^z \Omega_j \omega_k.$$

The shuffle relation (see e.g. [26, Appendix]) then gives

$$(92) \quad \Omega_j \Omega_k = \Omega_{jk} + \Omega_{kj}.$$

Let $()'$ and $()''$ denote the first and second order derivatives of a quantity with respect to t at $t = 0$. Then the following proposition holds.

Proposition 70. *The first order derivatives x'_j for $j = 1, 2, 3$ are polynomials of degree at most 2 in λ :*

$$x'_j = x'_{j,0} + x'_{j,1}\lambda + x'_{j,2}\lambda^2 = x'_{j,0} + (x'_j)^+$$

with the positive parts given by

$$(93) \quad \begin{cases} (x'_1)^+ = \frac{4i}{\pi} \text{Im}(\Omega_{21}(1) + \Omega_{31}(i)) (x_2 x_3)^+ \\ (x'_2)^+ = \frac{-4i}{\pi} \text{Im}(\Omega_{31}(i)) (x_1 x_3)^+ \\ (x'_3)^+ = \frac{-4i}{\pi} \text{Im}(\Omega_{21}(1)) (x_1 x_2)^+, \end{cases}$$

and the constant terms given by

$$(94) \quad \begin{cases} x'_{1,0} = \frac{-1}{\rho r^4} ((|u|^2 - |v|^2)X - 2\rho uvY) \\ x'_{2,0} = \frac{-1}{\rho r^4} (2\operatorname{Re}(u\bar{v})X - \rho(v^2 - u^2)Y) \\ x'_{3,0} = \frac{-1}{\rho r^4} (2\operatorname{Im}(u\bar{v})X - \rho i(u^2 + v^2)Y), \end{cases}$$

where $r^2 = |u|^2 + |v|^2$, ρ satisfying (63) and

$$(95) \quad \begin{aligned} X &= \sum_{j=1}^3 \underline{x}_{j,-1} x'_{j,1} \\ Y &= \sum_{j=1}^3 (\underline{x}_{j,-1} x'_{j,2} + \underline{x}_{j,0} x'_{j,1}), \end{aligned}$$

with $x'_{j,1}$ and $x'_{j,2}$ determined by (93).

Remark 71. The required Ω integrals are computed in Proposition 73 to be

$$\operatorname{Im}(\Omega_{21}(1)) = 2\pi \log \left| \frac{p^2 - 1}{2p} \right| \quad \text{and} \quad \operatorname{Im}(\Omega_{31}(i)) = -2\pi \log \left| \frac{p^2 + 1}{2p} \right|.$$

Proof. Let Φ be the fundamental solution of the equation $d\Phi_t = \Phi_t \eta_t$ with $\Phi_t(0) = \operatorname{Id}$. Recall that $\eta_{t=0} = 0$ and therefore $\Phi_{t=0} = \operatorname{Id}$. Differentiating the equation $d\Phi_t = \Phi_t \eta_t$ twice at $t = 0$ we thus obtain

$$d\Phi'' = \eta'' + 2\Phi' \eta',$$

hence

$$\Phi''(z) = \int_0^z (\eta'' + 2\Phi' \eta').$$

Then we have using the proof of Proposition 57

$$\eta' = \sum_{j=1}^3 \underline{x}_j \omega_j \mathbf{m}_j, \quad \Phi' = 2 \sum_{j=1}^3 \underline{x}_j \Omega_j \mathbf{m}_j, \quad \eta'' = 2 \sum_{j=1}^3 x'_j \omega_j \mathbf{m}_j.$$

This gives

$$\int_0^z \eta'' = 2 \begin{pmatrix} x'_1 \Omega_1 & x'_2 \Omega_2 + i x'_3 \Omega_3 \\ x'_2 \Omega_2 - i x'_3 \Omega_3 & -x'_1 \Omega_1 \end{pmatrix}$$

and

$$\int_0^z \Phi' \eta' = \begin{pmatrix} \sum_{j=1}^3 \underline{x}_j^2 \Omega_{jj} + i \underline{x}_2 \underline{x}_3 (\Omega_{32} - \Omega_{23}) & \underline{x}_1 \underline{x}_2 (\Omega_{12} - \Omega_{21}) + i \underline{x}_1 \underline{x}_3 (\Omega_{13} - \Omega_{31}) \\ \underline{x}_1 \underline{x}_2 (\Omega_{21} - \Omega_{12}) + i \underline{x}_1 \underline{x}_3 (\Omega_{13} - \Omega_{31}) & \sum_{j=1}^3 \underline{x}_j^2 \Omega_{jj} + i \underline{x}_2 \underline{x}_3 (\Omega_{23} - \Omega_{32}) \end{pmatrix}.$$

Using Leibniz rule, the shuffle relation (92) and that we solved Equation 75 we obtain

$$(96) \quad \begin{aligned} \mathfrak{p}'' &= \mathcal{P}_{12}'' - \mathcal{P}_{21}'' + 2(\mathcal{P}_{11}' \mathcal{P}_{21}' - \mathcal{P}_{12}' \mathcal{P}_{22}') \\ &= -4ix'_3 \Omega_3(1) + 4\underline{x}_1 \underline{x}_2 (\Omega_{21}(1) - \Omega_{12}(1)) + 4\underline{x}_1 \underline{x}_2 \Omega_1(1) \Omega_2(1) \\ &= 4\pi x'_3 + 8\underline{x}_1 \underline{x}_2 \Omega_{21}(1) \end{aligned}$$

$$(97) \quad \begin{aligned} \mathfrak{q}'' &= i(\mathcal{Q}_{21}'' + \mathcal{Q}_{12}'') + 2i(\mathcal{Q}_{11}' \mathcal{Q}_{21}' + \mathcal{Q}_{12}' \mathcal{Q}_{22}') \\ &= 4ix'_2 \Omega_2(i) - 4\underline{x}_1 \underline{x}_3 (\Omega_{13}(i) - \Omega_{31}(i)) + 4\underline{x}_1 \underline{x}_3 \Omega_1(i) \Omega_3(i) \\ &= 4\pi x'_2 + 8\underline{x}_1 \underline{x}_3 \Omega_{31}(i). \end{aligned}$$

Since we have solved $\mathbf{p}(t) = \mathbf{p}(t)^*$ for all t , we have $\mathbf{p}'' = (\mathbf{p}'')^*$. Moreover, $\underline{x}_j = \underline{x}_j^*$, hence

$$(98) \quad 4\pi x'_3 - 4\pi(x'_3)^* = -16i \underline{x}_1 \underline{x}_2 \operatorname{Im}(\Omega_{21}(1)).$$

Projecting onto \mathcal{W}^+ and remembering that $x' \in \mathcal{W}^{\geq 0}$, we obtain the formula for $(x'_3)^+$ stated in Equation (93), and $(\underline{x}_1 \underline{x}_2)^+$ is a degree-2 polynomial. In the same vein $\mathbf{q}'' = (\mathbf{q}'')^*$ gives

$$(99) \quad 4\pi x'_2 - 4\pi(x'_2)^* = -16i \underline{x}_1 \underline{x}_3 \operatorname{Im}(\Omega_{31}(i))$$

which determines $(x'_2)^+$.

