

# GLUING KARCHER–SCHERK SADDLE TOWERS I: TRIPLY PERIODIC MINIMAL SURFACES

HAO CHEN AND MARTIN TRAIZET

**ABSTRACT.** We construct minimal surfaces by gluing simply periodic Karcher–Scherk saddle towers along their wings. Such constructions were previously implemented assuming a horizontal reflection plane. We break this symmetry by prescribing phase differences between the saddle towers. It turns out that, in addition to the previously known horizontal balancing condition, the saddle towers must also be balanced under a subtle vertical interaction. This interaction vanishes in the presence of a horizontal reflection plane, hence was not perceived in previous works.

Our construction will be presented in a series of papers. In this first paper of the series, we will explain the background of the project and establish the graph theoretical setup that will be useful for all papers in the series. The main task of the current paper is to glue saddle towers into triply periodic minimal surfaces (TPMSs). Our construction expands many previously known TPMSs into new 5-parameter families, therefore significantly advances our knowledge on the space of TPMSs.

## 1. BACKGROUND

In the last decades, the node-opening technique has been very successful in gluing catenoids into minimal surfaces of finite or infinite topology in Euclidean space forms [Tra02b, Tra02a, Tra08, MT12, CT21].

In fact, the technique was first developed to glue Karcher–Scherk saddle towers. More specifically, the second named author desingularized arrangements of vertical planes into minimal surfaces by replacing the intersection lines with Scherk surfaces (saddle towers with four wings). This was first done by solving non-linear PDEs [Tra96] and later using the node-opening technique [Tra01]. In his thesis [You09], Rami Younes desingularized  $G \times \mathbb{R}$ , where  $G$  is a graph embedded in a flat 2-torus, into triply periodic minimal surfaces by placing saddle towers over the vertices of  $G$ . These constructions work only when all saddle towers are balanced under a horizontal interaction or, equivalently, when the graph embedding is a non-degenerate critical point of the length functional.

All these previous works assumed, however, that the surfaces have a horizontal reflection plane. It was wondered in [Tra96] whether this symmetry is necessary for simply periodic minimal surfaces (SPMSs) with ends of Scherk type (i.e. asymptotic to vertical planes). A positive answer was given for SPMSs of genus 0 [PT07], but Martín and Ramos Batista constructed the Karcher–Costa towers [MRB06] that provide non-symmetric examples of genus 1. The same question can be asked for triply periodic minimal surfaces (TPMSs). But the deformation families of the

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Gyroid, which were recently confirmed by the first named author [Che19], admit non-symmetric saddle tower limits.

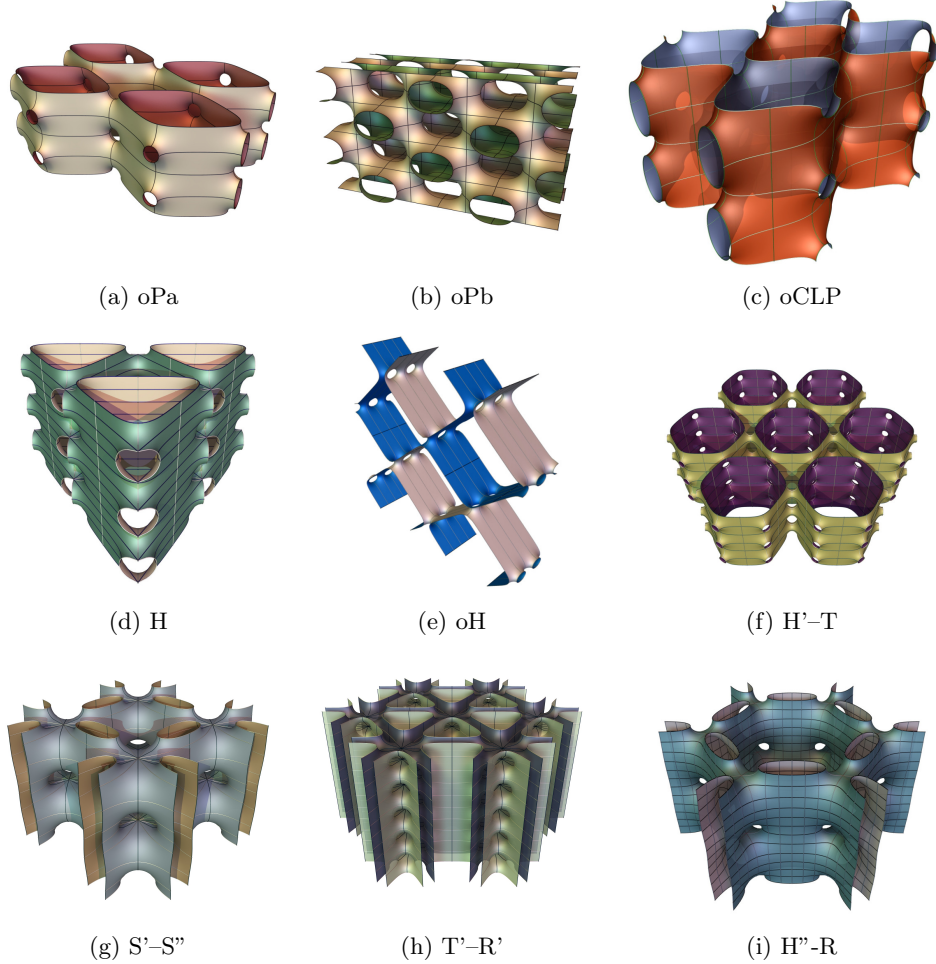


FIGURE 1. A gallery of known examples of saddle tower limits of triply periodic minimal surfaces (Source: Matthias Weber). Generalizations of them will be constructed in this paper.

The goal of this series of papers is to glue Karcher–Scherk saddle towers into minimal surfaces without the assumption of a horizontal reflection plane.

Let  $G$  be a graph in the complex plane. Formal definitions will be given in Section 3. At the moment, it suffices to understand that  $G$  is a set of straight segments (edges) and rays that intersect only at their endpoints (vertices).

For sufficiently small  $\varepsilon > 0$ , we want to construct a 1-parameter family  $(M_\varepsilon)_{\varepsilon>0}$  of minimal surfaces of vertical period  $2\pi\varepsilon^2$  that tends to  $G \times \mathbb{R}$  as  $\varepsilon \rightarrow 0$ . For this purpose, we place saddle towers over the vertices of  $G$  and glue their wings along the edges of the graph.

If a horizontal reflection plane is assumed, it must be a reflection plane for all saddle towers. To break this symmetry, we must vertically move the saddle towers so that they have different reflection planes. The “height differences” between the reflection planes, known as the phase differences, are prescribed on the edges of the graph.

Technically, the phase differences correspond to the complex arguments of the node-opening parameters; see Section 6.1 for details. Complex node-opening parameters were considered in previous works [Tra08], but never did the complex argument play such an important role.

In previous works, it was already established that the saddle towers must be balanced under a horizontal interaction. In other words, the graph embedding is a critical point of the length functional. In this paper, we reveal another very subtle vertical interaction between the saddle towers. Here, we give a preview of this interaction.

Assume that two saddle towers  $\mathcal{S}$  and  $\mathcal{S}'$  are glued along their wings. Let  $\ell$  be the length of the corresponding edge in the graph, and  $\phi$  be the prescribed phase difference of  $\mathcal{S}'$  over  $\mathcal{S}$ . Then, as  $\varepsilon \rightarrow 0$ , the vertical force exerted by  $\mathcal{S}'$  on  $\mathcal{S}$  is asymptotically of the form

$$K \sin \phi \exp(-\ell/\varepsilon^2).$$

Let us now give some explanation to the factors in this expression.

- The coefficient  $K$  is a positive real number that depends on the shape of the saddle towers and a first-order deformation of the graph. It is well-defined only when the saddle towers are already horizontally balanced.
- Because of the factor  $\sin \phi$ , the force vanishes whenever  $\phi = 0$  or  $\pi$ . This explains why this interaction was not perceived in previous works that assumed a reflection symmetry.
- Because of the exponential factor, the interaction dominates on shorter edges of the graph for small  $\varepsilon$ .

The vertical balancing condition requires, for every set of saddle towers, that the forces sum up to 0 over the edges that separate the set from other saddle towers. In the limit  $\varepsilon \rightarrow 0$ , it suffices to take the sum of  $K \sin \phi$  over the shortest separating edges.

Informally speaking, the paradigm of our main results in the series of papers is the following:

**Theorem 1.1** (Informal statement). *Let  $\mathbf{G}$  be an embedded graph  $\mathbf{G}$  with phase differences  $\phi$  prescribed on the edges. Then  $\mathbf{G} \times \mathbb{R}$  can be desingularized into a minimal surface that looks like saddle towers glued along their wings with the prescribed phase differences if*

- (1) *The graph can be consistently oriented in the sense that, for every face, the bounding edges are all in the same direction.*
- (2) *The graph is balanced and rigid in the sense that any perturbative external force can be canceled out by a slight deformation of the graph.*
- (3) *The phase differences is balanced and rigid in the sense that any non-trivial vertical perturbation destroys the vertical balance.*

In fact, in the neighborhood of a balanced and rigid graph, we obtain a family of minimal surfaces parameterized by  $\varepsilon$  and the first-order deformations of the graph that preserve the balance.

The current paper is organized as follows.

The first three sections are preparatory, meant to be relevant not only for the current manuscript but also for future papers in the series. After presenting the background in this section, we will introduce the saddle towers in Section 2. In particular, we will study the shape of the wings and the rigidity of the saddle towers. Section 3 aims to clarify the definition of graphs, which is crucial for formal statements of our main result.

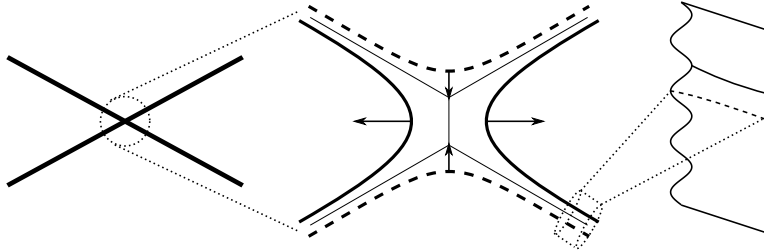


FIGURE 2. A four-wing saddle tower seen from a distance (left), then scaled at the “axis” (middle). The shape of a wing is illustrated on the right, featuring the undulation.

The remaining of the paper is dedicated to the construction of triply periodic minimal surfaces (TPMSs). TPMSs are given the privilege because many known families of TPMSs admit saddle tower limits, but most of them are symmetric in a horizontal plane; see Figure 1. So we expect to find many interesting new examples after breaking this symmetry. Indeed, after announcing the Main Theorem 4.16, we present in Section 5 examples of TPMSs that can be produced from our construction, including many new TPMSs of genus 3. The construction is proved in Section 6.

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## 2. THE KARCHER–SCHERK SADDLE TOWERS

**2.1. Construction through Jenkins–Serrin Theorem.** Recall that a *saddle tower* is an embedded SPMS with  $n$  Scherk-type ends, where  $n \geq 4$  is an even integer. Saddle towers with  $n = 4$  were discovered by Scherk [Sch35] in the 19th century. All other saddle towers were constructed by Karcher [Kar88]. Saddle towers are classified in [PT07] as the only embedded SPMS with  $n$  Scherk-type ends and genus zero in the quotient by the period.

We sketch Karcher’s construction as follows: Let  $P$  be a convex polygon with  $n$  sides of length one, where  $n \geq 4$  is even. We allow  $P$  to be non-strictly convex but exclude the *degenerate* case where  $P$  appears as a line segment of length  $n/2$ , and the *special* case where  $n \geq 6$  and  $P$  is a parallelogram with 2 sides of length one [PT07]. Then the Jenkins–Serrin theorem [JS66] guarantees the existence of a minimal graph over the interior of  $P$  that takes the values  $\pm\infty$ , alternately, on the edges of  $P$ , unique up to vertical translation. This graph is bounded by vertical lines over the vertices of  $P$ . Its conjugate minimal surface is bounded by  $n$  curvature lines that lie, alternately, in two horizontal planes. Reflections in these planes extend the conjugate surface into an SPMS called a saddle tower, which we denote by  $\mathcal{S}$ .

The edges of  $P$  are  $n$  unit vectors that sum up to 0. To be consistent with later notations, we label the edges by  $\mathbf{H} = \{0, \dots, n-1\}$  in the counterclockwise order, and define  $\varsigma : \mathbf{H} \rightarrow \mathbf{H}$  by  $\varsigma(h) = h+1 \pmod{n}$ . Then the corresponding unit vectors are  $u_h = \exp(i\theta_h) \in \mathbb{C} \simeq \mathbb{R}^2$ ,  $h \in \mathbf{H}$ . The corresponding ends of  $\mathcal{S}$  are called *wings*. Each wing is asymptotic to a vertical half-plane that is parallel to the corresponding edge. Wings extending in the same direction (same  $\theta_h$ ) are called *parallel*.

*Remark 2.1.* Seen from a distance,  $\mathcal{S}$  looks like  $n$  vertical half-planes sharing a vertical boundary; see Figure 2 (left). But a closer look reveals that the asymptotic planes of the wings do not intersect at a single vertical line; see Figure 2 (middle) for an illustration and Section 2.4.1 for a detailed analysis.

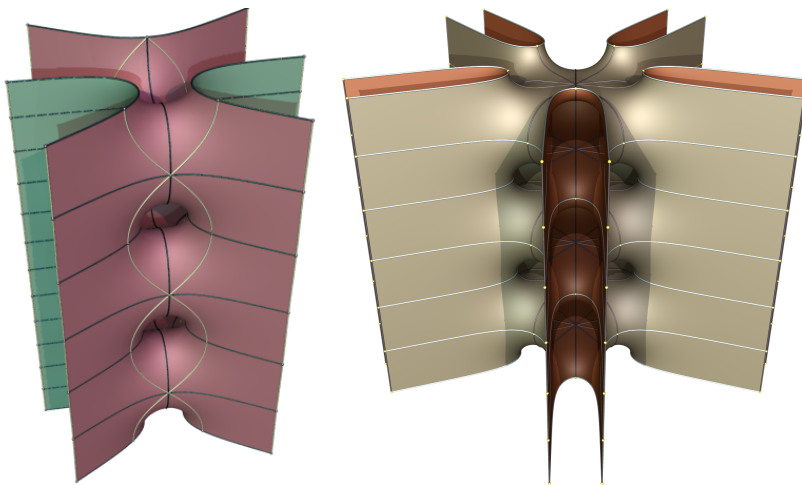


FIGURE 3. Two saddle towers with six and ten wings; see Section 2.4. The one on the right has five pairs of parallel wings. (Source: Matthias Weber)

**2.2. Weierstrass parameterization.** The quotient of a saddle tower by its vertical period is conformally equivalent to a Riemann sphere  $\mathbb{C} \cup \{\infty\}$  with  $n$  punctures  $p_h$ ,  $h \in \mathbb{H}$ , corresponding to the  $n$  wings. The punctures must lie on a circle  $C$  fixed by the anti-holomorphic involution  $\rho$  corresponding to the horizontal reflections. For convenience,  $C$  is often taken to be the unit circle or the real line.

By scaling, we may assume that the saddle tower has a vertical period  $2\pi$ . Then its Weierstrass data can be written on the punctured Riemann sphere as

$$(1) \quad \Phi_1 = \sum_{h \in \mathbb{H}} \frac{-\cos \theta_h}{z - p_h} dz, \quad \Phi_2 = \sum_{h \in \mathbb{H}} \frac{-\sin \theta_h}{z - p_h} dz, \quad \Phi_3 = \sum_{h \in \mathbb{H}} \frac{-i\sigma_h}{z - p_h} dz,$$

where  $\sigma_h = \pm 1$  and  $\sigma_{\varsigma(h)} = -\sigma_h$  for  $h \in \mathbb{H}$ . The conformality condition

$$\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0$$

determines the punctures  $p_h$  up to a Möbius transformation (although doing this explicitly can be difficult). The saddle tower is parameterized by the Weierstrass Representation formula

$$(2) \quad z \mapsto \operatorname{Re} \int_{z_0}^z \Phi = \operatorname{Re} \int_{z_0}^z (\Phi_1, \Phi_2, \Phi_3).$$

The stereographically projected Gauss map  $G = -(\Phi_1 + i\Phi_2)/\Phi_3$  extends holomorphically to the punctures with  $G(p_h) = i\sigma_h u_h$ . Then the Gauss map extends at the end  $p_h$  with

$$N(p_h) = \sigma_h(-\sin(\theta_h), \cos(\theta_h), 0).$$

We now make a distinction between the two horizontal reflection planes. The punctures  $p_h$  divide the circle  $C$  fixed by  $\rho$  into  $n$  arcs, corresponding to the curvature lines on the horizontal reflection planes. An arc between  $p_h$  and  $p_{\varsigma(h)}$  is called a 0-arc if  $\sigma_h = 1$ . The images of all 0-arcs lie on the same horizontal plane, which we call the 0-plane.

**Definition 2.2** (Phase). Let  $\mathcal{S}$  be an oriented saddle tower of vertical period  $2\pi$ . We say that the *phase* of  $\mathcal{S}$  is  $\phi \in \mathbb{R}/2\pi\mathbb{Z}$  if the horizontal plane  $x_3 = \phi$  is a 0-plane of  $\mathcal{S}$ .

**2.3. Shape of wings.** In this part, we define quantities that describe the geometry of the Scherk end at  $p_h$  and will play an important role in our construction.

Let  $w_h$  be a local complex coordinate in a neighborhood of  $p_h$  with  $w_h(p_h) = 0$ . We define

$$\begin{aligned}\Upsilon_h &= \sigma_h \left\langle N(p_h), \operatorname{Res} \left( \frac{\Phi}{w_h}, p_h \right) \right\rangle_{\mathbb{H}} \\ &= -\sin(\theta_h) \operatorname{Res} \left( \frac{\Phi_1}{w_h}, p_h \right) + \cos(\theta_h) \operatorname{Res} \left( \frac{\Phi_2}{w_h}, p_h \right),\end{aligned}$$

where  $\langle \cdot, \cdot \rangle_{\mathbb{H}}$  denotes the hermitian scalar product on  $\mathbb{C}^3$  (semi-linear on the left). Note that  $\Upsilon_h$  depends on the local coordinate  $w_h$ .

**Proposition 2.3.**

$$\Upsilon_h = i \frac{dG}{G dw_h}(p_h).$$

*Proof.* We may expand  $G$  and  $\Phi_3$  around  $p_h$  as

$$\begin{aligned}G &= i\sigma_h e^{i\theta_h} (1 + aw_h + O(w_h^2)), \\ G^{-1} &= -i\sigma_h e^{-i\theta_h} (1 - aw_h + O(w_h^2)), \\ \Phi_3 &= -i\sigma_h \left( \frac{1}{w_h} + b + O(w_h) \right) dw_h,\end{aligned}$$

where  $a, b \in \mathbb{C}$  and

$$a = \frac{dG}{G dw_h}(p_h).$$

This gives

$$\begin{aligned}\operatorname{Res} \left( \frac{\Phi_1}{w_h}, p_h \right) &= -\cos(\theta_h)b - i\sin(\theta_h)a, \\ \operatorname{Res} \left( \frac{\Phi_2}{w_h}, p_h \right) &= -\sin(\theta_h)b + i\cos(\theta_h)a,\end{aligned}$$

and

$$\Upsilon_h = -\sin(\theta_h) (-\cos(\theta_h)b - i\sin(\theta_h)a) + \cos(\theta_h) (-\sin(\theta_h)b + i\cos(\theta_h)a) = ia. \quad \square$$

We define the quantities

$$\begin{aligned}\mu_h &= \lim_{z \rightarrow p_h} \left( e^{i\theta_h} \log |w_h(z)| + \operatorname{Re} \int_{z_0}^z \Phi_1 + i \operatorname{Re} \int_{z_0}^z \Phi_2 \right), \\ \nu_h &= \lim_{z \rightarrow p_h} \left( -\sigma_h \arg(w_h(z)) + \operatorname{Re} \int_{z_0}^z \Phi_3 \right).\end{aligned}$$

It is easy to see that the limits exist, but these quantities depend on the local coordinate  $w_h$  and the choice of the base point  $z_0$ .

Recall that a saddle tower has a horizontal symmetry plane corresponding to an anti-holomorphic involution  $\rho$  of the Riemann sphere that fixes all punctures  $p_h$  for  $h \in \mathbb{H}$ .

**Definition 2.4.** The local coordinate  $w_h$  is said to be *adapted* if  $w_h \circ \rho = \overline{w_h}$  and  $w_h > 0$  near  $p_h$  on the arc between  $p_h$  and  $p_{\varsigma(h)}$ .

**Proposition 2.5.** *Assume that the coordinate  $w_h$  is adapted and let  $\phi$  be the phase of the saddle tower. Then  $\Upsilon_h > 0$  and*

$$\nu_h = \begin{cases} \phi & \pmod{2\pi} \quad \text{if } \sigma_h = +1, \\ \phi + \pi & \pmod{2\pi} \quad \text{if } \sigma_h = -1. \end{cases}$$

*Proof.* Assume that the coordinate  $w_h$  is adapted. We start by computing  $\nu_h$ . Consider  $z$  on the arc between  $p_h$  and  $p_{\zeta(h)}$ . We have  $\arg(w_h) = 0$  for  $z$  sufficiently close to  $p_h$ . Then, by the definition of the phase  $\phi$ , we have

$$\nu_h = \operatorname{Re} \int_{z_0}^z \Phi_3 = \begin{cases} \phi & \pmod{2\pi} & \text{if } \sigma_h = +1, \\ \phi + \pi & \pmod{2\pi} & \text{if } \sigma_h = -1. \end{cases}$$

Since the coordinate  $w_h$  is adapted, we have, for  $i = 1, 2$ ,

$$\rho^* \left( \frac{\Phi_i}{w_h} \right) = \frac{\overline{\Phi_i}}{w_h},$$

so  $\operatorname{Res}(\Phi_i/w_h, p_h) \in \mathbb{R}$  and  $\Upsilon_h \in \mathbb{R}$ .

To understand its sign we need to go back to the Jenkins-Serrin construction. The solution of the Jenkins-Serrin problem is a graph on the convex domain bounded by the polygon  $P$  so its Gauss map is non-horizontal in the interior of  $P$ . Recalling that the conjugate minimal surface has the same Gauss map, we have  $|G| \neq 1$  on the Riemann sphere minus the circle  $C$  fixed by the symmetry  $\rho$ , and  $|G| = 1$  on  $C$ . Hence in a neighborhood of  $p_h$ ,  $\log G$  is a well-defined holomorphic function and is pure imaginary only on  $C$ , so its zero at  $p_h$  is simple by the local behavior of holomorphic functions in a neighborhood of a zero. Hence  $\Upsilon_h \neq 0$  by Proposition 2.3.

On each vertical line above a vertex, the Gauss map of the Jenkins-Serrin graph is horizontal and varies monotonically. Hence each horizontal symmetry curve on the conjugate saddle tower is a convex curve. Since the unitary vectors  $u_h$  are ordered in the counterclockwise order, the argument of  $G$  is non-increasing on the arc between  $p_h$  and  $p_{\zeta(h)}$ . This implies  $\Upsilon_h \geq 0$  by Proposition 2.3.  $\square$

*Remark 2.6.* By a Theorem of R. Krust, the conjugate to a minimal graph over a convex domain is a graph. This is why the saddle towers are embedded. Moreover, the horizontal projection of a saddle tower is an unbounded concave domain, whose complement has  $n$  convex components, each bounded by the projection of a symmetry curve. The Gauss map on a 0-curve points towards the corresponding convex component.

For a geometric intuition of the quantities  $\Upsilon_h$ ,  $\mu_h$ , and  $\nu_h$ , let us expand the Weierstrass parametrization around  $p_h$ :

$$\begin{aligned} \operatorname{Re} \int_{z_0}^z \Phi_1 &= \overbrace{\operatorname{Re} \mu_h - \cos \theta_h \log |w_h(z)|}^{\text{planar terms}} + \overbrace{\operatorname{Re} \left( w_h(z) \operatorname{Res} \left( \frac{\Phi_1}{w_h}, p_h \right) \right)}^{\text{undulation terms}} + O(|w_h(z)|^2), \\ \operatorname{Re} \int_{z_0}^z \Phi_2 &= \operatorname{Im} \mu_h - \sin \theta_h \log |w_h(z)| + \operatorname{Re} \left( w_h(z) \operatorname{Res} \left( \frac{\Phi_2}{w_h}, p_h \right) \right) + O(|w_h(z)|^2), \\ \operatorname{Re} \int_{z_0}^z \Phi_3 &= \nu_h + \sigma_h \arg(w_h(z)) + O(|w_h(z)|). \end{aligned}$$

We then observe that

- The terms under the first brace describe a vertical half-plane that passes through  $(\mu_h, \nu_h)$  and extends in the direction  $\theta_h$  as  $w_h(z) \rightarrow 0$ .
- The terms under the second brace describe a sinusoidal undulation in the vertical coordinate that decays exponentially with the horizontal distance from  $(\mu_h, \nu_h)$ ; see Figure 2.
- The quantity  $\Upsilon_h$  describes the “initial” amplitude of the undulation on the 0-plane in the asymptotic normal direction of the wing.

*Remark 2.7.* The quantities  $\Upsilon_h$  and  $\mu_h$  behave as follows under change of coordinate: if  $\tilde{w}_h$  is another complex coordinate in a neighborhood of  $p_h$  and  $\tilde{\Upsilon}_h, \tilde{\mu}_h$  denote the corresponding quantities, we have, from the definitions,

$$\Upsilon_h = \tilde{\Upsilon}_h \frac{d\tilde{w}_h}{dw_h}(p_h) \quad \text{and} \quad \tilde{\mu}_h - \mu_h = e^{i\theta_h} \log \left| \frac{d\tilde{w}_h}{dw_h}(p_h) \right|.$$

Also observe that  $\frac{d\tilde{w}_h}{dw_h}(p_h) > 0$  if both coordinates are adapted.

**2.4. Examples.** In this section, we compute the quantities  $\Upsilon_h$  and  $\mu_h$  for certain saddle towers with explicitly known Weierstrass data. The punctures  $p_h$  will be placed on the unit circle in the clockwise order, so we use the adapted coordinate

$$w_h = i \frac{z - p_h}{z + p_h}.$$

With this choice, we have for  $h \in \mathbf{H}$

$$(3) \quad \mu_h = -e^{i\theta_h} \log 2 - \sum_{j \neq h} e^{i\theta_j} \log |p_h - p_j|.$$

**2.4.1. Symmetrically deformed saddle towers.** These examples are described in [Kar88, §2.4.1]. They have  $n = 2k$  ends with  $k \geq 2$ . Their Weierstrass data are given by

$$G = z^{k-1} \quad \text{and} \quad \Phi_3 = \frac{2k \sin(k\varphi)}{z^k + z^{-k} - 2 \cos(k\varphi)} \cdot \frac{dz}{z}$$

where  $0 < \varphi \leq \pi/n$ . Comparing to [Kar88, §2.4.1], we multiplied  $\Phi_3$  by  $2k \sin(k\varphi)$  so that the vertical period is  $2\pi$ .

The punctures are at

$$p_h = e^{i(-\lfloor \frac{h}{2} \rfloor \frac{2\pi}{k} + (-1)^h \varphi)}, \quad 1 \leq h \leq n,$$

and the direction of the wings are

$$\theta_h = \left\lfloor \frac{h}{2} \right\rfloor \frac{2\pi}{k} - (-1)^h \psi, \quad \psi - \frac{\pi}{n} = (k-1) \left( \frac{\pi}{n} - \varphi \right).$$

The case  $n = 4$  gives Scherk's surfaces. The case  $\psi = \varphi = \pi/n$  gives the most symmetric saddle towers. They are embedded for  $\pi/n \leq \psi \leq \pi/k$  (latter inequality strict for  $n = 4$ ). If  $n \geq 6$ , the limit case  $\psi = \pi/k$  gives a saddle tower with  $k$  pairs of parallel wings; an example with  $n = 10$  is illustrated on right side of Figure 3.

Given our choice of adapted coordinate and the simple form of the Gauss map, Proposition 2.3 immediately gives

$$\Upsilon_h = n - 2.$$

As for  $\mu_h$ , Equation (3) does not simplify very much in general, so we only present some special cases. When  $n = 4$ , (3) simplifies to

$$\mu_h = e^{-i\theta_h} \log \tan(\psi), \quad 1 \leq h \leq 4,$$

which is not collinear to  $e^{i\theta_h}$  unless  $\psi = \pi/4$  (see Remark 2.1). For arbitrary  $n$ , if  $\psi = \pi/n$ , then by symmetry,  $\mu_h$  is collinear to  $e^{i\theta_h}$ , and its norm (independent of  $h$ ) is tabulated below for small values of  $n$ .

$n$	$e^{-i\theta_h} \mu_h$
4	0
6	$\log \sqrt{3}$
8	$\sqrt{2} \log(1 + \sqrt{2})$
10	$\frac{1}{4} \log 5 + \frac{\sqrt{5}}{2} \log(2 + \sqrt{5})$
12	$\frac{1}{2} \log 3 + \sqrt{3} \log(2 + \sqrt{3})$



2.4.2. *Isosceles saddle tower with 6 wings.* These examples with  $n = 6$  wings are described in [Kar88, §2.5.1]. Their Weierstrass data are given by

$$G = \frac{z^2 + r}{1 + rz^2} \quad \text{and} \quad \Phi_3 = \frac{8 \cos(\varphi)^2}{(1-r)^2} \cdot \frac{1 + r^2 + r \cdot (z^2 + z^{-2})}{(z + z^{-1}) \cdot (z^2 + z^{-2} - 2 \cos(2\varphi))} \cdot \frac{dz}{z},$$

where

$$r = (\cos(\psi) - \cos(\phi)) \tan\left(\frac{\psi + \varphi}{2}\right)$$

with  $\sin \psi + \sin \varphi = 1$  and  $0 < \psi < \pi/2$ . Comparing to [Kar88, §2.5.2], we multiplied  $\Phi_3$  by  $8 \cos(\varphi)^2 / (1-r)^2$  so that the vertical period is  $2\pi$ .

The punctures are at

$$(p_0, \dots, p_5) = (e^{-i\varphi}, -i, -e^{i\varphi}, -e^{-i\varphi}, e^{i\varphi}),$$

and the directions of the wings are

$$(\theta_0, \dots, \theta_5) = (\psi, \pi/2, \pi - \psi, -\pi + \psi, -\pi/2, -\psi),$$

so the wings are parallel to the sides of an isosceles triangle. The most symmetric saddle tower is recovered by  $\psi = \pi/6$ . An example with  $\psi = \pi/3$  is illustrated on the left side of Figure 3. The Jenkins–Serrin polygon is degenerate in the limit  $\psi \rightarrow \pi/2$ , and special in the limit  $\psi \rightarrow 0$ .

We compute

$$\Upsilon_h = \begin{cases} 4 \cos(\psi) / \cos(\varphi) & h \equiv 1 \pmod{3}, \\ 4 \cos(\psi) / \sin(2\varphi) & h \not\equiv 1 \pmod{3}, \end{cases}$$

$$\mu_0 = i \log \frac{\cos(\varphi)}{1 - \sin(\varphi)} + \exp^{-i\psi} \log \cot(\varphi), \quad \mu_1 = 2i \sin(\psi) \log \frac{\cos(\varphi)}{1 - \sin(\varphi)},$$

and the others  $\mu_h$  can be obtained by symmetry, namely

$$\mu_0 = -\bar{\mu}_2 = -\mu_3 = \bar{\mu}_5 \quad \text{and} \quad \mu_1 = -\mu_4.$$

**2.5. Rigidity.** By a result of Cosin-Ros [CR01], all saddle towers are rigid, in the sense that the space of bounded Jacobi fields on a saddle tower is 3-dimensional and consists of translations. This means that when the angles  $\theta_h$  are fixed, a saddle tower admits no deformation other than translations. In this section, we reformulate this result in term of Weierstrass Representation, in a way that can be used in our gluing construction.