For $(x'_1)^+$ consider the equation $\mathcal{K} = 1$ which holds for all t . Therefore,

$$(100) \quad \mathcal{K}' = 0 = 2 \sum_{j=1}^3 \underline{x}_j x'_j.$$

Then $\mathcal{K}' = \mathcal{K}'^*$ and Equations (98), (99) give the equation

$$(101) \quad 4\pi \underline{x}_1 x'_1 - 4\pi \underline{x}_1 (x'_1)^* = 16i \underline{x}_1 \underline{x}_2 \underline{x}_3 \operatorname{Im}(\Omega_{21}(1) + \Omega_{31}(i)).$$

Dividing by $\underline{x}_1 \not\equiv 0$ and taking the positive part determines $(x'_1)^+$.

To compute the constant terms, we consider the coefficients of λ^{-1} , λ^0 and λ in \mathcal{K} :

$$\begin{aligned} \mathcal{K}'_{-1} &= 2 \sum_{j=1}^3 \underline{x}_{j,1} x'_{j,0} = 0 \\ \mathcal{K}'_0 &= 2 \sum_{j=1}^3 (\underline{x}_{j,0} x'_{j,0} + \underline{x}_{j,-1} x'_{j,1}) = 0 \\ \mathcal{K}'_1 &= 2 \sum_{j=1}^3 (\underline{x}_{j,1} x'_{j,0} + \underline{x}_{j,0} x'_{j,1} + \underline{x}_{j,-1} x'_{j,2}) \end{aligned}$$

which yield the system of equations

$$\begin{cases} \sum_{j=1}^3 \underline{x}_{j,-1} x'_{j,0} = 0 \\ \sum_{j=1}^3 \underline{x}_{j,0} x'_{j,0} = -X \\ \sum_{j=1}^3 \underline{x}_{j,1} x'_{j,0} = -Y \end{cases}$$

with X, Y as in Proposition 70. Its determinant simplifies to

$$\det(\underline{x}_{j,k})_{1 \leq j \leq 3, -1 \leq k \leq 1} = \frac{i}{2} \rho r^6.$$

Using the Cramer rule, we obtain (94) after simplification. \square

Remark 72. *The second derivative of \mathbf{r} is computed in the following. Through the character variety equation and the formulas for \mathbf{p}'' and \mathbf{q}'' it will give rise to an identity between Ω -integrals. Using Proposition 56 we have*

$$\begin{aligned} \mathbf{r}'' &= i[\mathcal{P}_{22}'' + 2\mathcal{P}_{22}' \mathcal{Q}_{11}' + \mathcal{Q}_{11}'' + 2\mathcal{P}_{12}' \mathcal{Q}_{21}' - 2\mathcal{P}_{21}' \mathcal{Q}_{12}' - \mathcal{P}_{11}'' - 2\mathcal{P}_{11}' \mathcal{Q}_{22}' - \mathcal{Q}_{22}'' \\ &\quad + (\mathcal{P}_{22}' + \mathcal{Q}_{11}')^2 + (\mathcal{P}_{12}' + \mathcal{Q}_{12}')^2 - (\mathcal{P}_{21}' + \mathcal{Q}_{21}')^2 - (\mathcal{P}_{11}' + \mathcal{Q}_{22}')^2] \\ &= i[\mathcal{P}_{22}'' - \mathcal{P}_{11}'' + \mathcal{Q}_{11}'' - \mathcal{Q}_{22}'' + 2(\mathcal{P}_{12}' - \mathcal{P}_{21}')(\mathcal{Q}_{12}' + \mathcal{Q}_{21}') + (\mathcal{P}_{12}')^2 - (\mathcal{P}_{21}')^2 + (\mathcal{Q}_{12}')^2 - (\mathcal{Q}_{21}')^2] \end{aligned}$$

where we used that $\mathcal{P}'_{11} + \mathcal{P}'_{22} = \mathcal{Q}'_{11} + \mathcal{Q}'_{22} = 0$. This gives

$$\begin{aligned} \mathfrak{r}'' &= i \left[-4x'_1 \Omega_1(1) + 4i\mathfrak{x}_2 \mathfrak{x}_3 (\Omega_{23}(1) - \Omega_{32}(1)) + 4x'_1 \Omega_1(i) - 4i\mathfrak{x}_2 \mathfrak{x}_3 (\Omega_{23}(i) - \Omega_{32}(i)) \right. \\ &\quad \left. + 8i\mathfrak{x}_2 \mathfrak{x}_3 \Omega_3(1) \Omega_2(i) + 4i\mathfrak{x}_2 \mathfrak{x}_3 \Omega_3(1) \Omega_2(1) + 4i\mathfrak{x}_2 \mathfrak{x}_3 \Omega_2(i) \Omega_3(i) \right] \\ (102) \quad &= 4\pi x'_1 - 8\mathfrak{x}_2 \mathfrak{x}_3 (\Omega_{23}(1) + \Omega_{32}(i) + \pi^2) \end{aligned}$$

using that we solved Equation 75 and the shuffle relation (92). Observe the similarity with (96) and (97). Remember from the proof of Proposition 58 that \mathfrak{p} , \mathfrak{q} and \mathfrak{r} satisfy the character variety equation for all t

$$(103) \quad 4 \cos(2\pi t)^2 + 4(\mathfrak{p}^2 + \mathfrak{q}^2 + \mathfrak{r}^2 - 1) + 8(-1)^j \mathfrak{p} \mathfrak{q} \mathfrak{r} = 0$$

with either $j = 1$ or $j = 2$. Taking the third order derivative of the above equation yields (since $\mathfrak{p}(0) = \mathfrak{q}(0) = \mathfrak{r}(0) = 0$)

$$\mathfrak{p}'\mathfrak{p}'' + \mathfrak{q}'\mathfrak{q}'' + \mathfrak{r}'\mathfrak{r}'' + 2(-1)^j \mathfrak{p}'\mathfrak{q}'\mathfrak{r}' = 0$$

Using Proposition 57 and Equations (96), (97), (102) we then obtain

$$8\pi^2(\mathfrak{x}_1 x'_1 + \mathfrak{x}_2 x'_2 + \mathfrak{x}_3 x'_3) + 16\pi \mathfrak{x}_1 \mathfrak{x}_2 \mathfrak{x}_3 (\Omega_{21}(1) + \Omega_{31}(i) - \Omega_{23}(1) - \Omega_{32}(i) - \pi^2 + (-1)^j \pi^2) = 0.$$

The first summand vanishes, as $x_1^2 + x_2^2 + x_3^2 = 1$ for all t , thus

$$\Omega_{21}(1) + \Omega_{31}(i) - \Omega_{23}(1) - \Omega_{32}(i) - \pi^2 + (-1)^j \pi^2 = 0.$$

In the most symmetric case of the 4-punctured sphere where $p = e^{i\pi/4}$ we have by symmetry

$$\Omega_{31}(i) = -\Omega_{21}(1) \quad \text{and} \quad \Omega_{32}(i) = -\Omega_{23}(i).$$

Hence $j = 2$ and we have proved the following identity that holds for all $p \in \mathbb{C}^+$:

$$(104) \quad \Omega_{23}(1) + \Omega_{32}(i) = \Omega_{21}(1) + \Omega_{31}(i).$$

Proposition 73. For $p \in \mathbb{C}^+$ we have

$$\Omega_{21}(1) = 2\pi i \log \left(\frac{p^2 - 1}{2ip} \right) \quad \text{and} \quad \Omega_{31}(i) = -2\pi i \log \left(\frac{p^2 + 1}{2p} \right),$$

where \log denote the principal valuation of the logarithm on $\mathbb{C} \setminus \mathbb{R}^-$.