The Weierstrass data of a saddle tower can always be written as in Equation (1). The equation to solve is  $Q = 0$ , where  $Q = \Phi_1^2 + \Phi_2^2 + \Phi_3^2$ . The angles  $\theta_h$  are fixed, and the unknowns are the poles  $p_h$  for  $h \in \mathbf{H}$ . We are given a solution, denoted by  $p_h^\circ$ , and we want to study its infinitesimal deformations. The corresponding Weierstrass data is denoted  $(\Phi_1^\circ, \Phi_2^\circ, \Phi_3^\circ)$ .

We formulate the equation  $Q = 0$  as follows. Without loss of generality, we may assume by rotation that  $\cos(\theta_h) \neq 0$  for all  $h \in \mathbf{H}$  and all zeros of  $\Phi_1^\circ$  are simple. Then for  $p$  in a neighborhood of  $p^\circ$ ,  $\Phi_1$  has  $n - 2$  simple zeros  $\zeta_1, \dots, \zeta_{n-2}$  which depend holomorphically on  $p$ . The meromorphic 1-form  $Q/\Phi_1$  is holomorphic at  $p_h$ ,  $h \in \mathbf{H}$ , and has (at most) simple poles at  $\zeta_1, \dots, \zeta_{n-2}$ . We define

$$\Lambda(p) = \left( \text{Res} \left( \frac{Q}{\Phi_1}, \zeta_i \right) \right)_{1 \leq i \leq n-3}.$$

By the Residue Theorem,  $Q = 0$  is equivalent to  $\Lambda(p) = 0$ . By a Möbius transformation, we may fix the value of three points  $p_h$ , so the parameter  $p$  lies in a space of complex dimension  $n - 3$ .

**Theorem 2.8.** *The differential  $D\Lambda(p^\circ)$  is an isomorphism.*

This theorem is proved in Annexe A of [You09], which is unfortunately not published. So we include a proof in Appendix A.

### 3. GRAPHS

Graph theory plays a central role in our construction. Combinatorially, we use a graph to describe how saddle towers (vertices) are glued along wings (edges) into a minimal surface. Geometrically, the degenerate limit of our surface projects to a horizontal plane as a geometric “representation” of the graph.

In this paper, a “graph” actually refers to a multigraph, possibly with multiple edges and loops. This maximizes the generality of our construction. Such a multigraph is best described in terms of rotation systems. The standard references for this section are [GR01, MT01, Die17].

**3.1. Intuitive picture.** For readers not familiar with rotation systems, we give here an intuitive picture which, although informal, should suffice for understanding our construction.

A graph  $G$  in a flat 2-torus  $\mathbb{T}^2$  can be understood as a set of points (vertices) and straight segments (edges) whose endpoints are (not necessarily distinct) vertices and whose interiors are disjoint. The connected components of  $\mathbb{T}^2 \setminus G$  are called faces. The set of faces is denoted  $F$ . By Euler formula,

$$|V| - |E| + |F| = 0.$$

Each edge has two possible orientations, i.e. a choice of “initial vertex”. The set of oriented edges of  $G$  is denoted  $H$ , so  $|H| = 2|E|$ . For  $h \in H$ , we denote  $e(h) \in E$  the corresponding unoriented edge and  $-h$  the same edge with opposite orientation. The initial vertex of an oriented edge is denoted  $v(h)$ , so its terminal vertex is  $v(-h)$ .

*Remark 3.1.* To agree with the language of rotation systems, we call elements of  $H$  half-edges and the adjacency relation is denoted  $h \in v$ . This terminology is preferred because vertices correspond to saddle towers and half-edges correspond to wings of saddle towers.

The number of half-edges adjacent to a vertex  $v$  is called the degree of  $v$  and denoted  $\deg(v)$ . The wings of a saddle tower are naturally ordered by a cyclic permutation. Accordingly, we define a permutation  $\varsigma$  on  $H$  as follows: if  $h \in v$ ,  $\varsigma(h) \in v$  is the half-edge which comes after  $h$  when traveling around  $v$  in the counterclockwise direction.

**3.2. Formal definition.** In the case that a saddle tower has parallel wings, they project to the same straight segment in the degenerate limit. Then the intuitive picture above, which defines edges geometrically, cannot distinguish them properly. To include this situation, it is necessary to use the language of rotation system.

A *rotation system* consists of a set  $H$  and two permutations  $\iota$  and  $\varsigma$  acting on  $H$ , such that  $\iota$  is an involution without fixed points and the group generated by  $\iota$  and  $\varsigma$  acts transitively on  $H$ . The elements of  $H$  are called half-edges. To ease notations and be consistent with the intuitive picture in Section 3.1, we write  $-h$  for  $\iota(h)$ .

**3.2.1. The multigraph.** The rotation system  $(H, \iota, \varsigma)$  defines a connected multigraph  $(H, V, E)$ , where the vertex set  $V$  consists of the orbits of  $\varsigma$  and the edge set  $E$  consists of the orbits of  $\iota$ . As we have mentioned before, the vertices correspond to saddle towers, the half-edges correspond to wings, and the edges correspond to glued wing pairs. In this sense, the graph describes the gluing pattern for our construction.

Note that vertices and edges are identified with subsets of  $H$ , so  $h \in v$  means that the half-edge  $h$  is adjacent to the vertex  $v$ . For a half-edge  $h \in H$ , we use  $v(h)$

and  $e(h)$  to denote the unique vertex and edge associated to  $h$ . We say that  $e(h)$  is a *loop* if  $v(h) = v(-h)$ .

The orbits of  $\varsigma\iota$  are called (combinatorial) *faces* of the graph, and the set of faces is denoted by  $F$ .

**Assumption 3.2.** All faces have at least two elements.

Among the edges, we define an equivalence relation known as “parallel”, such that  $e(h)$  is *parallel* to  $e(h')$  whenever  $\{h, h'\}$  is a face. Because an equivalence relation is transitive, we have  $e$  is parallel to  $e'$  whenever there is a sequence  $h_0, \dots, h_n$  such that  $e = e(h_0)$ ,  $e' = e(h_n)$ , and  $\{h_i, h_{i+1}\}$  is a face for all  $0 \leq i < n$ .

**3.2.2. Topological embedding.** The rotation system  $(H, \iota, \varsigma)$  also defines, up to homeomorphism, a 2-cell embedding of the multigraph on a closed oriented surface.

Recall that an embedding represents vertices by distinct points and edges by curves that do not intersect in their interiors. The half-edges then correspond to the curves in small neighborhoods of the vertices. The permutation  $\varsigma$  sends a half-edge to the next half-edge around the same vertex in the counterclockwise direction, and  $\iota$  sends a half-edge to the other half-edge of the same edge.

The connected components of the complement of a 2-cell embedding are all homeomorphic to an open disk, and are called (topological) *faces* of the embedding. They are in correspondence with the combinatorial faces: Half-edges on the boundary of a topological face in the clockwise direction form a combinatorial face of the graph. Note that two edges are parallel if and only if their representatives are homologous.

If the graph is finite, the genus of the oriented surface can be calculated as

$$g = 1 - \frac{1}{2}(|V| - |E| + |F|)$$

and we call this number the *genus* of the graph.

**Assumption 3.3.** Graphs in the current paper are of genus 1, so they are embedded in a 2-torus  $\mathbb{T}^2$ .

*Remark 3.4.* Graphs of genus 0 arise when constructing SPMS instead of TPMS. Higher genus graphs are certainly interesting and could be used to glue saddle towers in  $\mathbb{H}^2 \times \mathbb{R}$ .

**3.2.3. Geometric representation.** We have seen that the rotation system  $(H, \iota, \varsigma)$  determines a homeomorphism class of 2-cell embeddings. We choose a geometric representation  $\varrho$  from the closure of this homeomorphism class. More specifically,  $\varrho$  maps vertices to distinct points in a flat torus  $\mathbb{T}^2$  and maps each edge to a segment between the images of its endpoints, so that parallel edges are mapped to the same segment, and non-parallel edges are mapped to segments with disjoint interiors.

Such a geometric representation always exists. Indeed, let  $G'$  be the graph obtained from  $G$  by merging parallel edges. We may apply the genus-one version of Tutte’s embedding theorem (see [GGT06] for instance) which implies that  $G'$  admits a straight-edge representation on a flat torus. We then obtain a straight-edge representation of  $G$  by mapping parallel edges to the same segment that represents the corresponding edge of  $G'$ .

**3.2.4. Orientation.** An *orientation* of the graph  $G$  is a function  $\sigma: H \rightarrow \{\pm 1\}$  such that  $\sigma(-h) = -\sigma(h)$  for all  $h$ .

**Definition 3.5.** An orientation  $\sigma$  on  $G$  is *consistent* if  $\sigma \circ \varsigma = -\sigma$ . A graph is *orientable* if it admits a consistent orientation. Once a consistent orientation is fixed, we say that the graph is *oriented*.

Note that an orientable graph only has vertices of even degrees. Vertices of degree 2 are not relevant for us, since there is no saddle tower with two wings.

**Assumption 3.6.** Graphs in this paper have only vertices of degree at least 4.

### 3.3. Vector spaces on graphs.

**3.3.1. Cycles and cuts.** A (simple) *cycle* is a set of half-edges  $c \subset \mathbf{H}$  that can be ordered into a sequence  $(h_1, \dots, h_n)$  such that  $v(-h_i) = v(h_{i+1})$  for  $1 \leq i < n$ ,  $v(-h_n) = v(h_1)$ , and  $v(h_i) \neq v(h_j)$  whenever  $i \neq j$ . We use  $-c$  to denote the reversed cycle  $\{-h: h \in c\}$ . The set of cycles is denoted by  $\mathbf{C}$ . In particular, combinatorial faces are all cycles.

For some partition  $\mathbf{V} = \mathbf{V}_1 \sqcup \mathbf{V}_2$  of the vertices, the *cut* between  $\mathbf{V}_1$  and  $\mathbf{V}_2$  is the set of half-edges  $b \subset \mathbf{H}$  such that  $v(h) \in \mathbf{V}_1$  and  $v(-h) \in \mathbf{V}_2$ . We use  $-b$  to denote the reversed cut  $\{-h: h \in b\}$ . The set of cuts is denoted<sup>1</sup> by  $\mathbf{B}$ . In particular, for any vertex  $v$ , the set

$$b(v) = \{h \in v: v(-h) \neq v\},$$

is a cut between  $\{v\}$  and  $\mathbf{V} \setminus \{v\}$ . We call  $b(v)$  the *vertex cut* at  $v$ .

**3.3.2. Functions on half-edges.** Let  $\mathcal{H}$  be the space of functions  $f: \mathbf{H} \rightarrow \mathbb{R}$ . We say that  $f \in \mathcal{H}$  is *symmetric* if  $f_{-h} = f_h$ , and *antisymmetric* if  $f_{-h} = -f_h$ . They can be seen as edge labelings on, respectively, undirected and directed graphs. The orientation  $\sigma$  is an example of antisymmetric function. We use  $\mathcal{S}$  and  $\mathcal{A}$  to denote, respectively, the space of symmetric and antisymmetric functions.

We denote  $e_h$  the characteristic function of  $\{h\}$ , so  $(e_h)_{h \in \mathbf{H}}$  is the canonical basis of  $\mathcal{H}$ . We equip  $\mathcal{H}$  with the inner product  $(\cdot, \cdot)$  defined by  $(e_h, e_{h'}) = \delta_{h, h'}$ . Then  $\mathcal{S}$  and  $\mathcal{A}$  are orthogonal complementary  $|\mathbf{E}|$ -dimensional subspaces of  $\mathcal{H}$ , i.e.  $\mathcal{H} = \mathcal{A} \oplus \mathcal{S}$ . More specifically, an orthogonal basis for  $\mathcal{A}$  is given by

$$a_h = e_h - e_{-h}$$

and an orthogonal basis for  $\mathcal{S}$  is given by

$$s_h = e_h + e_{-h}.$$

So any  $f \in \mathcal{H}$  can be decomposed into  $f = (f^a + f^s)/2$ , where  $f^a \in \mathcal{A}$  and  $f^s \in \mathcal{S}$  are defined by

$$f_h^a = f_h - f_{-h} \quad \text{and} \quad f_h^s = f_h + f_{-h}.$$

**3.3.3. Cut space and cycle space.** We use  $\mathcal{C}$  to denote the subspace of  $\mathcal{A}$  generated by the character functions

$$a_c = \sum_{h \in c} a_h \quad \text{for } c \in \mathbf{C},$$

known as the *cycle space*. We use  $\mathcal{B}$  to denote the subspace of  $\mathcal{A}$  generated by the character functions

$$a_b = \sum_{h \in b} a_h \quad \text{for } b \in \mathbf{B},$$

known as the *cut space*. It is well known [GR01, Chapter 14] that the cut and cycle spaces are orthogonal complementary subspaces of  $\mathcal{A}$ , i.e.  $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$ . The dimension of  $\mathcal{C}$  is  $|\mathbf{E}| - |\mathbf{V}| + 1$ , and the dimension of  $\mathcal{B}$  is  $|\mathbf{V}| - 1$ .

A cut basis is a set  $\mathbf{B}^* \subset \mathbf{B}$  such that  $(a_b)_{b \in \mathbf{B}^*}$  is a basis of the cut space  $\mathcal{B}$ . An explicit cut basis is given as follows: For  $v \in \mathbf{V}$ , define

$$a_v = a_{b(v)} = \sum_{h \in v} a_h.$$

<sup>1</sup>Inclusion-wise minimal cuts are called *bonds*. Hence the cut space is sometimes referred to as the bond space, therefore our notation.

Let  $V^*$  be the set of all but one vertices. Then  $B^* = \{a_v : v \in V^*\}$  is a canonical cut basis.

A cycle basis is a set  $C^* \subset C$  such that  $(a_c)_{c \in C^*}$  is a basis of the cycle space  $C$ . In case  $G$  has genus one, the cycle space has dimension  $|F| + 1$  by Euler formula. An explicit cycle basis is given as follows. Let  $F^*$  be the set of all but one faces. For  $i = 1, 2$ , let  $c_i \in C$  be a cycle which is homologous in  $\mathbb{T}^2$  to the segment  $[0, T_i]$ . Then  $C^* = F^* \cup \{c_1, c_2\}$  is a canonical cycle basis.

**3.3.4. Discrete differential operators.** Let  $\mathcal{V}$  be the vector space of functions  $f : V \rightarrow \mathbb{R}$  such that  $\sum_{v \in V} f_v = 0$ . For  $f \in \mathcal{V}$ , define

$$\text{grad}(f) = - \sum_{v \in V} f_v a_v = \sum_{h \in H} (f_{v(-h)} - f_{v(h)}) e_h.$$

Then  $\text{grad} : \mathcal{V} \rightarrow \mathcal{B}$  is an isomorphism.