Proof. We first prove the Proposition for $p = e^{i\varphi}$ with $0 < \varphi < \pi/2$. In that case, we have by [26, Proposition 35]

$$\begin{aligned} (105) \quad \Omega_{21}(1) - \Omega_{12}(1) &= 4\pi i \log(\sin \varphi) - i(\pi - 2\varphi) \log \left(\frac{1 - \cos(\varphi)}{1 + \cos(\varphi)} \right) \\ \Omega_{31}(i) - \Omega_{13}(i) &= -4\pi i \log(\cos(\varphi)) + 2i\varphi \log \left(\frac{1 - \sin(\varphi)}{1 + \sin(\varphi)} \right). \end{aligned}$$

On the other hand, by the shuffle product formula and using (75)

$$\begin{aligned} \Omega_{21}(1) + \Omega_{12}(1) &= \Omega_1(1) \Omega_2(1) = i(\pi - 2\varphi) \log \left(\frac{1 - \cos(\varphi)}{1 + \cos(\varphi)} \right) \\ \Omega_{31}(i) + \Omega_{13}(i) &= \Omega_1(i) \Omega_3(i) = -2i\varphi \log \left(\frac{1 - \sin(\varphi)}{1 + \sin(\varphi)} \right). \end{aligned}$$

Hence

$$\begin{aligned} \Omega_{21}(1) &= 2\pi i \log(\sin \varphi) \\ \Omega_{31}(i) &= -2\pi i \log(\cos(\varphi)) \end{aligned}$$

proving the result for $p = e^{i\varphi}$.

For $p \in \mathbb{C}^+$, both $\frac{p^2-1}{2ip}$ and $\frac{p^2+1}{2p}$ are in $\mathbb{C} \setminus \mathbb{R}^-$ and both sides of the formulas of Proposition 73 are well-defined holomorphic functions in $p \in \mathbb{C}^+$ which coincide when $p \in \mathbb{C}^+ \cap \mathbb{S}^1$, therefore they are equal. \square

6.3.2. First order derivative of the metric. By Lemma 68 we have $\varpi = \frac{dx_2 \wedge dx_3}{x_1}$ for all t . Recall from Proposition 32 that the open dense subset of the Higgs bundle moduli space specified by trivial holomorphic bundles is independent of the weight t . Thus the complex structure I is independent of t and also the holomorphic symplectic form $\omega_J + i\omega_K$ are independent of t . The same holds true for the holomorphic symplectic form $\omega_J - i\omega_K$ for the complex structure $-I$. This gives that the first order derivatives of the twisted holomorphic symplectic form is given by the derivative of its constant term

$$(106) \quad \varpi_0 = -2\omega_I = \frac{1}{x_{1,-1}} \left(\frac{-x_{1,0}}{x_{1,-1}} d\bar{x}_{2,-1} \wedge dx_{3,-1} + dx_{2,0} \wedge d\bar{x}_{3,-1} + d\bar{x}_{2,-1} \wedge x_{3,0} \right).$$

Using the formulas for $x'_{1,0}$, $x'_{2,0}$ and $x'_{3,0}$, we obtain

$$(107) \quad \begin{aligned} \varpi'_0 = & \frac{4i \operatorname{Im}(\Omega_{21}(1))}{\pi r^8} [(-r^4 + 3(\bar{u}v + u\bar{v})^2) |v|^2 + r^8 (|u|^2 - |v|^2)] du \wedge d\bar{u} \\ & + \frac{4i \operatorname{Im}(\Omega_{31}(i))}{\pi r^8} [(r^4 + 3(\bar{u}v - u\bar{v})^2) |v|^2 + r^8 (|v|^2 - |u|^2)] du \wedge d\bar{u} \\ & + \frac{4i \operatorname{Im}(\Omega_{21}(1))}{\pi r^8} [(r^4 - 3(\bar{u}v + u\bar{v})^2) \bar{u}v + r^8 (-\bar{u}v - 3u\bar{v})] du \wedge d\bar{v} \\ & + \frac{4i \operatorname{Im}(\Omega_{31}(i))}{\pi r^8} [(-r^4 - 3(\bar{u}v - u\bar{v})^2) \bar{u}v + r^8 (\bar{u}v - 3u\bar{v})] du \wedge d\bar{v} \\ & + \frac{4i \operatorname{Im}(\Omega_{21}(1))}{\pi r^8} [(r^4 - 3(\bar{u}v + u\bar{v})^2) u\bar{v} + r^8 (-u\bar{v} - 3\bar{u}v)] dv \wedge d\bar{u} \\ & + \frac{4i \operatorname{Im}(\Omega_{31}(i))}{\pi r^8} [(-r^4 - 3(\bar{u}v - u\bar{v})^2) u\bar{v} + r^8 (u\bar{v} - 3\bar{u}v)] dv \wedge d\bar{u} \\ & + \frac{4i \operatorname{Im}(\Omega_{21}(1))}{\pi r^8} [(-r^4 + 3(\bar{u}v + u\bar{v})^2) |u|^2 + r^8 (|v|^2 - |u|^2)] dv \wedge d\bar{v} \\ & + \frac{4i \operatorname{Im}(\Omega_{31}(i))}{\pi r^8} [(r^4 + 3(\bar{u}v - u\bar{v})^2) |u|^2 + r^8 (|u|^2 - |v|^2)] dv \wedge d\bar{v}. \end{aligned}$$

where the Ω -integrals are given by Proposition 73. Note that the $(2,0)$ and $(0,2)$ terms $du \wedge dv$ and $d\bar{u} \wedge d\bar{v}$ vanish in accordance with the twistorial description of hyper-Kähler metrics [30].

6.4. Energy. The (Dirichlet) energy of a map $f: \Sigma \rightarrow (M, g)$ from a Riemann surface Σ to a Riemannian manifold (M, g) is given by

$$\mathcal{E}(f) = -\frac{1}{2} \int_{\Sigma} g(df \wedge *df)$$

(where $*$ denotes the Riemann surface $*$ -operator). If f is equivariant (with respect to a discrete group acting holomorphically on Σ and isometrically on M), the energy remains well-defined. If (∇, Φ, h) is a solution to the self-duality equations, the equivariant harmonic map is given by the harmonic metric $f = h$, see for example [13, 33]. Furthermore, the energy is then

$$(108) \quad \mathcal{E}(h) = 2i \int_{\Sigma} \operatorname{tr}(\Phi \wedge \Phi^*),$$

which is L^2 -norm of the Higgs field Φ . We refer to $\mathcal{E}(h)$ in (108) as the energy of the self-duality solution (∇, Φ, h) and note that it is a Kähler potential for the hyperkähler metric on the moduli space of solutions with respect to the complex structure J , see [29].