For  $f = (f_h)_{h \in H} \in \mathcal{A}$ , define

$$\begin{aligned} \text{div}_b(f) &= (f, a_b)/2 = \sum_{h \in b} f_h, & \text{div}(f) &= (\text{div}_b(f))_{b \in B}; \\ \text{curl}_c(f) &= (f, a_c)/2 = \sum_{h \in c} f_h, & \text{curl}(f) &= (\text{curl}_c(f))_{c \in C}. \end{aligned}$$

Because of the orthogonality between  $\mathcal{B}$  and  $\mathcal{C}$ , we have

$$\ker(\text{div}) = \mathcal{C} \quad \text{and} \quad \ker(\text{curl}) = \mathcal{B}.$$

Hence we can identify

$$\text{im}(\text{div}) \simeq \mathcal{B} \quad \text{and} \quad \text{im}(\text{curl}) \simeq \mathcal{C}.$$

*Remark 3.7.* Let  $\pi_B : \mathbb{R}^{|B|} \rightarrow \mathbb{R}^{|V|-1}$  be the projection  $(x_b)_{b \in B} \mapsto (x_b)_{b \in B^*}$ . Then  $\pi_B : \text{im}(\text{div}) \rightarrow \mathbb{R}^{|V|-1}$  is an isomorphism. Indeed,  $(a_v)_{v \in V^*}$  is a basis of the cut space so the Gram matrix  $(a_v, a_{v'})_{v, v' \in V^*}$  is invertible. In the same way, the projection  $\pi_C : \mathbb{R}^{|C|} \rightarrow \mathbb{R}^{|F|+1}$ ,  $(x_c)_{c \in C} \mapsto (x_c)_{c \in C^*}$  restricts to an isomorphism from  $\text{im}(\text{curl})$  to  $\mathbb{R}^{|C|+1}$ .

**3.4. A divergence over shortest edges.** For each half-edge  $h$ , let  $\ell_h^\circ$  be the length of the segment  $\varrho(e(h))$ . For  $b \in B$ , define

$$\ell_b^\circ = \min_{h \in b} \ell_h^\circ \quad \text{and} \quad m(b) = \{h \in b \mid \ell_h^\circ = \ell_b^\circ\}.$$

We define the operator

$$\text{mdiv}_b(\phi) = (a_{m(b)}, \phi)/2 = \sum_{h \in m(b)} \phi_h, \quad \text{mdiv}(\phi) = (\text{mdiv}_b(\phi))_{b \in B}.$$

In general,  $\mathcal{B}_m := \text{im}(\text{mdiv})$  is different from  $\mathcal{B}$ , but the following proposition asserts that they have the same dimension.

**Proposition 3.8.** *mdiv has rank  $|V| - 1$ . Moreover, there exists a cut basis  $B_m^*$  such that  $(\text{mdiv}_b)_{b \in B_m^*}$  has rank  $|V| - 1$ .*

*Proof.* Let  $\phi \in \mathcal{B}$  such that  $\text{mdiv}(\phi) = 0$ . We can write  $\phi = \text{grad}(f)$  with  $f \in \mathcal{V}$ . Assume that  $f$  is not constant. Let  $V_1$  be the set of vertices where  $f$  achieves maximum and let  $b$  be the corresponding cut. Then  $\text{mdiv}_b(\phi) < 0$ , a contradiction. So  $f$  is constant and  $\phi = 0$ . Hence  $\text{mdiv}$  is injective on  $\mathcal{B}$  and

$$\text{rank}(\text{mdiv}) \geq |V| - 1.$$

For the reverse inequality, consider for  $\varepsilon > 0$  the operator  $\text{mdiv}^\varepsilon : \mathcal{A} \rightarrow \mathbb{R}^{|B|}$  defined by

$$\text{mdiv}_b^\varepsilon(\phi) = \sum_{h \in b} e^{\varepsilon^{-2}(\ell_b^\circ - \ell_h^\circ)} \phi_h, \quad b \in B.$$

Then

$$\text{mdiv} = \lim_{\varepsilon \rightarrow 0} \text{mdiv}^\varepsilon.$$

But we can write for  $\varepsilon > 0$

$$\text{mdiv}^\varepsilon = \Phi^\varepsilon \circ \text{div} \circ \Psi^\varepsilon,$$

where

$$\Psi^\varepsilon(\phi) = (e^{-\varepsilon^{-2}\ell_h^\circ} \phi_h)_{h \in \mathbf{H}} \quad \text{and} \quad \Phi^\varepsilon(X) = (e^{\varepsilon^{-2}\ell_b^\circ} X_b)_{b \in \mathbf{B}}.$$

Hence

$$\text{rank}(\text{mdiv}^\varepsilon) \leq \text{rank}(\text{div}) = |\mathbf{V}| - 1.$$

Since the rank is lower semi-continuous, it follows that

$$\text{rank}(\text{mdiv}) \leq |\mathbf{V}| - 1.$$

Hence  $\text{mdiv}$  has rank  $|\mathbf{V}| - 1$ . There exists a subset  $\mathbf{B}_m^* \subset \mathbf{B}$  with cardinal  $|\mathbf{V}| - 1$  such that  $(\text{mdiv}_b)_{b \in \mathbf{B}_m^*}$  has rank  $|\mathbf{V}| - 1$ . Then for  $\varepsilon > 0$ ,  $(\text{mdiv}_b^\varepsilon)_{b \in \mathbf{B}_m^*}$  has rank  $|\mathbf{V}| - 1$  by continuity, so  $(\text{div}_b)_{b \in \mathbf{B}_m^*}$  has rank  $|\mathbf{V}| - 1$  and  $\mathbf{B}_m^*$  is a cut basis.  $\square$

#### 4. MAIN RESULT

In this part, we assume that  $\mathbf{G} = (\mathbf{H}, \iota, \varsigma, \varrho)$  is a finite graph represented in a flat torus  $\mathbb{T}^2 = \mathbb{C}/\langle T_1, T_2 \rangle$ .

**4.1. Horizontal balance and rigidity.** To each half-edge  $h$  is associated the unit tangent vector  $u_h^\circ = e^{i\theta_h^\circ}$  of the segment  $\varrho(e(h))$  at  $\varrho(v(h))$ . Recall that  $\ell_h^\circ$  denotes the length of the segment  $\varrho(e(h))$  and set  $x_h^\circ = \ell_h^\circ u_h^\circ$ . As a general rule, we use an superscript  $\circ$  to denote quantities associated to the given graph  $\mathbf{G}$ , which are to be perturbed as parameters in the construction.

*Remark 4.1.* In the intuitive picture in Section 3.1,  $u_h^\circ$  is simply the unit vector in the direction of the oriented edge  $h$  and  $\ell_h^\circ$  is its length.

Obviously,  $u^\circ = (u_h^\circ)_{h \in \mathbf{H}} \in \mathcal{A}^2$ ,  $\ell^\circ = (\ell_h^\circ)_{h \in \mathbf{H}} \in \mathcal{S}$  and  $x^\circ = (x_h^\circ)_{h \in \mathbf{H}} \in \mathcal{A}^2$ . For  $x \in \mathcal{A}^2$  in a neighborhood of  $x^\circ$ , we define

$$u_h(x) = \frac{x_h}{\|x_h\|} \quad \text{and} \quad u(x) = (u_h(x))_{h \in \mathbf{H}} \in \mathcal{A}^2.$$

We define the horizontal forces as the function

$$\begin{aligned} F^{\text{hor}}: \mathcal{A}^2 &\rightarrow \mathcal{B}^2 \\ x &\mapsto \text{div}(u(x)). \end{aligned}$$

More explicitly, for any cut  $b \in \mathbf{B}$

$$F_b^{\text{hor}}(x) = \sum_{h \in b} \frac{x_h}{\|x_h\|}.$$

**Definition 4.2.** The graph  $\mathbf{G}$  is balanced if  $F^{\text{hor}}(x^\circ) = 0$ .

*Remark 4.3.* By remark 3.7, the graph is balanced if and only if

$$\sum_{h \in v} \frac{x_h^\circ}{\|x_h^\circ\|} = 0$$

for all  $v \in V^*$ . In other words, it suffices to consider horizontal forces on a canonical cut basis consisting of all but one vertex cuts.

*Remark 4.4.* A balanced graph is a weak local minimal network in the sense of Ivanov and Tuzhilin [IT94]; see Proposition B.1 in the Appendix.

Now define the horizontal periods as the operator

$$P^{\text{hor}}: \mathcal{A}^2 \rightarrow \mathcal{C}^2$$

$$x \mapsto \text{curl}(x).$$

More explicitly, for any cycle  $c \in \mathbf{C}$

$$P_c^{\text{hor}}(x) = \sum_{h \in c} x_h.$$

The graph  $\mathbf{G}$  is represented on the torus  $\mathbb{T}^2$  so we have on any canonical cycle basis  $\mathbf{C}^*$

$$(4) \quad P_c^{\text{hor}}(x^\circ) = \begin{cases} 0, & c \in \mathbf{F}^*; \\ T_i, & c = c_i, \quad i = 1, 2. \end{cases}$$

*Remark 4.5.* The balance and period equations can be compared to Kirchhoff's current and voltage laws of electrical networks. More specifically,  $u^\circ$ ,  $x^\circ$ , and  $\ell^\circ$  play, respectively, the roles of currents, voltages and resistance.

Recall from Section 3.3 that  $\ker(P^{\text{hor}}) = \mathcal{B}^2$ .

**Definition 4.6.** The graph  $\mathbf{G}$  is rigid if the differential  $DF^{\text{hor}}(x^\circ)$  restricted to  $\ker(P^{\text{hor}})$  is an isomorphism from  $\mathcal{B}^2$  to  $\mathcal{B}^2$ . Equivalently, the graph is rigid if

$$(DF^{\text{hor}}(x^\circ), P^{\text{hor}}): \mathcal{A}^2 \rightarrow \mathcal{A}^2$$

is an isomorphism.

A computation reveals that

$$(5) \quad DF_b^{\text{hor}}(x^\circ) \cdot \chi = \sum_{h \in b} \frac{1}{\|x_h^\circ\|^3} (\langle x_h^\circ, x_h^\circ \rangle \chi_h - \langle x_h^\circ, \chi_h \rangle x_h^\circ).$$

Next we need to assign a saddle tower to each vertex.

**Definition 4.7.** Let  $v \in \mathbf{V}$ . Consider the convex closed polygon  $P$  whose edges are the unit vectors  $u_h^\circ$  for  $h \in v$  in the order given by the permutation  $\varsigma$ . We say that  $v$  is *ordinary* if  $P$  is neither *degenerate* (appears as a line segment) nor *special* (appears as a parallelogram with two sides of length 1 and two sides of length  $\geq 2$ ).

Note that non-ordinary vertices occur only in the presence of parallel edges. If  $v$  is ordinary, there exists a saddle tower  $\mathcal{S}_v$ , unique up to translations, with  $\deg(v)$  wings which are in correspondence with the half-edges  $h \in v$ , so that the direction of the wing corresponding to  $h$  is  $u_h^\circ$  and the natural order on the wings is given by the permutation  $\varsigma$  (see Section 2.1).

We want to glue the wings of these saddle towers along the edges of  $\mathbf{G}$ . This construction is achieved if the saddle towers share a horizontal reflection plane, as stated in the following theorem due to Younes [You09].

**Theorem 4.8** (TPMSs with horizontal symmetry). *Let  $\mathbf{G}$  be a graph represented in  $\mathbb{T}^2 = \mathbb{C}/\langle T_1, T_2 \rangle$ , and assign  $\phi_v \in \{0, \pi\}$  to each vertex  $v \in \mathbf{V}$ . If  $\mathbf{G}$  is orientable, balanced, rigid, and all vertices are ordinary, then for sufficiently small  $\varepsilon > 0$ , there is a family  $\mathcal{M}_\varepsilon$  of embedded minimal surfaces of genus  $|\mathbf{F}| + 1$  in the flat 3-torus*

$$\mathbb{T}_\varepsilon^3 = \mathbb{R}^3 / \langle (T_1 \varepsilon^{-2}, 0), (T_2 \varepsilon^{-2}, 0), (0, 0, 2\pi) \rangle.$$

*They lift to triply periodic minimal surfaces  $\widetilde{\mathcal{M}}_\varepsilon$  in  $\mathbb{R}^3$  such that*

- (1)  $\widetilde{\mathcal{M}}_\varepsilon$  converges, after scaling by  $\varepsilon^2$ , to  $\widetilde{\mathbf{G}} \times \mathbb{R}$  as  $\varepsilon \rightarrow 0$ , where  $\widetilde{\mathbf{G}}$  is the lift of  $\mathbf{G}$  to  $\mathbb{R}^2$ .

- (2) For each vertex  $v$  of  $\widetilde{\mathcal{G}}$ , there exists a horizontal vector  $X_v(\varepsilon)$  such that  $\widetilde{\mathcal{M}}_\varepsilon - X_v(\varepsilon)$  converges on compact subset of  $\mathbb{R}^3$  to a saddle tower  $\mathcal{S}_v$  as  $\varepsilon \rightarrow 0$ . Moreover,  $\varepsilon^2 X_v(\varepsilon) \rightarrow \widetilde{\varrho}(v)$  as  $\varepsilon \rightarrow 0$ , where  $\widetilde{\varrho}$  is the lift of  $\varrho$ .
- (3) Each limit saddle tower  $\mathcal{S}_v$  has phase  $\phi_v$ . Moreover,  $\widetilde{\mathcal{M}}_\varepsilon$  is symmetric with respect to a horizontal plane.

**4.2. Vertical balance and rigidity.** To break the horizontal reflection symmetry, we want to prescribe arbitrary phase differences between saddle towers. This is done through an antisymmetric *phase function*  $\phi^\circ: \mathbf{H} \rightarrow \mathbb{R}/2\pi\mathbb{Z}$  that assigns a phase difference  $\phi_h^\circ$  to each half-edge  $h$ . We say that  $\phi^\circ$  is trivial if  $\phi^\circ = 0$  or  $\phi^\circ = \pi$  on every half-edge, which is the case in Theorem 4.8.

Define the vertical periods as the function

$$P^{\text{ver}}: \mathcal{A} \rightarrow \mathcal{C}$$

$$\phi \mapsto \text{curl}(\phi).$$

We require that the periods of the phase function are given on a canonical cycle basis  $\mathbf{C}^*$  as

$$(6) \quad P_c^{\text{ver}}(\phi^\circ) = \sum_{h \in c} \phi_h^\circ = \begin{cases} 0, & c \in \mathbf{F}; \\ \Psi_i, & c = c_i, \quad i = 1, 2. \end{cases}$$

for some  $\Psi_1, \Psi_2 \in \mathbb{R}/2\pi\mathbb{Z}$ . We call  $\Psi_1$  and  $\Psi_2$  the *fundamental shifts*. We want to construct minimal surfaces in the flat 3-torus

$$\mathbb{T}_\varepsilon^3 = \mathbb{R}^3 / \langle (\Lambda_1 + T_1 \varepsilon^{-2}, \Psi_1), (\Lambda_2 + T_2 \varepsilon^{-2}, \Psi_2), (0, 0, 2\pi) \rangle,$$

Here  $\Lambda_1, \Lambda_2$  are fixed complex numbers that prescribe a first order horizontal deformation of the lattice as  $\varepsilon$  varies.

The phase function must satisfy a balancing condition, which we now explain.

For each vertex  $v \in \mathbf{V}$ , let  $\widetilde{\mathbb{C}}_v$  be the punctured Riemann sphere on which the saddle tower  $\mathcal{S}_v$  is parametrized. Fix an adapted local coordinate  $w_h$  in a neighborhood of the puncture  $p_h \in \mathbb{C}_{v(h)}$  for every  $h \in \mathbf{H}$ . Recall the definition of the numbers  $\Upsilon_h, \mu_h$  in Section 2 and the notation  $\mu_h^a = \mu_h - \mu_{-h}$ . If the graph  $\mathbf{G}$  is balanced and rigid, the system

$$(7) \quad \begin{cases} DF_b^{\text{hor}}(x^\circ) \cdot \xi = 0, & b \in \mathbf{B}; \\ P_c^{\text{hor}}(\xi) = -P_c^{\text{hor}}(\mu^a), & c \in \mathbf{F}^*; \\ P_{c_i}^{\text{hor}}(\xi) = -P_{c_i}^{\text{hor}}(\mu^a) + \Lambda_i, & i = 1, 2 \end{cases}$$

has a unique solution  $\xi \in \mathcal{A}^2$  by Definition 4.6 and Remark 3.7.