Let $\eta = \eta_t$ be given by the parameters (x_1, x_2, x_3) provided by Theorem 8. Assume that $t = \frac{l}{k}$ and let $\pi : \Sigma_k \rightarrow \mathbb{C}P^1$ be the k -fold covering as in (37) totally branched over p_1, \dots, p_4 . A twistor line provided by Theorem 10 then gives rise to a solution (∇, Φ, h) of the self-duality equations on Σ_k .

Analogously to [25, Corollary 4.3] and similar to the proof of Lemma 68, the energy of the self-duality solution (∇, Φ, h) corresponding to the real section provided by $\eta = \eta_{x_1, x_2, x_3}$ on the covering surface Σ_k is given by

$$(109) \quad \mathcal{E}(h) = -4\pi \sum_j \text{Res}_{q_j} \text{tr} (\eta_{-1} G_{j,1} G_{j,0}^{-1}).$$

In this equation $G_j = G_{j,0} + G_{j,1}\lambda + \dots$ is a desingularizing gauge, i.e., a holomorphic family of gauge transformations G_j that removes the singularity of the pull-back potential $\pi^*\eta$ at $\hat{p}_j = \pi^{-1}(p_j)$ on Σ_k and extends holomorphically to $\lambda = 0$ (compare with (38) for fixed λ). The reason for the different sign in (109) compared to [25, Corollary 4.3] is that we are considering (equivariant) harmonic maps into hyperbolic 3-space here, with associated family of the form $\nabla + \lambda^{-1}\Phi + \lambda\Phi^*$, while in [25] we considered harmonic maps into the 3-sphere with associated family of the form $\nabla + \lambda^{-1}\Phi - \lambda\Phi^*$, see also [6, Theorem 2.4].

A desingularizing gauge in a neighborhood of \tilde{p}_1 is given by

$$G_1 = \begin{pmatrix} 1 & 0 \\ \frac{1-x_1}{x_2+ix_3} & 1 \end{pmatrix} \begin{pmatrix} w^{-l} & 0 \\ 0 & w^l \end{pmatrix}$$

with w the local coordinate with $w^k = z - p_1$. Then

$$\frac{1}{l} \text{Res}_{\tilde{p}_1} \text{tr} (\tilde{\eta}_{-1} G_{1,1} G_{1,0}^{-1}) = 1 - x_{1,0} + \frac{x_{1,-1}(x_{2,0} + ix_{3,0})}{x_{2,-1} + ix_{3,-1}} = 1 - x_{1,0} - \frac{(x_{2,-1} - ix_{3,-1})(x_{2,0} + ix_{3,0})}{x_{1,-1}}.$$

For the residues at the other punctures, we substitute $(x_1, x_2, x_3) \rightarrow (-x_1, -x_2, x_3)$ to obtain

$$\frac{1}{l} \text{Res}_{\tilde{p}_2} \text{tr} (\tilde{\eta}_{-1} G_{2,1} G_{2,0}^{-1}) = 1 + x_{1,0} + \frac{(x_{2,-1} + ix_{3,-1})(x_{2,0} - ix_{3,0})}{x_{1,-1}}$$

and substitute $(x_1, x_2, x_3) \rightarrow (x_1, -x_2, -x_3)$ to obtain for $k = 1, 2$

$$\text{Res}_{\tilde{p}_{k+2}} \text{tr} (\tilde{\eta}_{-1} G_{k+2,1} G_{k+2,0}^{-1}) = \text{Res}_{\tilde{p}_k} \text{tr} (\tilde{\eta}_{-1} G_{k,1} G_{k,0}^{-1}).$$

This gives the following formula for the energy of the equivariant harmonic map on Σ_k

$$\mathcal{E} = -8\pi l \left(1 + \frac{i}{x_{1,-1}} (-x_{2,-1}x_{3,0} + x_{2,0}x_{3,-1}) \right).$$

Therefore, we define for $t > 0$ the *renormalized energy* by

$$(110) \quad \underline{\mathcal{E}}_t := -8\pi \left(1 + \frac{i}{x_{1,-1}} (-x_{2,-1}x_{3,0} + x_{2,0}x_{3,-1}) \right).$$

Using the central value of the parameters we find that \mathcal{E} extends to $t = 0$ with

$$(111) \quad \underline{\mathcal{E}} = -8\pi(1 - \rho r^2) = 8\pi(\sqrt{1 + r^4} - 1),$$

where $r^2 = |u|^2 + |v|^2$. The first order derivatives of the parameters then yield

$$(112) \quad \begin{aligned} \mathcal{E}' &= -8 \text{Im}(\Omega_{21}(1)) (|u|^4 + |v|^4 - 3\bar{u}^2 v^2 - 3u^2 \bar{v}^2 - 4|u|^2 |v|^2) \\ &\quad + 8 \text{Im}(\Omega_{31}(i)) (|u|^4 + |v|^4 + 3\bar{u}^2 v^2 + 3u^2 \bar{v}^2 - 4|u|^2 |v|^2). \end{aligned}$$

For the most symmetric case of $p = e^{i\pi/4}$, using $\Omega_{21}(1) = -\pi i \log(2)$ from Appendix A of [26] and $\Omega_{31}(i) = -\Omega_{21}(1)$, this simplifies to

$$(113) \quad \mathcal{E}' = 16\pi \log(2) (|u|^4 + |v|^4 - 4|u|^2 |v|^2).$$

Remark 74. *The energy \mathcal{E} can also be used to give a different proof of Lemma 65, because for $\rho < 0$, the (rescaled) energy in the limit*

$$\mathcal{E} = -8\pi(1 + \sqrt{1 + r^4}) < 0$$

is negative (instead of (111)), but twistor lines must have non-negative energy. In the case of nilpotent Higgs fields, the (negative) energy can be interpreted as (the negative of) the Willmore energy of a Willmore surface, see [6, Section 4].

6.5. Another hyper-Kähler metric. Recall that the implicit function theorem (Theorem 8) also applies for the choice of negative ρ in the initial condition (62). The corresponding real holomorphic sections are not twistor lines by Lemma 65. On the other hand, we can still apply the general twistor space theory [30] to obtain a hyper-Kähler metric locally. In fact, the rescaled metric at $t = 0$ is still the Eguchi-Hanson metric g_{EH} , and therefore, for small t and small Higgs fields, the metric (and the smooth structure) remains non-degenerate. Using Lemma 63 we can compare the t -families of hyper-Kähler metrics g^\pm for both choices of sign in the initial condition (62) and obtain

$$g^\pm = g_{EH} + \sum_{k \geq 1} (\pm 1)^k g_k t^k.$$

6.6. Convergence of Hitchin metrics on symmetric components. The aim of this section is to show Theorem 2, which we restate here for the convenience of the reader.