*Remark 4.9.* The system (7) is invariant by horizontal translations of the saddle towers. Indeed, if  $\mathcal{S}_v$  is translated by a horizontal vector  $X_v$ , then  $X_{v(h)}$  is added to  $\mu_h$ , so  $\text{curl}(\text{grad}(X)) = 0$  is added to the right-hand side of (7).

We define a symmetric function

$$(8) \quad K_h = \Upsilon_h \Upsilon_{-h} e^{-\text{Re}(\xi_h \overline{u_h^\circ})}.$$

By Proposition 2.5, we have  $K_h > 0$ . We will see in Proposition 4.19 that  $K_h$  is independent of the choice of adapted coordinates  $w_h$ . Both  $\xi$  and  $K_h$  depend on  $\Lambda_1$  and  $\Lambda_2$ , but the dependence is omitted for simplicity. When the values of  $\Lambda_1$  and  $\Lambda_2$  matter, but are not specified in the context, it is implied that  $\Lambda_1 = \Lambda_2 = 0$ .

We define the vertical forces as the function

$$F^{\text{ver}}: \mathcal{A} \rightarrow \mathcal{B}_m$$

$$\phi \mapsto \text{mdiv}((K_h \sin \phi_h)_{h \in \mathbf{H}}).$$



**Definition 4.10.** The phase function  $\phi^\circ$  is balanced if  $F^{\text{ver}}(\phi^\circ) = 0$ .

*Remark 4.11.* The above definition of vertical balance may seem strange, as it only uses edges of shortest length in every cut. We will see in the proof that when  $\varepsilon > 0$ , the vertical force actually contains a factor  $\exp(-\ell_h^\circ/\varepsilon^2)$ , which dominates on shortest edges as  $\varepsilon \rightarrow 0$ .

*Remark 4.12.* Trivial phase functions are trivially balanced.

*Remark 4.13.* Unlike horizontal balancing (see Remark 4.3), the equation  $F^{\text{ver}}(\phi^\circ) = 0$  is in general not equivalent to  $F_{b(v)}^{\text{ver}}(\phi^\circ) = 0$  for  $v \in V^*$ : it is not enough to consider vertex cuts; see Example 5.6. This is the reason why it is necessary to introduce the whole cut space to define vertical balancing.

*Remark 4.14.* In general, the vertical forces do not depend continuously on the horizontal periods  $T_1$  and  $T_2$ , but they depend continuously on the deformations  $\Lambda_1$  and  $\Lambda_2$ .

**Definition 4.15.** The phase function  $\phi^\circ$  is rigid if the differential  $DF^{\text{ver}}(\phi^\circ)$  restricted to  $\mathcal{B}$  is an isomorphism between  $\mathcal{B}$  and  $\mathcal{B}_m$ . Equivalently, the phase function is rigid if  $(DF^{\text{ver}}(\phi^\circ), P^{\text{ver}})$  is an isomorphism.

We call the pair  $(\mathbf{G}, \phi^\circ)$  a *configuration*, and say that the configuration is balanced (resp. rigid) if both the graph and the phase function are balanced (resp. rigid). Our main result for TPMSs is the following.

**Theorem 4.16** (TPMSs). *Let  $(\mathbf{G}, \phi^\circ)$  be a configuration, where the graph  $\mathbf{G}$  is represented in  $\mathbb{T}^2 = \mathbb{C}/\langle T_1, T_2 \rangle$ , and the fundamental shifts of  $\phi^\circ$  is  $\Psi_1$  and  $\Psi_2$ . Assume that  $\mathbf{G}$  is orientable, that the configuration is balanced and rigid, and that all vertices are ordinary. Then for sufficiently small  $\varepsilon > 0$ , there is a family  $\mathcal{M}_\varepsilon$  of embedded minimal surfaces of genus  $|\mathbf{F}| + 1$  in the flat 3-torus*

$$\mathbb{T}_\varepsilon^3 = \mathbb{R}^3 / \langle (\Lambda_1 + T_1\varepsilon^{-2}, \Psi_1), (\Lambda_2 + T_2\varepsilon^{-2}, \Psi_2), (0, 0, 2\pi) \rangle.$$

*They lift to triply periodic minimal surfaces  $\widetilde{\mathcal{M}}_\varepsilon$  in  $\mathbb{R}^3$  such that*

- (1)  *$\widetilde{\mathcal{M}}_\varepsilon$  converges, after a scaling by  $\varepsilon^2$ , to  $\widetilde{\mathbf{G}} \times \mathbb{R}$  as  $\varepsilon \rightarrow 0$ , where  $\widetilde{\mathbf{G}}$  is the lift of  $\mathbf{G}$  to  $\mathbb{R}^2$ .*
- (2) *For each vertex  $v$  of  $\widetilde{\mathbf{G}}$ , there exists a horizontal vector  $X_v(\varepsilon)$  such that  $\widetilde{\mathcal{M}}_\varepsilon - X_v(\varepsilon)$  converges on compact subset of  $\mathbb{R}^3$  to a saddle tower  $\mathcal{S}_v$  as  $\varepsilon \rightarrow 0$ . Moreover,  $\varepsilon^2 X_v(\varepsilon) \rightarrow \widetilde{\varrho}(v)$  as  $\varepsilon \rightarrow 0$ , where  $\widetilde{\varrho}$  is the lift of  $\varrho$ .*
- (3) *For each half-edge  $h$ , the phase difference of  $\mathcal{S}_{v(-h)}$  over  $\mathcal{S}_{v(h)}$  is equal to  $\phi_h^\circ$ .*

*Remark 4.17.* Unfortunately, Theorem 4.16 does not contain Theorem 4.8 as a particular case. A trivial phase function is trivially balanced, but it is not necessarily rigid. However, Proposition 4.22 below implies that the phase function is rigid when  $\phi_h^\circ$ ,  $h \in \mathbf{H}$ , are all 0 or all  $\pi$ . So Theorem 4.8 follows from Theorem 4.16 when all saddle towers are in-phase, or all adjacent saddle towers are anti-phase.

*Remark 4.18.* In fact, we construct a continuous family locally parameterized by  $\varepsilon$ ,  $\Lambda_{1,2}$ , and  $\Psi_{1,2}$ . By Proposition 4.20 below, we may assume that  $\Lambda_1 = 0$  up to a scaling and a horizontal rotation. This is therefore a 5-parameter family up to Euclidean rotations and scalings, in correspondence with the deformations of the lattice.

### 4.3. Some auxiliary results.

**Proposition 4.19.** *The constant  $K_h$  defined in Equation (8) is independent of the adapted local coordinates  $w_h$ .*

*Proof.* Consider another adapted local coordinates  $\tilde{w}_h$  in a neighborhood of  $p_h$  for each  $h \in \mathbf{H}$ . We use a tilde for all quantities associated to the coordinate  $\tilde{w}_h$ . By Remark 2.7, we have, writing  $\kappa_h = \frac{d\tilde{w}_h}{dw_h}(p_h) > 0$ , that

$$\tilde{\mu}_h - \mu_h = u_h^\circ \log \kappa_h \quad \text{and} \quad \tilde{\Upsilon}_h = \Upsilon_h / \kappa_h.$$

Therefore, since  $u_{-h}^\circ = -u_h^\circ$

$$\tilde{\mu}_h^a - \mu_h^a = u_h^\circ \log(\kappa_h \kappa_{-h}).$$

Observe that  $\tilde{\xi} - \xi$  is the solution of

$$\begin{cases} DF^{\text{hor}}(x^\circ) \cdot (\tilde{\xi} - \xi) = 0; \\ P^{\text{hor}}(\tilde{\xi} - \xi) = -P^{\text{hor}}(\tilde{\mu}^a - \mu^a). \end{cases}$$

By Equation (5), we have

$$DF^{\text{hor}}(x^\circ) \cdot (\tilde{\mu}_h^a - \mu_h^a) = 0.$$

So the solution is trivially

$$\tilde{\xi}_h - \xi_h = -\tilde{\mu}_h^a + \mu_h^a = -u_h^\circ \log(\kappa_h \kappa_{-h}), \quad h \in \mathbf{H}.$$

Therefore,

$$\frac{\tilde{K}_h}{K_h} = \frac{\tilde{\Upsilon}_h \tilde{\Upsilon}_{-h}}{\Upsilon_h \Upsilon_{-h}} \exp\left(-\operatorname{Re}\left((\tilde{\xi}_h - \xi_h) \overline{u_h^\circ}\right)\right) = \frac{e^{\log(\kappa_h \kappa_{-h})}}{\kappa_h \kappa_{-h}} = 1.$$

□

**Proposition 4.20.** *The vertical balance condition is invariant under the transform  $\Lambda_i \mapsto \tilde{\Lambda}_i = \Lambda_i + \lambda T_i$ ,  $\lambda \in \mathbb{C}$ .*

*Proof.* We use a tilde for all quantities associated to  $\tilde{\Lambda}_i$ . For the solutions of (7), we have  $\tilde{\xi} - \xi = \lambda x^\circ$ . Hence  $\tilde{K}_h = K_h \exp(-\ell_h^\circ \operatorname{Re} \lambda)$  and  $\tilde{F}_b^{\text{ver}} = F_b^{\text{ver}} \exp(-\ell_b^\circ \operatorname{Re} \lambda)$ . □

The following horizontal rigidity result is proved in [You09, Theorem 10].

**Theorem 4.21.** *Assume that all faces of  $\mathbf{G}$  have 2 or 3 edges. Then  $\mathbf{G}$  is rigid.*

As [You09] is not published, we include a proof in Appendix B. We conclude this section with the following vertical rigidity result:

**Proposition 4.22.** *Let  $\phi^\circ$  be a phase function such that  $\cos(\phi_h^\circ)$ ,  $h \in \mathbf{H}$ , are all positive or all negative. Then  $\phi^\circ$  is rigid. In particular, the zero phase function is always balanced and rigid.*

*Proof.* It suffices to prove the case where  $\cos(\phi_h^\circ)$  are all positive. Let  $\dot{\phi} \in \mathcal{B}$  such that  $DF_b^{\text{ver}}(\phi^\circ) \cdot \dot{\phi} = 0$ . We can write  $\dot{\phi} = \operatorname{grad}(f)$  with  $f \in \mathcal{V}$ . Then

$$0 = dF_b^{\text{ver}}(\phi^\circ) \cdot \dot{\phi} = \sum_{h \in m(b)} K_h \cos(\phi_h^\circ) (f_{v(-h)} - f_{v(h)}).$$

Since  $K_h \cos(\phi_h^\circ) > 0$  for all  $h \in \mathbf{H}$ , we conclude that  $\dot{\phi} = 0$  by the same argument as in the proof of Proposition 3.8, considering the maximum of  $f$ . □

## 5. EXAMPLES

In this part, examples are sketched in the form of diagrams. Edges are decorated with arrows to illustrate a consistent orientation  $\sigma$ . Unless otherwise specified (e.g. Example 5.7), for each  $h$  such that  $\sigma(h) = 1$ , we label the phase difference  $\phi_h^\circ$  on the edge  $e(h)$ . The fundamental parallelogram of the torus spanned by  $T_1$  and  $T_2$  is illustrated by dotted lines.

For all the examples presented below, the computation of  $K_h$  is either trivial because of symmetry, or not necessary (e.g. Examples 5.2).

**5.1. Genus three.** The genus of a TPMS is at least three, hence the graph for our construction has at least two faces. We notice four families of balanced configurations with two faces. They are illustrated in Figure 4. Theorem C.1 in the Appendix asserts that these are the only balanced configurations whose graphs are orientable with two faces. Hence they are the only possible configurations that give rise to TPMSs of genus 3.

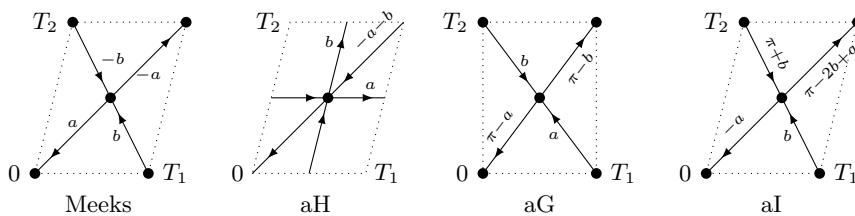


FIGURE 4. The four balanced configurations that could give rise to TPMSs of genus three.

**Example 5.1** (Meeks family). The first diagram illustrates a 4-parameter family (parameterized by  $T_1$ ,  $T_2$ ,  $a$ , and  $b$ ). It actually describes the Scherk limit of Meeks' family [Mee90]. To see this, note that the configurations are invariant under the translation  $(T_1 + T_2)/2$ . This implies an orientation-reversing translational symmetry in the corresponding TPMSs, which characterizes Meeks' surfaces. See Figures 1(a–c) for examples in this family.

A generic configuration in the family is rigid, hence gives rise to a 5-parameter family of TPMSs. If  $\arg(T_2/T_1) < \pi/2$  (resp.  $> \pi/2$ ), the configuration is not rigid when  $\cos(b) = 0$  (resp.  $\cos(a) = 0$ ); see Example 5.5. If  $\arg(T_2/T_1) = \pi/2$ , the configuration is not rigid when  $\cos(a) + \cos(b) = 0$ ; see Example 5.4.  $\square$

**Example 5.2** (aH). The second diagram is again a 4-parameter family. We name it aH because, when  $|T_1| = |T_2|$  and  $a = b = 0$ , it gives Scherk limits of the oH family [CW21]; see Figure 1(d). Another special case in this family is the Scherk limit of the rhombohedral deformation family rGL of the Gyroid [Che19], given by  $T_2/T_1 = \exp(2i\pi/3)$  and  $a = b = 2\pi/3$ . All configurations in the family are rigid, hence give rise to a new 5-parameter family of TPMSs, generalizing H and rGL.  $\square$

Previously, we knew that both the Gyroid and the H surfaces can be continuously deformed to Meeks surfaces [Che19, CW21]. The aH family implies a deformation path between them that does not pass through the Meeks family.

**Corollary 5.3.** *The Gyroid can be continuously deformed to an H surface along a path in the space of TPMSs of genus 3 that stays outside the Meeks family.*

**Example 5.4** (aG). The third diagram is constrained to  $\arg(T_2/T_1) = \pi/2$ , hence a 3-parameter family. One of the fundamental shift must be  $\pi$ . We name it aG

as it includes the Scherk limit of the tetragonal deformation family tG of the Gyroid [Che19], given by  $|T_1| = |T_2|$  and  $a = b = \pi/2$ . It intersects Meeks family when  $a = b = 0$ . Moreover, when  $a = 0$  and  $b = \pi$ , we recognize alternative Scherk limits of the oH family [CW21]; see Figure 1(e).

Unfortunately, configurations in this family are not vertically rigid, hence our construction is inconclusive for them. To see this, note that adding a common constant to  $a$  and  $b$  does not change the fundamental shifts. For the corresponding TPMSs, this seems to suggest that one can vertically slide one Scherk tower with respect to the other without changing the lattice. Numerical experiments suggest that these configurations do give rise to TPMSs, but a vertical sliding between the towers must be accompanied by a very slight deformation of the horizontal lattice, which becomes undetectable in the Scherk limit.  $\square$

**Example 5.5 (aI).** The fourth diagram is again a 4-parameter family, but constrained to  $\arg(T_2/T_1) \neq \pi/2$ . The fundamental shifts satisfy  $\Psi_2 - \Psi_1 = \pi$ . When  $\Psi_1 = \pi$ , it tends to aG configurations as  $\arg(T_2/T_1) \rightarrow \pi/2$ . Otherwise, the dependence of the vertical balancing condition on the horizontal lattice is not continuous; see Remark 4.14.

Configurations in this family are not vertically rigid: adding a common constant to  $a$  and  $b$  does not change the fundamental shifts. Our construction is inconclusive and, apart from the aG limit, we are not aware of any known TPMS that admits this kind of Scherk limit. But by Theorem 4.8, the trivial phase functions given by  $a, b = 0$  or  $\pi$  should give rise to a family of TPMSs. For a lack of a better name, we call this family aI (following the pattern of aG, aH).  $\square$

## 5.2. Technical examples with triangular lattice.

**Example 5.6.** The purpose of this example is to demonstrate the necessity to define vertical balancing on the whole cut space.

Figure 5 illustrates a graph with four vertices. If only vertex cuts are considered, solving the vertical balancing equation would only determine phase differences on the four vertical edges. This gives a false illusion that one may vertically slide the saddle towers with respect to each other. In fact, to determine phase differences on the remaining edges, we must acknowledge that they form a cut and must also be balanced.  $\square$

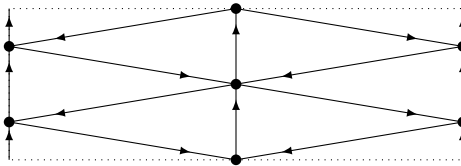


FIGURE 5. For vertical balancing on this graph, it is not enough to consider only vertex cuts.

**Example 5.7.** The purpose of this example is to demonstrate the diversity of balanced phase functions.

The graph in Figure 6, which is a  $3 \times 3$  block of the triangular lattice, is obviously balanced and rigid. Unlike other diagrams in this section, the phases of the saddle towers are labeled on the vertices up to the addition of a common constant. This is possible because the fundamental shifts are 0.

On this small graph, we look for phase functions symmetric in the points marked with empty circles in the figure. That is,

$$\phi_1 = -\phi_2, \quad \phi_3 = -\phi_6, \quad \phi_4 = -\phi_8, \quad \phi_5 = -\phi_7.$$

Under these strict restrictions, we still find two non-trivial, balanced, and rigid phase functions, namely

- $-\phi_1 = \phi_2 = 2\pi/3$ ,  $-\phi_3 = \phi_5 = \phi_6 = -\phi_7 = \pi/3$ , and  $\phi_4 = -\phi_8 = \pi$ ;
- $\phi_1 = -\phi_2 = -\phi_3 = \phi_5 = \phi_6 = -\phi_7 = 2 \arctan \sqrt{5/7}$  and  $\phi_4 = -\phi_8 = \pi$ .

For readers who are interested in double checking: the rigidities are confirmed numerically by computing the determinant of an  $8 \times 8$  Jacobian matrix, which is the derivative of vertical forces on 8 vertices with respect to their phases. If we assume  $K_h = 1$  on all half-edges (up to scaling of local coordinates), the Jacobian determinant is  $-3/4$  for the first phase function, and  $-315/4$  for the second.

We certainly did not find all non-trivial, balanced, and rigid phase functions on this graph. It can be imagined that, if we take a larger block of triangular lattice or relax the inversion symmetry, there will be more balanced and rigid phase functions.  $\square$

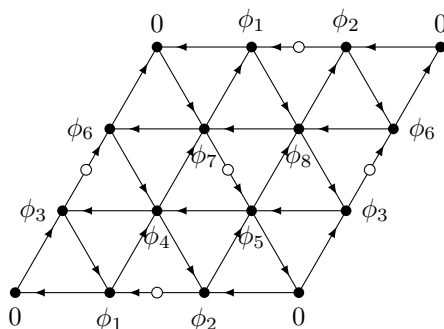


FIGURE 6. A configuration with the graph of the triangular lattice. Phase functions are labeled on the vertices instead of the edges. We look for phase functions symmetric in the points marked with the empty circles.

**5.3. Generalizing some known examples.** We generalize here some interesting known examples to demonstrate the power of our construction. Many known families of TPMSs admit saddle tower limits. In addition to those discussed below, examples also include Schoen's so-called RII, RIII, I-6, I-8, I-9<sup>2</sup>, and GW surfaces [Sch70, Bra], and many more constructed in [FK87, Kar89]. Brakke's webpage [Bra] is a great source of examples. We certainly do not plan to discuss all of them.

If a graph contains a pair of parallel edges, they must be adjacent, and their orientations and phase differences must all be opposite. In fact, given any graph  $G = (H, \iota, \varsigma, \varrho)$  with only simple edges, we may construct a *doubling graph*  $\bar{G} = (\bar{H}, \bar{\iota}, \bar{\varsigma}, \bar{\varrho})$  with only parallel edges as follows (we use a bar for all objects associated to  $\bar{G}$ ).

- For each  $h \in H$ , we have two half-edges  $\bar{h}_+, \bar{h}_- \in \bar{H}$ ;
- We define  $\bar{\iota}$  by  $\bar{\iota}(h_+) = (\iota(h))_-$  and  $\bar{\iota}(h_-) = (\iota(h))_+$ ;
- We define  $\bar{\varsigma}$  by  $\bar{\varsigma}(h_+) = h_-$  and  $\bar{\varsigma}(h_-) = (\varsigma(h))_+$ ;
- We define  $\bar{\varrho}$  by  $\bar{\varrho}(\bar{v}(h_\pm)) = \varrho(v(h))$  and  $\bar{\varrho}(\bar{e}(h_\pm)) = \varrho(e(h))$ .

<sup>2</sup>These names are coined by Brakke for they are the 6th, 8th, and 9th surface on Page I of a note by Schoen.

Clearly, the doubling graph  $\bar{G}$  is always orientable, even if the original graph  $G$  is not. This observation expands the power of our constructions: Even if the graph is not orientable, as long as a graph is balanced and rigid, its doubling would be an orientable, balanced, and rigid graph.

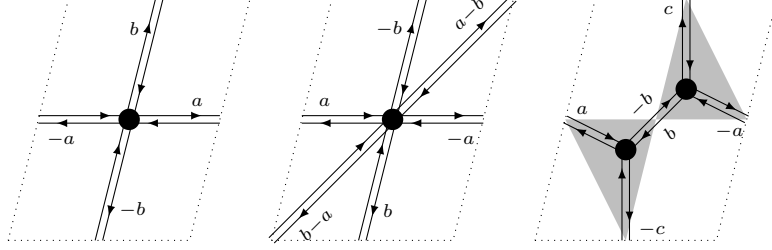


FIGURE 7. Configurations doubling the graphs of parallelogram, triangle and hexagonal tilings.

**Example 5.8.** In Figure 7, we illustrate three configurations that double the graphs of tilings of the Euclidean plane. Each diagram describes a 4-parameter family of balanced configurations. They generalize, respectively, the Scherk limits of Schoen's  $S'$ - $S''$ ,  $T'$ - $R'$ , and  $H'$ - $R$  families [Sch70]; see Figures 1(g-i). The first two configurations in the figure are trivially balanced and rigid, hence give rise to 5-parameter families of TPMSs.

As for the last one that doubles hexagonal tilings, the graph is balanced if and only if the vertices are at the unique Fermat-Torricelli points of the gray triangles spanned by the 2-division points of the torus. This is well defined only when these triangles do not have an angle  $\geq 2\pi/3$ . Let  $l_a$ ,  $l_b$  and  $l_c$  be the lengths of the edges labeled by  $\pm a$ ,  $\pm b$  and  $\pm c$  in the figure, respectively. We may assume that  $l_a \leq l_b \leq l_c$ . Then the balance condition for the phase function is

$$\left. \begin{array}{l} e^{ia} \\ e^{ia} + e^{ib} \\ e^{ia} + e^{ib} + e^{ic} \end{array} \right\} \text{ being real, if } \left\{ \begin{array}{l} l_a < l_b \leq l_c; \\ l_a = l_b < l_c; \\ l_a = l_b = l_c. \end{array} \right.$$

The phase function is rigid if the left hand side is non-zero.

We see again that balancing conditions do not depend continuously on the horizontal lattice. But the dependence on  $\Lambda_1$  and  $\Lambda_2$  is continuous, so the rigid configurations still give rise to 5-parameter families of TPMSs. See Remark 4.14.  $\square$

We end the section with the saddle tower limit of a known TPMS family, which is however not horizontally rigid.

**Example 5.9.** On the left of Figure 8 is a configuration that generalizes the Scherk limit of Schoen's  $H'$ - $T$  family [Sch70] (with  $T_2/T_1 = \exp(i\pi/3)$  and  $a = b = 0$ ; see Figure 1(f)). The graph is obviously balanced but, unfortunately, not rigid. To see this, notice that the graph is represented as the union of three lines. Any of the lines can be moved parallelly yet the graph remains balanced. As a consequence, our construction does not work directly on this graph.

But we may impose an inversion symmetry in the vertices. Under this imposed symmetry, the phase function must have the form as shown in the figure. The configurations form a 4-parameter family. They are trivially balanced and rigid modulo symmetries. Our construction works with little modification, and gives rise to a 5-parameter family of TPMSs of genus 4 generalizing the  $H'$ - $T$  surfaces. We do not plan to write down the details.  $\square$

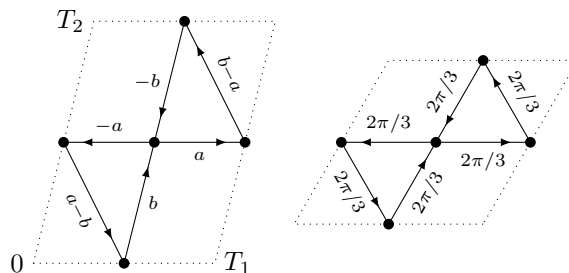


FIGURE 8. Left: Configurations generalizing the Scherk limit of Schoen's H'-T family. Right: Scherk limit of the QTZ-QZD family, a chiral generalization of H'-T.

*Remark 5.10.* Recently, a chiral variation of H'-T was discovered and named QTZ-QZD [MSSTM18]. It arises from the same graph as H'-T, but has a non-trivial phase function; see the right side of Figure 8.

## 6. CONSTRUCTION

In this section, we prove Theorem 4.16. The given phase function is denoted  $\phi^\circ$ . All parameters will vary in a neighborhood of a central value denoted with a superscript  $\circ$ , which depends on the given configuration  $(\mathbf{G}, \phi^\circ)$ . The Implicit Function Theorem will be applied at  $\varepsilon = 0$  and the central value of all parameters.

Without loss of generality, we assume as in Section 2.5 that  $\cos(\theta_h^\circ) \neq 0$  for all  $h \in \mathbf{H}$  and that the zeros of  $\Phi_1$  are simple for all saddle towers involved in the construction. If this is not the case, we could always apply a horizontal rotation to the configuration.

**6.1. Opening nodes.** For each vertex  $v \in \mathbf{V}$ , we want to place a saddle tower  $\mathcal{S}_v$  determined by the angles  $\theta_h^\circ$ ,  $h \in v$ , and glue them along the wings.

Recall that a saddle tower is conformally a sphere with punctures corresponding to the wings. Hence our initial surface at  $\varepsilon = 0$  is a singular Riemann surface consisting of  $|\mathbf{V}|$  spheres identified at their punctures. More specifically: to each vertex  $v \in \mathbf{V}$  of the oriented graph  $\mathbf{G}$ , we associate a Riemann sphere  $\hat{\mathbb{C}}_v$ . To each half-edge  $h \in v$ , we associate a complex number  $p_h^\circ \in \hat{\mathbb{C}}_v$ , so that  $\hat{\mathbb{C}}_v$  punctured at  $p_h^\circ$ ,  $h \in v$ , provides a conformal model for  $\mathcal{S}_v$ . Then we identify  $p_h^\circ$  and  $p_{-h}^\circ$ . The resulting singular Riemann surface with nodes is denoted  $\Sigma_0$ .

As  $\varepsilon$  increases, we want to desingularize the nodes into necks. This is done as follows. For each  $h \in \mathbf{H}$ ,  $p_h$  is a complex parameter in a neighborhood of  $p_h^\circ$ . Consider a local coordinate  $w_h$  in a neighborhood of  $p_h \in \hat{\mathbb{C}}_{v(h)}$  such that  $w_h(p_h) = 0$ . The local coordinate  $w_h$  depends holomorphically on  $p_h$ , but the dependence is omitted for simplicity. Moreover, we assume that the local coordinate  $w_h^\circ$  associated to  $p_h^\circ$  is adapted. We denote  $p = (p_h)_{h \in \mathbf{H}}$ .

Since the graph is finite, it is possible to fix a small number  $\delta > 0$  independent of  $v$  such that the disks  $|w_h^\circ| < 2\delta$  for  $h \in v$  are disjoint in each Riemann sphere  $\hat{\mathbb{C}}_v$ . Then for  $p$  close enough to  $p^\circ$ , the disks  $|w_h| < \delta$  for  $h \in v$  are disjoint.

Consider a symmetric complex parameter  $t = (t_h)_{h \in \mathbf{H}}$  in the neighborhood of 0 with  $|t_h| < \delta^2$ . For every  $h \in \mathbf{H}$ , if  $t_h \neq 0$ , we remove the disk

$$|w_h| < |t_h|/\delta,$$

and identify the annuli

$$|t_h|/\delta \leq |w_h| \leq \delta \quad \text{and} \quad |t_{-h}|/\delta \leq |w_{-h}| \leq \delta$$

by

$$w_h w_{-h} = t_h.$$

When  $t_h = 0$ ,  $p_h$  and  $p_{-h}$  are simply identified to form a node. This produces a Riemann surface, possibly with nodes, denoted by  $\Sigma_t$ . Note that  $\Sigma_t$  also depends on the parameter  $p$ , but the dependence is not written for simplicity. When  $t_h \neq 0$  for all  $h \in \mathbf{H}$ ,  $\Sigma_t$  is a regular Riemann surface of genus  $g = |\mathbf{F}| + 1$  which provides the conformal model for our construction.

We consider the following fixed domains in all  $\Sigma_t$ :

$$U_{v,\delta} = \{z \in \hat{\mathbb{C}}_v : \forall h \in v, |w_h^o| > \delta/2\} \quad \text{and} \quad U_\delta = \bigsqcup_{v \in \mathbf{V}} U_{v,\delta}$$

For  $v \in \mathbf{V}$ , we denote  $\infty_v$  the point at infinity in the Riemann sphere  $\hat{\mathbb{C}}_v$  and fix an origin  $O_v \in U_{v,\delta} \setminus \{\infty_v\}$  as the starting point of the integration defining the Weierstrass parameterization of the saddle tower  $\mathcal{S}_v$ .

**6.2. Regular 1-forms.** Let  $\Sigma_t$  be a family of Riemann surfaces defined by opening nodes as above. A *regular 1-form*  $\omega$  on  $\Sigma_t$  is a differential 1-form that is holomorphic away from the nodes and, whenever two points  $p$  and  $q$  are identified to form a node, has simple poles of opposite residues at  $p$  and  $q$ . Regular 1-forms extend the notion of holomorphic 1-forms to noded Riemann surfaces. By [Mas76, Proposition 4.1], there is a basis  $\omega_{1,t}, \dots, \omega_{g,t}$  for the space of regular 1-forms on  $\Sigma_t$  which “depends holomorphically” on  $t$  in a neighborhood of 0. More formally, in our case, this means that the restriction of  $\omega_{j,t}$  to  $U_\delta$  depends holomorphically on  $(z, t)$ .

One can also consider regular 1-forms with simple poles away from the nodes: by [Mas76, Proposition 4.2], if  $p, q$  are two points on  $\Sigma_0$  minus the nodes, there exists a unique 1-form  $\omega_{p,q,t}$  on  $\Sigma_t$  which has simple poles at  $p$  and  $q$  with residues 1 and  $-1$ , is otherwise regular in the sense above, and has suitably normalized periods (a normalized differential of the third kind). Moreover,  $\omega_{p,q,t}$  depends holomorphically on  $t$  in a neighborhood of 0.

More specifically, we shall use the following result. For each half-edge  $h \in \mathbf{H}$ , let  $A_h$  denote a small anticlockwise circle in  $U_{v(h),\delta}$  around  $p_h$ ; it is then homologous in  $\Sigma_t$  to a clockwise circle in  $U_{v(-h),\delta}$  around  $p_{-h}$ .