Theorem. *Let $C > 0$ and $l \in \mathbb{N}^{>0}$ be fixed. Consider the compact subspaces*

$$\mathcal{C}_k^l := \{[\nabla, \Phi] \in \mathcal{M}_{SD}^l(\Sigma_k) \mid \mathcal{E}(\nabla, \Phi) \leq C\}$$

of \mathcal{M}_{SD}^l with the induced hyper-Kähler metric g_k . Then for $k \rightarrow \infty$, we obtain smooth convergence

$$(\mathcal{C}_k^l, g_k) \longrightarrow (\{v \in T^*\mathbb{C}P^1 \mid \mathcal{E}_{EH}(v) \leq C\}, 32\pi l g_{EH})/(\mathbb{Z}_2 \times \mathbb{Z}_2).$$

For the setup of the proof, let $\pi: \Sigma_k \rightarrow \mathbb{C}P^1$ be the totally branched covering defined in (37). Consider the moduli space of polystable Higgs bundles, and the moduli space of solutions to the self-duality equations $\mathcal{M}_{SD}(\Sigma_k)$, respectively, equipped with the hyper-Kähler metric on its smooth locus $\mathcal{M}_{SD}^{irr}(\Sigma_k)$. Consider the natural \mathbb{Z}_k -action (generated by σ) on $\mathcal{M}_{SD}(\Sigma_k)$. The fixed point set of σ in $\mathcal{M}_{SD}^{irr}(\Sigma_k)$ has multiple connected components, which can be distinguished as follows. A fixed point (∇, Φ, h) of the \mathbb{Z}_k action can be represented by a \mathbb{Z}_k -equivariant solution of the self-duality equations, i.e., there exists a gauge transformation $g_\sigma: \Sigma_k \rightarrow \text{SU}(2)$ such that

$$\sigma^*(\nabla, \Phi, h) = (\nabla, \Phi, h).g_\sigma.$$

By assumption, (∇, Φ, h) lies in the smooth locus and in particular ∇ is irreducible. Hence g_σ is unique up to sign. Moreover, at every fixed point $q \in \Sigma_k$ of σ , we have $g_\sigma^k(q) = \pm \text{Id}$. Hence, the eigenvalues of $g_\sigma(q)$ are of the form $\exp(\pm 2\pi i \frac{l_q}{2k})$ for some integer $l_q \in \{0, \dots, k\}$. The integers l_{p_1}, \dots, l_{p_4} are then the invariants of a connected component \mathcal{C} of the fixed point set. In fact, for $l_{p_1}, \dots, l_{p_4} \in \{1, \dots, k-1\}$, this component is diffeomorphic to $\mathcal{M}_{SD}(\mathbb{C}P^1, p_1, \dots, p_4, \frac{l_{p_1}}{2k}, \dots, \frac{l_{p_4}}{2k})$. For even l_{p_1}, \dots, l_{p_4} , this directly follows from the discussion in Section 3.2.1. For odd l_{p_j} , the gauge (38) has to be modified to involve square roots, see for example [38], or [24] for the specific Σ_k . Likewise, one can first go to the double covering $\Sigma_{2k} \rightarrow \Sigma_k$ and then prove that the solutions on Σ_{2k} are actually obtained as pull-backs of solutions on Σ_k .

Note that the component \mathcal{C} is a hyper-Kähler submanifold of $\mathcal{M}_{SD}^{irr}(\Sigma_k)$ (see for example [28, Theorem 8]). In fact, this easily follows from the uniqueness of solutions to the self-duality equations. These components can be ‘completed’ by adding orbifold points consisting of gauge classes of reducible connections.

For fixed $l \in \mathbb{N}^{>0}$ and every $k \in \mathbb{N}^{>2l}$ we denote by $\mathcal{M}_{SD}^l(\Sigma_k)$ the component of equivariant solutions corresponding to the integers

$$l_{p_1} = l_{p_2} = l_{p_3} = l_{p_4} = l.$$

It then follows that this space is isomorphic to the hyper-Kähler space $\mathcal{M}_{SD}(\mathbb{CP}^1, p_1, \dots, p_4, \frac{l}{2k})$ up to the scaling constant $32\pi k$, see Proposition 47. We consider the energy (of the solution of the self-duality equations) \mathcal{E} on $\mathcal{M}_{SD}^l(\Sigma_k)$, which corresponds up to a constant with the energy $\underline{\mathcal{E}}_{\frac{l}{2k}}$ on $\mathcal{M}_{SD}(\mathbb{CP}^1, p_1, \dots, p_4, \frac{l}{2k})$ by Section 6.4.

We consider the Eguchi-Hanson metric g_{EH} on $T^*\mathbb{CP}^1$. By identifying $T^*\mathbb{CP}^1$ with the blow up of $(\mathbb{C}^2 \setminus \{(0,0)\})/\mathbb{Z}_2$ at the origin, we have the energy

$$\mathcal{E}_{EH}(u, v) = 8\pi(\sqrt{1 + (u\bar{u} + v\bar{v})^2} - 1)$$

defined in terms of the coordinates $(u, v) \in \mathbb{C}^2$. Consider also the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action on $T^*\mathbb{CP}^1$ generated by

$$\zeta \in \mathbb{CP}^1 \mapsto -\zeta \in \mathbb{CP}^1 \quad \text{and} \quad \zeta \in \mathbb{CP}^1 \mapsto \zeta^{-1} \in \mathbb{CP}^1.$$

This defines an isometric action on $(T^*\mathbb{CP}^1, g_{EH})$, and has 6 fixed points contained in the zero section $\mathbb{CP}^1 \subset T^*\mathbb{CP}^1$. Their images in $T^*\mathbb{CP}^1/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ represent the 3 strictly semi-stable parabolic bundles (with vanishing Higgs field), see Proposition 25. Note that the energy \mathcal{E}_{EH} is invariant under this action, and furthermore \mathcal{E}_{EH} is a Kähler potential for $32\pi g_{EH}$ on $T^*\mathbb{CP}^1$ with respect to the complex structure J (where I is the natural complex structure induced from the cotangent bundle of \mathbb{CP}^1).

Proof of Theorem 2. We first show that for k large enough, any $[\nabla, \Phi] \in \mathcal{C}_k^l$ has (semi-)stable underlying holomorphic structure $\bar{\partial}^\nabla$. Recall that every equivariant stable Higgs pair $(\bar{\partial}^\nabla, \Phi) \in \mathcal{M}_{SD}^l$ gives rise to stable strongly parabolic Higgs pair on \mathbb{CP}^1 with 4 singular points and parabolic weight $t = \frac{l}{2k}$. If the underlying holomorphic structure $\bar{\partial}^\nabla$ is unstable, then also the parabolic structure is unstable. By Lemma 27, there is only a complex affine line of corresponding stable Higgs pairs with unstable underlying holomorphic structure. The Higgs pair with least energy in this affine line has vanishing determinant, and it remains to estimate its energy. This energy is given by $2\pi d$, where d is the degree of the destabilizing holomorphic line bundle L , see for example the proof of [29, Proposition 7.1]. It therefore remains to estimate the degree d , which depends on k . Note that the equivariant nilpotent Higgs field is determined by a holomorphic section

$$\phi \in H^0(\Sigma_k, KL^{-2}).$$

Since ϕ is equivariant with respect to σ as well, it must have a zero of order $(l-1)$ at all 4 singular points p_1, \dots, p_4 , compare with [24, Theorem 3.3]. Hence, the degree of KL^{-2} is $4(l-1)$ and since the genus of Σ_k is $(k-1)$ we obtain

$$d = k - 2l > \frac{C}{2\pi}$$

for all k large enough.