**Proposition 6.1.** *Given an antisymmetric function  $(\alpha_h)_{h \in \mathbf{H}}$ , there exists a unique regular 1-form  $\omega_t$  on  $\Sigma_t$ , possibly with simple poles at  $\infty_v$ ,  $v \in \mathbf{V}$ , such that*

$$\forall h \in \mathbf{H}, \quad \int_{A_h} \omega_t = 2\pi i \alpha_h.$$

*Moreover, the restriction of  $\omega_t$  to  $U_\delta$  depends holomorphically on  $t$  in a neighborhood of 0.*

*Proof.* this follows from [Mas76, Proposition 4.2]. See also [Tra13, Theorem 8.2] for a constructive proof.  $\square$

Note that by the Residue Theorem in  $\hat{\mathbb{C}}_v$ ,

$$\text{Res}(\omega_t, \infty_v) = - \sum_{h \in v} \alpha_h.$$

We will also need the following results.

**Lemma 6.2** ([Tra08, Lemma 3]). *The derivative  $\partial\omega_t/\partial t_h$  at  $t = 0$ , restricted to  $U_\delta$ , coincides with a meromorphic 1-form on  $\Sigma_0$  with double poles at the nodes, holomorphic elsewhere, and vanishing  $A$ -periods. In term of the local complex coordinates  $w_h$  used to open nodes, the principal part at  $p_h$  is*

$$-\frac{dw_h}{w_h^2} \text{Res}\left(\frac{\omega_0}{w_{-h}}, p_{-h}\right).$$



For every half-edge  $h$  and  $t_h \neq 0$ , let  $B_h$  be the concatenation of

- (1) a path in  $U_{v(h),\delta}$  from  $O_{v(h)}$  to  $w_h = \delta$ ,
- (2) the path parameterized by  $w_h = \delta^{1-2s} t_h^s$  for  $s \in [0, 1]$ , from  $w_h = \delta$  to  $w_h = t_h/\delta$ , which is identified with  $w_{-h} = \delta$ , and
- (3) a path in  $U_{v(-h),\delta}$  from  $w_{-h} = \delta$  to  $O_{v(-h)}$ .

**Lemma 6.3** ([Tra02b, Lemma 1]). *The difference*

$$(9) \quad \left( \int_{B_h} \omega_t \right) - \alpha_h \log t_h$$

extends holomorphically to  $t_h = 0$ . Moreover, its value at  $t = 0$  is equal to

$$\lim_{z \rightarrow p_h} \left[ \left( \int_{O_{v(h)}}^z \omega_0 \right) - \alpha_h \log w_h(z) \right] - \lim_{z \rightarrow p_{-h}} \left[ \left( \int_{O_{v(-h)}}^z \omega_0 \right) - \alpha_{-h} \log w_{-h}(z) \right].$$

Lemma 6.3 was essentially proved in [Tra02b, Lemma 1]. As the lemma has been and will be used in similar constructions, we consider it a good time to refurbish the proof in Appendix D.

**6.3. Weierstrass data.** We construct a conformal minimal immersion using the Weierstrass parameterization in the form

$$z \mapsto \operatorname{Re} \int^z (\Phi_1, \Phi_2, \Phi_3),$$

where  $\Phi_i$  are meromorphic 1-forms on  $\Sigma_t$  satisfying the conformality equation

$$(10) \quad Q := \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0.$$

Observe that  $Q$  is a meromorphic quadratic differential on  $\Sigma_t$ .

**6.3.1. A-periods.** We need to solve the following A-period problem

$$\operatorname{Re} \int_{A_h} (\Phi_1, \Phi_2, \Phi_3) = (0, 0, 2\pi\sigma_h), \quad \forall h \in \mathbb{H}.$$

We define  $\Phi_1$ ,  $\Phi_2$ , and  $\Phi_3$ , using Proposition 6.1, as the unique regular 1-forms on  $\Sigma_t$  with (at most) simple poles at  $\infty_v$  for  $v \in \mathbb{V}$  and the A-periods

$$\int_{A_h} (\Phi_1, \Phi_2, \Phi_3) = 2\pi i(\alpha_h, \beta_h, \gamma_h - i\sigma_h), \quad \forall h \in \mathbb{H},$$

where  $(\alpha, \beta, \gamma) \in \mathcal{A}^3$  are antisymmetric parameters. This way, the A-period problems are solved by definition. We choose the following central value for the parameters:

$$\alpha_h^\circ = -\cos(\theta_h^\circ), \quad \beta_h^\circ = -\sin(\theta_h^\circ), \quad \gamma_h^\circ = 0.$$

Then at  $\varepsilon = 0$  and the central value of all parameters, we have in  $\hat{\mathbb{C}}_v$

$$\Phi_1^\circ = \sum_{h \in v} \frac{-\cos(\theta_h^\circ)}{z - p_h^\circ} dz, \quad \Phi_2^\circ = \sum_{h \in v} \frac{-\sin(\theta_h^\circ)}{z - p_h^\circ} dz \quad \text{and} \quad \Phi_3^\circ = \sum_{h \in v} \frac{-i\sigma_h}{z - p_h^\circ} dz.$$

In other words,  $(\Phi_1^\circ, \Phi_2^\circ, \Phi_3^\circ)$  is precisely the Weierstrass data of the saddle tower  $\mathcal{S}_v$  as we want.

Note that  $\sigma \in \ker(\operatorname{div})$ , that is

$$\sum_{h \in v} \sigma_h = 0,$$

so we have by Residue Theorem in  $\hat{\mathbb{C}}_v$

$$\operatorname{Res}(\Phi, \infty_v) = - \sum_{h \in v} (\alpha_h, \beta_h, \gamma_h) = - \operatorname{div}_{b(v)}(\alpha, \beta, \gamma).$$

We want  $\infty_v$  to be regular points, so we need to solve

$$(11) \quad \operatorname{div}_{b(v)}(\alpha, \beta, \gamma) = \sum_{h \in v} (\alpha_h, \beta_h, \gamma_h) = 0, \quad \text{for all } v \in \mathbf{V}.$$

If the graph is balanced, then the central values solve (11) at  $\varepsilon = 0$ . We call (11) the *balance equations*.

*Remark 6.4.* If we see the surface as a soap film, then the saddle tower at  $v(h)$  is pulled by a surface tension force along the wing of  $h$ , which can be calculated (up to a physical coefficient) as

$$-\operatorname{Im} \int_{A_h} (\Phi_1, \Phi_2, \Phi_3) = -2\pi(\alpha_h, \beta_h, \gamma_h).$$

**6.3.2. B-periods.** For any cycle  $c = (h_1, \dots, h_n)$  of the graph, let  $B_c$  denote the concatenation  $B_{h_1} * \dots * B_{h_n}$  which is a cycle in  $\Sigma_t$ . Recall from Section 3.3.3 the cycle basis  $\mathbf{C}^* = \mathbf{F}^* \cup \{c_1, c_2\}$ . We need to solve the following B-period problem:

$$(12) \quad \varepsilon^2 \left( \operatorname{Re} \int_{B_c} \Phi_1 + i \operatorname{Re} \int_{B_c} \Phi_2 \right) = \begin{cases} 0 & c \in \mathbf{F}^*, \\ T_i + \varepsilon^2 \Lambda_i & c = c_i, i = 1, 2; \end{cases}$$

$$(13) \quad \operatorname{Re} \int_{B_c} \Phi_3 = \begin{cases} 0 \pmod{2\pi} & c \in \mathbf{F}^*, \\ \Psi_i \pmod{2\pi} & c = c_i, i = 1, 2. \end{cases}$$

**6.3.3. Conformality.** At  $\varepsilon = 0$  and the central value of all parameters,  $\Phi_1^\circ$  has  $\deg(v)$  simple poles in  $\hat{\mathbb{C}}_v$ , hence  $\deg(v) - 2$  zeros denoted  $\zeta_{v,j}^\circ$  for  $1 \leq j \leq \deg(v) - 2$ . Recall that these zeros are simple (see the beginning of Section 6). We can also assume that they are not  $\infty_v$ . When the parameters are close to their central value,  $\Phi_1$  has a simple zero  $\zeta_{v,j}$  close to  $\zeta_{v,j}^\circ$  in  $\hat{\mathbb{C}}_v$  for  $1 \leq j \leq \deg(v) - 2$ .

*Remark 6.5.* If the balancing equations are not solved,  $\Phi_1$  may have a pole at  $\infty_v$  and consequently, an extra zero near  $\infty_v$ . We may ignore this zero, as it disappears when the balancing equations are solved. See Proposition 6.6 below.

We will solve the following equations:

$$(14) \quad \int_{A_h} \frac{Q}{\Phi_1} = 0, \quad h \in \mathbf{H},$$

$$(15) \quad \operatorname{Res} \left( \frac{Q}{\Phi_1}, \zeta_{v,j} \right) = 0, \quad 1 \leq j \leq \deg(v) - 3, v \in \mathbf{V}.$$

**Proposition 6.6.** *The conformality equation (10) is solved if the equations (11), (14), and (15) are solved.*

*Proof.* If Equation (11) is solved, then  $Q$  and  $\Phi_1$  are holomorphic at  $\infty_v$  and  $\Phi_1$  has no extra zero in  $\hat{\mathbb{C}}_v$ . By the Residue Theorem in  $\hat{\mathbb{C}}_v$

$$\sum_{h \in v} \int_{A_h} \frac{Q}{\Phi_1} + 2\pi i \sum_{j=1}^{\deg(v)-2} \operatorname{Res} \left( \frac{Q}{\Phi_1}, \zeta_{v,j} \right) = 0.$$

Hence if Equations (14) and (15) are solved, the residue of  $Q/\Phi_1$  at the last zero  $\zeta_{v, \deg(v)-2}$  must also vanish. So the 1-form  $Q/\Phi_1$ , being holomorphic on  $\Sigma$  with vanishing A-periods, must be 0.  $\square$

6.3.4. *Dimension count.* Let us perform a dimension count before proceeding further.

We have  $3|E|$  real parameters  $(\alpha, \beta, \gamma)$ . The complex parameters  $t$  comprises  $|E|$  complex numbers. For each  $v \in V$ , Möbius transforms on  $\hat{C}_v$  do not change the conformal structure of  $\Sigma_t$ . So we can fix the positions of three of the punctures  $(p_h)_{h \in v}$ , leaving  $\deg(v) - 3$  free complex parameters for each  $v \in V$ . So, together with the parameter  $\varepsilon$  of the family, we have  $9|E| - 6|V| + 1$  real parameters.

Let us now count the equations. The B-period problems is  $3g = 3(|F| + 1)$  real equations. The conformality equations contain  $|E| + 2|E| - 3|V| = 3|F|$  complex equations. (This is of course the dimension of the space of holomorphic quadratic differentials on  $\Sigma_t$ .) Moreover, we have  $3|V| - 3$  balancing equations (11). Hence there are  $9|F| + 3|V| = 9|E| - 6|V|$  real equations. This is one less than the number of parameters, so 1-parameter families are expected out of our construction.

#### 6.4. Solving the conformality problem.

**Proposition 6.7.** *For  $(t, \alpha, \beta, \gamma)$  in a neighborhood of  $(0, \alpha^\circ, \beta^\circ, 0)$ , there exists  $p = (p_h)_{h \in H}$  depending analytically on  $(t, \alpha, \beta, \gamma)$ , such that the conformality equations (15) are solved. Moreover,  $p_h(0, \alpha^\circ, \beta^\circ, 0) = p_h^\circ$ .*

*Proof.* As in Section 2.5, for each  $v \in V$ , we may fix the position of three points  $p_h$  in  $\hat{C}_v$  using a Möbius transformation. At  $(t, \alpha, \beta, \gamma) = (0, \alpha^\circ, \beta^\circ, 0)$ ,  $(\Phi_1, \Phi_2, \Phi_3)$  restricted to  $\hat{C}_v$  is the Weierstrass data considered in Section 2.5, depending on the parameters  $(p_h)_{h \in v}$ . By Theorem 2.8, the partial differential of

$$\left( \text{Res} \left( \frac{Q}{\Phi_1}, \zeta_{v,j} \right) \right)_{1 \leq j \leq \deg(v) - 3}$$

with respect to  $(p_h)_{h \in v}$  is an isomorphism. So Proposition 6.7 follows from the Implicit Function Theorem.  $\square$

From now on, we assume that  $p_h$  are given by Proposition 6.7. We make the change of parameters

$$\alpha_h + i\beta_h = -\rho_h \exp(i\theta_h).$$

Clearly,  $\rho = (\rho_h)_{h \in H} \in \mathcal{S}$ , and  $(\exp(i\theta_h))_{h \in H} \in \mathcal{A}^2$ . The central values of  $\rho_h$  is  $\rho_h^\circ = 1$  and the central value of  $\theta_h$  is  $\theta_h^\circ$  with  $u_h^\circ = e^{i\theta_h^\circ}$ .

**Proposition 6.8.** *For  $(t, \theta)$  in a neighborhood of  $(0, \theta^\circ)$ , there exist unique values of  $\rho$  and  $\gamma$ , depending real-analytically on  $(t, \theta)$ , such that the equations (14) are solved. At  $t_h = 0$  we have, no matter the values of  $\theta$ ,  $(t_k)_{k \neq \pm h}$ , and other parameters, that*

$$(16) \quad \rho_h = 1 \quad \text{and} \quad \gamma_h = 0.$$

*In particular, we have  $|\gamma_h| \leq C|t_h|$  for a uniform constant  $C$ . Moreover, at  $(t, \theta) = (0, \theta^\circ)$ , we have the Wirtinger derivatives*

$$(17) \quad \frac{\partial \rho_h}{\partial t_h} = -\frac{1}{2} \Upsilon_h \Upsilon_{-h} \quad \text{and} \quad \frac{\partial \gamma_h}{\partial t_h} = -\frac{i}{2} \sigma_h \Upsilon_h \Upsilon_{-h}.$$

*Proof.* Define for  $h \in H$

$$\mathcal{E}_h(t, \rho, \gamma, \theta) = \frac{1}{2\pi i} \int_{A_h} \frac{Q}{\Phi_1}.$$

Assume that  $t_h = 0$ . Then  $\Phi_1$ ,  $\Phi_2$ , and  $\Phi_3$  have a simple pole at  $p_h$ , so  $Q/\Phi_1$  has a simple pole at  $p_h$  and, by the Residue Theorem

$$\mathcal{E}_h |_{t_h=0} = \frac{\alpha_h^2 + \beta_h^2 + (\gamma_h - i\sigma_h)^2}{\alpha_h} = \frac{\rho_h^2 + \gamma_h^2 - 1 - 2i\sigma_h\gamma_h}{\alpha_h}.$$

So the solution of (14) at  $t_h = 0$  is  $(\rho_h, \gamma_h) = (1, 0)$ , no matter the values of the other parameters. This proves (16). We compute the partial derivatives of  $\mathcal{E}_h$  with respect to  $\rho_h$  and  $\gamma_h$  at  $(t, \rho, \gamma, \theta) = (0, 1, 0, \theta^\circ)$ :

$$(18) \quad \frac{\partial \mathcal{E}_h}{\partial \rho_h} = \frac{2}{\alpha_h^\circ}, \quad \frac{\partial \mathcal{E}_h}{\partial \gamma_h} = \frac{-2i\sigma_h}{\alpha_h^\circ}.$$

So the existence and uniqueness statement of the proposition follows from the Implicit Function Theorem. To prove the last point, we need to compute the partial derivative of  $\mathcal{E}_h$  with respect to  $t_h$  at  $(t, \rho, \gamma, \theta) = (0, 1, 0, \theta^\circ)$ . We use the following elementary results: if  $f = a_{-1}z^{-1} + a_0 + O(z)$  and  $g = b_{-1}z^{-1} + b_0 + O(z)$  are meromorphic functions with a simple pole at  $z = 0$ , then

$$\operatorname{Res}\left(\frac{f}{z^2g}, 0\right) = \frac{a_0b_{-1} - a_{-1}b_0}{(b_{-1})^2}, \quad \operatorname{Res}(f^2, 0) = 2a_{-1}a_0, \quad \text{and} \quad \operatorname{Res}(f^2z, 0) = (a_{-1})^2.$$