By Theorem 8 there exists for every $\tilde{C} > 0$ an $\epsilon > 0$ such that for all $(u, v) \in \mathbb{C}^2$ with $0 < |(u, v)|^2 < \tilde{C}$ and all $\epsilon > t > 0$, we have constructed a real holomorphic section in $\mathcal{M}_{DH}^{par}(\mathbb{CP}^1, p_1 + \dots + p_4, t)$ with parabolic Higgs field $t\Psi$ for Ψ as in (33) with residues (32) and (34). By Theorem 9 and Theorem 10, these real holomorphic sections are twistor lines and our construction extends to polystable parabolic structures with vanishing Higgs fields. Denote this subset of strongly parabolic Higgs fields (with trivial underlying holomorphic bundle) by $\tilde{\mathcal{C}}_t$, and (by a slight abuse of notation) use (u, v) as coordinates on it.

From the results of Section 6.4 and particularly (111), we can chose for all $C > 0$ a constant $\tilde{C} > 0$ and $\epsilon > 0$ such that

$$\mathcal{E}_t(u, v) < \frac{C}{t} \implies (u, v) \in \tilde{\mathcal{C}}_t.$$

Let $t = \frac{l}{2k} < \epsilon$. By the definition of \mathcal{E}_t and by the results of Section 6.4 we have

$$\mathcal{E}(\nabla, \Phi, h) = l \underline{\mathcal{E}}_{\frac{l}{2k}}(u, v)$$

for the twisted lift (∇, Φ, h) corresponding to the strongly parabolic Higgs field given by (u, v) .

Recall that unless (∇, ϕ) is a fixed point of the \mathbb{C}^* -action, the energy along rays $r \in \mathbb{R}^{\geq 0} \mapsto \mathcal{E}(\nabla, r\phi)$ is strictly increasing, see for example the proof of [29, Proposition 9.1]. For k large enough, we therefore obtain that \mathcal{C}_k is completely contained in $\tilde{\mathcal{C}}_{\frac{l}{2k}}$ (after taking the twisted lift), and we are therefore in the domain covered by our implicit function theorem. Thus, the theorem then directly follows from Proposition 69 and the results from Section 6.2. \square

7. HIGHER ORDER DERIVATIVES

7.1. The algorithm. Just as in [26, Section 5] we give an iterative algorithm for computing higher order derivatives of the parameters in terms of the multiple polylogarithm (MPL) function. The difference to the minimal surface case [26] is that we have no extrinsic closing conditions but complex parameters here.

The computation of the higher order derivatives of the parameters involves the iterated integral Ω_{i_1, \dots, i_n} defined recursively by

$$(114) \quad \Omega_{i_1, \dots, i_n}(z) = \int_0^z \Omega_{i_1, \dots, i_{n-1}} \omega_{i_n},$$

where ω_i is as in (48) for $i = 1, 2, 3$. It is shown in [26, A.3] that (and how) these iterated integrals can be expressed in terms of multiple polylogarithm $\text{Li}_{n_1, \dots, n_d}$. For positive integers $n_1, \dots, n_d \in \mathbb{Z}_{>0}$, and $z_i \in \mathbb{C}^d$ in the region given by $|z_1 \dots z_d| < 1$, the multiple polylogarithm $\text{Li}_{n_1, \dots, n_d}$ is defined by

$$\text{Li}_{n_1, \dots, n_d}(z_1, \dots, z_d) = \sum_{0 < k_1 < k_2 < \dots < k_d} \frac{z_1^{k_1} \dots z_d^{k_d}}{k_1^{n_1} \dots k_d^{n_d}}.$$

This function is extended to a multivalued function by analytic continuation. Here the *depth* d counts the number of indices n_1, \dots, n_d , and the *weight* is given by the sum $n_1 + \dots + n_d$ of the indices. The functions $\Omega_{i_1, \dots, i_n}(z)$ can be expressed in terms of multiple polylogarithms of depth n and weight n .

In the following we denote by $x_i^{(n)}$, $\mathcal{P}^{(n)}$, $\mathcal{Q}^{(n)}$ the n -th order derivatives of x_i , \mathcal{P} , \mathcal{Q} with respect to t at $t = 0$ and we suppress the dependence of (u, v) and p .

Proposition 75. *For $n \geq 1$ and $1 \leq i \leq 3$, $x_i^{(n)}$ are polynomials (with respect to λ) of degree at most $n + 1$ and the coefficients of $\mathcal{P}^{(n+1)}$, $\mathcal{Q}^{(n+1)}$ are Laurent polynomials of degree at most $n + 1$, which can be expressed explicitly in terms of multiple-polylogarithms of depth $n + 1$ and weight $n + 1$.*

Proof. The proof is by induction on n . Let H_n be the statement of the proposition. We have already proved H_1 in Section 6.3. Fix $n \geq 1$ and assume that H_k is true for all $1 \leq k \leq n - 1$ and let the index ‘lower’ denote all terms of a quantity that depends only on derivatives of lower order. As in [26, Proposition 37] we have:

$$\mathcal{P}^{(n+1)} = \sum_{\ell=1}^{n+1} \frac{(n+1)!}{(n+1-\ell)!} \sum_{i_1, \dots, i_\ell} x_{i_1, \dots, i_\ell}^{(n+1-\ell)} \mathbf{m}_{i_1, \dots, i_\ell} \Omega_{i_1, \dots, i_\ell}(1)$$

with

$$x_{i_1, \dots, i_\ell} = \prod_{j=1}^{\ell} x_{i_j} \quad \text{and} \quad \mathbf{m}_{i_1, \dots, i_\ell} = \prod_{j=1}^{\ell} \mathbf{m}_{i_j}.$$

We rewrite this as

$$\mathcal{P}^{(n+1)} = (n+1) \sum_{i=1}^3 x_i^{(n)} \mathbf{m}_i \Omega_i(1) + \mathcal{P}_{\text{lower}}^{(n+1)}$$

with

$$\mathcal{P}_{\text{lower}}^{(n+1)} = \sum_{\ell=2}^{n+1} \frac{(n+1)!}{(n+1-\ell)!} \sum_{i_1, \dots, i_\ell} x_{i_1, \dots, i_\ell}^{(n+1-\ell)} \mathbf{m}_{i_1, \dots, i_\ell} \Omega_{i_1, \dots, i_\ell}(1).$$

Similar formula holds for $\mathcal{Q}^{(n+1)}$ with $\Omega_{i_1, \dots, i_\ell}(1)$ replaced by $\Omega_{i_1, \dots, i_\ell}(i)$. Using Leibniz rule we have

$$\mathfrak{p}^{(n+1)} = 2\pi(n+1)x_3^{(n)} + \mathfrak{p}_{\text{lower}}^{(n+1)}$$

with

$$\mathfrak{p}_{\text{lower}}^{(n+1)} = \sum_{k=1}^n \binom{n+1}{k} \left(\mathcal{P}_{11}^{(k)} \mathcal{P}_{21}^{(n+1-k)} - \mathcal{P}_{12}^{(k)} \mathcal{P}_{22}^{(n+1-k)} \right) + \mathcal{P}_{\text{lower}, 21}^{(n+1)} - \mathcal{P}_{\text{lower}, 12}^{(n+1)}$$

and from $\mathfrak{p}^{(n+1)} = (\mathfrak{p}^{(n+1)})^*$ we obtain by taking the positive part:

$$(x_3^{(n)})^+ = \frac{1}{2\pi(n+1)} \left((\mathfrak{p}_{\text{lower}}^{(n+1)})^{-*} - (\mathfrak{p}_{\text{lower}}^{(n+1)})^+ \right).$$

In the same way

$$\mathfrak{q}^{(n+1)} = 2\pi(n+1)x_2^{(n)} + \mathfrak{q}_{\text{lower}}^{(n+1)}$$

with

$$\mathfrak{q}_{\text{lower}}^{(n+1)} = i \sum_{k=1}^n \binom{n+1}{k} \left(\mathcal{Q}_{11}^{(k)} \mathcal{Q}_{21}^{(n+1-k)} + \mathcal{Q}_{12}^{(k)} \mathcal{Q}_{22}^{(n+1-k)} \right) + i\mathcal{Q}_{\text{lower}, 21}^{(n+1)} + i\mathcal{Q}_{\text{lower}, 12}^{(n+1)}$$

and we obtain

$$(x_2^{(n)})^+ = \frac{1}{2\pi(n+1)} \left((\mathfrak{q}_{\text{lower}}^{(n+1)})^{-*} - (\mathfrak{q}_{\text{lower}}^{(n+1)})^+ \right).$$

By inspection and the induction hypothesis, $(x_2^{(n)})^+$ and $(x_3^{(n)})^+$ are polynomials of degree at most $n+1$.

Remark 76. We could also determine $\text{Im}(x_{3,0}^{(n)})$ and $\text{Im}(x_{2,0}^{(n)})$ from the zero part of $\mathfrak{p}^{(n)} = (\mathfrak{p}^{(n)})^*$ and $\mathfrak{q}^{(n)} = (\mathfrak{q}^{(n)})^*$, but it is simpler to determine the three complex parameters $x_{i,0}^{(n)}$ by solving a complex linear system, see below.

Using Leibniz rule:

$$0 = \mathcal{K}^{(n)} = 2 \sum_{i=1}^3 \underline{x}_i x_i^{(n)} + \mathcal{K}_{\text{lower}}^{(n)}$$

with

$$\mathcal{K}_{\text{lower}}^{(n)} = \sum_{i=1}^3 \sum_{k=1}^{n-1} \binom{n}{k} x_i^{(k)} x_i^{(n-k)}.$$

We multiply by λ and obtain (recall that $P_j = \lambda \underline{x}_j$):

$$(115) \quad P_1 x_1^{(n)} + P_2 x_{2,0}^{(n)} + P_3 x_{3,0}^{(n)} = -\frac{\lambda}{2} \mathcal{K}_{\text{lower}}^{(n)} - P_2 (x_2^{(n)})^+ - P_3 (x_3^{(n)})^+.$$

The right hand side of (115) is already known and is a polynomial in λ of degree at most $n+3$. Hence $P_1 x_1^{(n)}$ is a polynomial of degree at most $n+3$. When $u \sim v$, then both roots of P_1 are in \mathbb{D}_a , so since $x_1^{(n)}$ cannot have poles in \mathbb{D}_a , it must be a polynomial of degree at most $n+1$. This remains true for all (u, v) by analyticity. Let Q, R be the quotient and remainder of the division of the right side of (115) with respect to P_1 . Then

$$(x_1^{(n)})^+ = Q^+$$

and looking at the coefficients of λ^0, λ^1 and λ^2 in (115), we obtain a system of three complex equations with unknowns $x_{1,0}^{(n)}, x_{2,0}^{(n)}$ and $x_{3,0}^{(n)}$, whose determinant is

$$\det(\underline{x}_{i,j})_{\substack{-1 \leq i \leq 1 \\ 1 \leq j \leq 3}} = \frac{1}{2} i \rho r^6 \neq 0$$

.

□

As a direct corollary we can express the hyper-Kähler metric of the moduli space $\mathcal{M}(t)$ in terms of multiple polylogarithms.

Proof of Theorem 3. By twistor theory, we can compute the hyper-Kähler metric explicitly in terms of the relative holomorphic symplectic form $\varpi = 32\pi \frac{dx_2 \wedge dx_3}{x_1}$. Since x_1, x_2, x_3 depend real analytic on t the theorem follows from Proposition 75. \square

The algorithm has been implemented and using Mathematica we obtain, for example, for $p = e^{i\pi/4}$

$$(116) \quad \mathcal{E}'' = \frac{32\pi}{\rho r^6} (|u|^2 - |v|^2)^2 (3r^8 + 2r^4 + 4|u|^2|v|^2) \log(2)^2.$$

7.2. Results for the nilpotent cone of the most symmetric case. All following computations are conducted and simplified using Mathematica. The fully documented Mathematica notebook can be downloaded on the webpage

<https://www.idpoisson.fr/traizet/>

We restrict to the case $p = e^{i\pi/4}$ and assume $v = pu$, i.e., by (64) we are in the nilpotent cone. The computations of higher order derivatives then simplify to:

$$\mathcal{E}'' = 0$$

$$(117) \quad \mathcal{E}''' = -192\pi |u|^4 (127|u|^4 + 20) \zeta(3)$$

where ζ is the Riemann ζ -function. If $\tilde{\varpi}$ denotes the restriction of ϖ to the nilpotent cone, we obtain

$$\tilde{\varpi}' = 8i|u|^2 \log(2) du \wedge d\bar{u}$$

$$\tilde{\varpi}'' = 0$$

$$(118) \quad \tilde{\varpi}''' = 96i \zeta(3) (127|u|^6 + 10|u|^2) du \wedge d\bar{u}.$$

7.3. New Ω -identities. Write

$$\varpi = \sum_{k=0}^{\infty} \varpi_k \lambda^k.$$

We can compute ϖ_k'' for $k = 0, 1, 2, \dots$ from the derivatives of the parameters. On the other hand, we know from section 2.7 and Lemma 68 that $\varpi_k'' = 0$ for $k \geq 1$, and also

$$\varpi_{0,u,v}'' = \varpi_{0,\bar{u},\bar{v}}'' = 0.$$

Note that $\varpi_{0,\bar{u},\bar{v}}'' = 0$ is trivial because of (106) and because $d\bar{x}_{2,-1}$ and $d\bar{x}_{3,-1}$ are holomorphic 1-forms so vanish on $(\frac{\partial}{\partial \bar{u}}, \frac{\partial}{\partial \bar{v}})$. On the other hand, $\varpi_{0,u,v}'' = 0$ is not trivial and gives identities involving Ω -integrals. In this section, we use the notation

$$\Omega_{i_1, \dots, i_n} = \Omega_{i_1, \dots, i_n}(1) \quad \text{and} \quad \Theta_{i_1, \dots, i_n} = \Omega_{i_1, \dots, i_n}(i).$$

From

$$\varpi_0'' = \frac{1}{\underline{x}_{1,-1}} \left(\frac{-x_{1,0}''}{\underline{x}_{1,-1}} d\bar{x}_{2,-1} \wedge dx_{3,-1} + dx_{2,0}'' \wedge d\bar{x}_{3,-1} + d\bar{x}_{2,-1} \wedge dx_{3,0}'' \right)$$

we obtain

$$(119) \quad \begin{aligned} \varpi_{0,u,v}'' &= \frac{\rho}{\pi^2} [6(u\bar{u}^2\bar{v} - \bar{u}v\bar{v}^2)(I_1 + \bar{I}_1) + 8(\bar{u}^3v - u\bar{v}^3)(I_2 + \bar{I}_2) \\ &\quad + \left(\frac{u\bar{u}^6}{\bar{v}^3} - \frac{v\bar{v}^6}{\bar{u}^3} + \frac{3\bar{u}^5v}{\bar{v}^2} - \frac{3u\bar{v}^5}{\bar{u}^2} \right) (I_3 + \bar{I}_3)] \end{aligned}$$

with

$$\begin{cases} I_1 = 6\pi(\Omega_{333} - \Theta_{222}) + i((\Omega_{21})^2 + (\Theta_{31})^2) + 2\pi(\Omega_{223} - \Theta_{332}) - 8\pi(\Omega_{311} - \Theta_{211}) + 10i\Omega_{21}\Theta_{31} \\ I_2 = i((\Omega_{21})^2 - (\Theta_{31})^2) + 2\pi(\Omega_{223} + \Theta_{332} + \Omega_{311} + \Theta_{211}) - 4\pi(\Omega_{333} + \Theta_{222}) \\ I_3 = -i((\Omega_{21})^2 + (\Theta_{31})^2) + 2\pi(\Omega_{333} - \Theta_{222} - \Omega_{223} + \Theta_{332}) - 2i\Omega_{21}\Theta_{31}. \end{cases}$$

Since $\varpi''_{0,u,v} = 0$, I_1 , I_2 and I_3 are pure imaginary. Since they are holomorphic functions of $p \in \mathbb{C}^+$, they must be constant. We find the constants by evaluating at $p = e^{i\pi/4}$, where all integrals are known from [26, Appendix], and find

$$\begin{cases} I_1 = -i\frac{\pi^4}{3} \\ I_2 = 0 \\ I_3 = -i\pi^4. \end{cases}$$

Using the elementary values

$$\Omega_{333} = \frac{1}{6}(\Omega_3)^3 = \frac{-i\pi^3}{6} \quad \text{and} \quad \Theta_{222} = \frac{1}{6}(\Theta_2)^2 = \frac{i\pi^3}{6}$$

we obtain the following identities for all $p \in \mathbb{C}^+$:

$$(120) \quad \begin{cases} \Omega_{223} + \Omega_{311} = -\frac{i}{2\pi}(\Omega_{21})^2 \\ \Theta_{211} + \Theta_{332} = \frac{i}{2\pi}(\Theta_{31})^2 \\ \Omega_{223} - \Theta_{332} = -\frac{i}{2\pi}(\Omega_{21} + \Theta_{31})^2 + \frac{i\pi^3}{6}. \end{cases}$$

From higher order derivatives in t and higher order terms ϖ_k we expect a hierarchy of identities for Ω -values. Another source of identities is the character variety of the 4-punctured sphere. Analogously to Remark (72), taking the 4th-order derivative of (103) gives rise to the following three identities, which express linear combinations of Ω -integrals of depth 3 as a function of Ω -integrals of depth at most 2, and are non-trivial in the sense that they do not follow from shuffle product relations alone:

$$(121) \quad \begin{aligned} & \Omega_{2,1,2} - \Theta_{1,2,1} - \Theta_{2,1,2} = \\ & -\frac{1}{2}\Omega_2\Omega_{1,2} + \frac{1}{2}\Omega_2\Omega_{2,1} - \frac{i\Omega_{1,2}^2}{4\pi} + \frac{3i\Omega_{2,1}^2}{4\pi} - \frac{i\Omega_{1,2}\Omega_{2,1}}{2\pi} + \frac{1}{2}\Theta_1\Theta_{1,2} + \frac{1}{2}i\pi\Theta_{1,2} - \frac{1}{2}\Theta_1\Theta_{2,1} \\ & - \frac{1}{2}i\pi\Theta_{2,1} + \frac{i\Omega_1^2\Omega_2^2}{4\pi} + \frac{1}{2}\Omega_1\Omega_2^2 - \pi^2\Omega_1 + \frac{1}{2}i\pi\Theta_1^2 - \frac{1}{4}\Theta_1\Theta_2^2 + \frac{1}{4}\pi^2\Theta_1 + \frac{2i\pi^3}{3} \end{aligned}$$

$$(122) \quad \begin{aligned} & \Omega_{1,3,1} + \Omega_{3,1,3} - \Theta_{3,1,3} = \\ & -\frac{1}{2}\Omega_1\Omega_{1,3} + \frac{1}{2}\Omega_1\Omega_{3,1} + \frac{1}{2}i\pi\Omega_{1,3} - \frac{1}{2}i\pi\Omega_{3,1} - \frac{i\Theta_{1,3}^2}{4\pi} + \frac{3i\Theta_{3,1}^2}{4\pi} + \frac{1}{2}\Theta_3\Theta_{1,3} - \frac{1}{2}\Theta_3\Theta_{3,1} - \frac{i\Theta_{1,3}\Theta_{3,1}}{2\pi} \\ & + \Omega_1^2\Theta_1 - \Omega_1\Theta_1^2 - \frac{1}{3}\Omega_1^3 + \frac{1}{2}i\pi\Omega_1^2 + \frac{1}{4}\Omega_3^2\Omega_1 - \frac{1}{4}\pi^2\Omega_1 + \frac{1}{3}\Theta_1^3 + \frac{i\Theta_1^2\Theta_3^2}{4\pi} - \frac{1}{2}\Theta_1\Theta_3^2 + \pi^2\Theta_1 + \frac{i\pi^3}{3} \end{aligned}$$

$$(123) \quad \begin{aligned} & \Omega_{232} - \Theta_{323} = \\ & \frac{1}{2}\Omega_2\Omega_{23} - \frac{1}{2}\Omega_2\Omega_{32} + \frac{1}{2}\Theta_3\Omega_{32} - \frac{1}{2}\Theta_3\Omega_{21} + \Theta_3\Omega_{23} + \frac{1}{2}\Theta_3\Theta_{23} - \frac{1}{2}\Theta_3\Theta_{31} \\ & - \frac{i\Omega_{21}\Theta_{23}}{2\pi} - \frac{i\Omega_{32}\Theta_{23}}{2\pi} + \frac{3i\Omega_{21}\Theta_{31}}{2\pi} - \frac{i\Omega_{32}\Theta_{31}}{2\pi} - \frac{i\Omega_{21}\Omega_{32}}{2\pi} - \frac{i\Theta_{23}\Theta_{31}}{2\pi} \\ & + \frac{3i\Omega_{21}^2}{4\pi} - \frac{i\Omega_{32}^2}{4\pi} - \frac{i\Theta_{23}^2}{4\pi} + \frac{3i\Theta_{31}^2}{4\pi} + \frac{i\pi\Omega_2^2}{4} + \frac{i\pi\Theta_3^2}{4} - i\pi\Omega_{21} - i\pi\Omega_{32} \\ & - i\pi\Theta_{23} - i\pi\Theta_{31} - \pi^2\Omega_2 + \pi^2\Theta_3 - \frac{i\pi^3}{3}. \end{aligned}$$

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