We have

$$(19) \quad \begin{aligned} \frac{\partial \mathcal{E}_h}{\partial t_h} &= \operatorname{Res}\left(\sum_{j=1}^3 2 \frac{\Phi_j^\circ}{\Phi_1^\circ} \frac{\partial \Phi_j}{\partial t_h}, p_h^\circ\right) \quad \text{by the Residue Theorem} \\ &= -2 \sum_{j=1}^3 \operatorname{Res}\left(\frac{\Phi_j^\circ}{\Phi_1^\circ} \frac{dw_h}{w_h^2}, p_h^\circ\right) \operatorname{Res}\left(\frac{\Phi_j^\circ}{w_{-h}}, p_{-h}^\circ\right) \quad \text{by Lemma 6.2} \\ &= -\frac{2}{\alpha_h^\circ} \sum_{j=1}^3 \operatorname{Res}\left(\frac{\Phi_j^\circ}{w_h}, p_h^\circ\right) \operatorname{Res}\left(\frac{\Phi_j^\circ}{w_{-h}}, p_{-h}^\circ\right) \end{aligned}$$

$$(20) \quad + \frac{2}{(\alpha_h^\circ)^2} \operatorname{Res}\left(\frac{\Phi_1^\circ}{w_h}, p_h^\circ\right) \sum_{j=1}^3 \operatorname{Res}\left(\Phi_j^\circ, p_h^\circ\right) \operatorname{Res}\left(\frac{\Phi_j^\circ}{w_{-h}}, p_{-h}^\circ\right).$$

Since  $Q^\circ = 0$ , we have

$$(21) \quad \operatorname{Res}\left(\frac{w_h Q^\circ}{dw_h}, p_h^\circ\right) = \sum_{j=1}^3 \operatorname{Res}\left(\Phi_j^\circ, p_h^\circ\right)^2 = 0,$$

$$(22) \quad \operatorname{Res}\left(\frac{Q^\circ}{dw_h}, p_h^\circ\right) = 2 \sum_{j=1}^3 \operatorname{Res}\left(\Phi_j^\circ, p_h^\circ\right) \operatorname{Res}\left(\frac{\Phi_j^\circ}{w_h}, p_h^\circ\right) = 0.$$

Because  $\operatorname{Res}\left(\Phi_j^\circ, p_{-h}^\circ\right) = -\operatorname{Res}\left(\Phi_j^\circ, p_h^\circ\right)$ , Eq. (22) implies that the term (20) vanishes. Note that

$$\frac{1}{\sqrt{2}} \operatorname{Res}\left(\Phi^\circ, p_h^\circ\right), \quad \frac{1}{\sqrt{2}} \operatorname{Res}\left(\overline{\Phi^\circ}, p_h^\circ\right), \quad N^\circ(p_h^\circ)$$

form an orthonormal basis of  $\mathbb{C}^3$  for the standard hermitian product  $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ . After recalling the definition of  $\Upsilon_h$  in Section 2.3, we decompose in this basis

$$\operatorname{Res}\left(\frac{\Phi^\circ}{w_h}, p_h^\circ\right) = \frac{1}{2} \left\langle \operatorname{Res}\left(\Phi^\circ, p_h^\circ\right), \operatorname{Res}\left(\frac{\Phi^\circ}{w_h}, p_h^\circ\right) \right\rangle_{\mathbb{H}} \operatorname{Res}\left(\Phi^\circ, p_h^\circ\right) + \sigma_h \Upsilon_h N^\circ(p_h^\circ).$$

Here, the component on  $\operatorname{Res}\left(\overline{\Phi^\circ}, p_h^\circ\right)$  vanishes because of (22). In the same way, after recalling that  $N^\circ(p_{-h}^\circ) = N^\circ(p_h^\circ)$  and  $\sigma_{-h} = -\sigma_h$ ,

$$\operatorname{Res}\left(\frac{\Phi^\circ}{w_{-h}}, p_{-h}^\circ\right) = \frac{1}{2} \left\langle \operatorname{Res}\left(\Phi^\circ, p_h^\circ\right), \operatorname{Res}\left(\frac{\Phi^\circ}{w_{-h}}, p_{-h}^\circ\right) \right\rangle_{\mathbb{H}} \operatorname{Res}\left(\Phi^\circ, p_h^\circ\right) - \sigma_h \Upsilon_{-h} N^\circ(p_h^\circ).$$

Hence by Equation (19)

$$\frac{\partial \mathcal{E}_h}{\partial t_h} = \frac{-2}{\alpha_h^\circ} \left\langle \overline{\operatorname{Res}\left(\frac{\Phi^\circ}{w_h}, p_h^\circ\right)}, \operatorname{Res}\left(\frac{\Phi^\circ}{w_{-h}}, p_{-h}^\circ\right) \right\rangle_{\mathbb{H}} = \frac{2}{\alpha_h^\circ} \Upsilon_h \Upsilon_{-h}.$$

If  $k \neq \pm h$ , we have  $\partial \mathcal{E}_h / \partial t_k = 0$  as  $\partial \Phi_j / \partial t_k$  is holomorphic at  $p_h^\circ$ .

We differentiate  $\mathcal{E}_h(t, \rho(t, \theta), \gamma(t, \theta), \theta) = 0$  with respect to  $t_h$  and obtain, using Equation (18),

$$(23) \quad \frac{2}{\alpha_h^\circ} \left( \Upsilon_h \Upsilon_{-h} + \frac{\partial \rho_h}{\partial t_h} - i \sigma_h \frac{\partial \gamma_h}{\partial t_h} \right) = 0.$$

Since  $\mathcal{E}_h$  is holomorphic in  $t_h$ , we obtain by differentiation with respect to  $\bar{t}_h$  and conjugation

$$(24) \quad \frac{2}{\alpha_h^\circ} \left( \frac{\partial \rho_h}{\partial t_h} + i \sigma_h \frac{\partial \gamma_h}{\partial t_h} \right) = 0.$$

Solving the system (23), (24) gives Equation (17).  $\square$

From now on, we assume that  $\rho$  and  $\gamma$  are given by Proposition 6.8.

**6.5. Solving horizontal balance and period problems.** We make the change of parameters

$$(25) \quad t_h = -\exp \left( -\ell_h \varepsilon^{-2} + i \sigma_h \phi_h \right),$$

where  $\ell = (\ell_h)_{h \in \mathbb{H}} \in \mathcal{S}$ , and  $\phi = (\phi_h)_{h \in \mathbb{H}} \in \mathcal{A}$ . The central value of  $\ell_h$  is  $\ell_h^\circ$ , the length of the edge  $e(h)$ . The central value of  $\phi_h$  is the given phase function  $\phi_h^\circ$ . We combine  $\ell$  and  $\theta$  into

$$x_h = \ell_h e^{i\theta_h},$$

whose central value is  $x_h^\circ$  as given by the graph.

*Remark 6.9.* To understand the negative sign on the right-hand side of (25), recall from Section 2 that, if  $\sigma_h = -1$  (resp.  $+1$ ), then the positive (resp. negative) real line in the neighborhood of  $p_k$  is mapped onto the 0-reflection plane. Therefore,  $t_h$  must be real negative when two saddle towers are in phase ( $\phi_h = 0$ ).

**Proposition 6.10.** *Assume that the graph  $\mathbb{G}$  is rigid and balanced. For  $(\varepsilon, \phi)$  in a neighborhood of  $(0, \phi^\circ)$ , there exist unique values for  $(x_h)_{h \in \mathbb{H}}$ , depending smoothly on  $(\varepsilon, \phi)$ , such that  $x(0, \phi) = x^\circ$  for all  $\phi$  and the horizontal  $B$ -period equations (12) as well as the  $\alpha$  and  $\beta$  components of the balance equations (11) are solved. Moreover,  $x$  is an even function of  $\varepsilon$  and at  $\varepsilon = 0$ , we have for all  $\phi$*

$$\frac{\partial^2 x_h}{\partial \varepsilon^2}(0, \phi) = 2\xi_h$$

where  $\xi_h$  is the solution to (7).

*Proof.* Define for  $(\varepsilon, x, \phi)$  in a neighborhood of  $(0, x^\circ, \phi^\circ)$  and  $h \in \mathbb{H}$

$$\mathcal{P}_h^{\text{hor}}(\varepsilon, x, \phi) = \varepsilon^2 \left( \operatorname{Re} \int_{B_h} \Phi_1 + i \operatorname{Re} \int_{B_h} \Phi_2 \right).$$

By Lemma 6.3

$$\lambda_h^{\text{hor}}(\varepsilon, x, \phi) := \left[ \left( \operatorname{Re} \int_{B_h} \Phi_1 \right) - \alpha_h \log |t_h| \right] + i \left[ \left( \operatorname{Re} \int_{B_h} \Phi_2 \right) - \beta_h \log |t_h| \right]$$

extends at  $\varepsilon = 0$  to a smooth function of remaining parameters  $(\varepsilon, x, \phi)$ .

Recall the definition of  $\mu_h$  in Section 2.3 and that  $z_0 = O_v$  is chosen as the origin of the parameterization of the saddle tower  $\mathcal{S}_v$ . By the last statement of Lemma 6.3, we have at the central value

$$\lambda_h^{\text{hor}}(0, x^\circ, \phi) = \mu_h - \mu_{-h}.$$

We then have

$$\begin{aligned}\mathcal{P}_h^{\text{hor}}(\varepsilon, x, \phi) &= \varepsilon^2 [(\alpha_h + i\beta_h) \log |t_h| + \lambda_h^{\text{hor}}(\varepsilon, x, \phi)] \\ &= -\varepsilon^2 \rho_h e^{i\theta_h} (-\ell_h \varepsilon^{-2}) + \varepsilon^2 \lambda_h^{\text{hor}}(\varepsilon, x, \phi) \\ &= \rho_h x_h + \varepsilon^2 \lambda_h^{\text{hor}}(\varepsilon, x, \phi).\end{aligned}$$

Note that by definition,  $\mathcal{P}_h^{\text{hor}}$  is an even function of  $\varepsilon$ . Since  $\rho_h$  is an analytic function of  $t_h$  with value 1 at  $t_h = 0$  and all derivatives of  $t_h$  with respect to  $\varepsilon$  vanish at  $\varepsilon = 0$ , we have at  $\varepsilon = 0$

$$(26) \quad \mathcal{P}_h^{\text{hor}}(0, x, \phi) = x_h \quad \text{and} \quad \frac{\partial^2 \mathcal{P}_h^{\text{hor}}}{\partial \varepsilon^2}(0, x^\circ, \phi) = 2(\mu_h - \mu_{-h}).$$

Now define for  $c$  in the cycle basis  $\mathbf{C}^*$

$$\mathcal{P}_c^{\text{hor}}(\varepsilon, x, \phi) = \varepsilon^2 \left( \operatorname{Re} \int_{B_c} \Phi_1 + i \operatorname{Re} \int_{B_c} \Phi_2 \right) = \sum_{h \in c} \mathcal{P}_h^{\text{hor}}(\varepsilon, x, \phi).$$

Then, at  $\varepsilon = 0$ , we have

$$\begin{aligned}\mathcal{P}_c^{\text{hor}}(0, x, \phi) &= \sum_{h \in c} x_h = P_c^{\text{hor}}(x) \quad \text{and} \\ \frac{\partial^2 \mathcal{P}_c^{\text{hor}}}{\partial \varepsilon^2}(0, x^\circ, \phi) &= 2 \sum_{h \in c} (\mu_h - \mu_{-h}) = 2P_c^{\text{hor}}(\mu^a).\end{aligned}$$

Regarding the horizontal balance equation (11), we define for  $b$  in the cut-basis  $\mathbf{B}^*$

$$\mathcal{F}_b^{\text{hor}} = -\operatorname{div}_b(\alpha + i\beta) = \operatorname{div}_b(\rho_h e^{i\theta_h}) = \operatorname{div}_b(\rho_h u_h(x)).$$

Again,  $\mathcal{F}_b^{\text{hor}}$  is an even function of  $\varepsilon$  and at  $\varepsilon = 0$ , we have since  $\rho = 1$

$$\mathcal{F}_b^{\text{hor}}(0, x, \phi) = F_b^{\text{hor}}(x) \quad \text{and} \quad \frac{\partial^2 \mathcal{F}_b^{\text{hor}}}{\partial \varepsilon^2}(0, x^\circ, \phi) = 0.$$

We want to solve the system

$$\begin{cases} \mathcal{P}_c^{\text{hor}}(\varepsilon, x, \phi) = 0, & c \in \mathbf{F}^*, \\ \mathcal{P}_{c_i}^{\text{hor}}(\varepsilon, x, \phi) = T_i + \varepsilon^2 \Lambda_i, & i = 1, 2, \\ \mathcal{F}_b^{\text{hor}}(\varepsilon, x, \phi) = 0, & b \in \mathbf{B}^*. \end{cases}$$

If  $\mathbf{G}$  is balanced, the system is solved at  $\varepsilon = 0$  with  $x(0, \phi) = x^\circ$  for all  $\phi$ . If  $\mathbf{G}$  is rigid, then by Implicit Function Theorem, for  $\varepsilon$  in a neighborhood of 0, there exists a unique solution  $x(\varepsilon, \phi)$  depending smoothly on  $(\varepsilon, \phi)$  such that  $x(0, \phi) = x^\circ$ .

Moreover, since the system is even in  $\varepsilon$ , so is the unique solution  $x(\varepsilon, \phi)$ . Taking the second derivative of the system with respect to  $\varepsilon$  at  $\varepsilon = 0$  gives

$$\begin{cases} 2P_c^{\text{hor}}(\mu^a) + P_c^{\text{hor}}\left(\frac{\partial^2 x}{\partial \varepsilon^2}(0, \phi)\right) = 0, & c \in \mathbf{F}^*, \\ 2P_{c_i}^{\text{hor}}(\mu^a) + P_{c_i}^{\text{hor}}\left(\frac{\partial^2 x}{\partial \varepsilon^2}(0, \phi)\right) = 2\Lambda_i, & i = 1, 2, \\ DF_b^{\text{hor}}(x^\circ) \cdot \frac{\partial^2 x}{\partial \varepsilon^2}(0, \phi) = 0, & b \in \mathbf{B}^*. \end{cases}$$

Its unique solution is, by definition,  $\frac{\partial^2 x}{\partial \varepsilon^2}(0, \phi) = 2\xi$ .  $\square$

From now on, we assume that the parameters  $x_h$  are given by Proposition 6.10.

### 6.6. Solving vertical balance and period problems.

**Proposition 6.11.** *Assume that the phase function  $\phi^\circ$  is balanced and rigid. For  $\varepsilon$  in a neighborhood of 0, there exist unique values for  $(\phi_h)_{h \in \mathbb{H}}$ , depending smoothly on  $\varepsilon$ , such that  $\phi_h(0) = \phi_h^\circ$  and the vertical  $B$ -period problems (13) as well as the  $\gamma$  component of (11) are solved.*

*Proof.* Similarly to the proof of Proposition 6.10, we define for  $h \in \mathbb{H}$

$$\mathcal{P}_h^{\text{ver}}(\varepsilon, \phi) = \text{Re} \int_{B_h} \Phi_3.$$

By Lemma 6.3,

$$\lambda_h^{\text{ver}}(\varepsilon, \phi) := \text{Re} \left[ \left( \int_{B_h} \Phi_3 \right) - (\gamma_h - i\sigma_h) \log t_h \right]$$

extends smoothly at  $\varepsilon = 0$ . By Proposition 6.8, we have  $\gamma_h = O(t_h)$  so  $\gamma_h \log t_h$  extends smoothly at  $\varepsilon = 0$  with value 0. Hence by Lemma 6.3, we have at  $\varepsilon = 0$

$$\begin{aligned} \lambda_h^{\text{ver}}(0, \phi) &= \text{Re} \lim_{z \rightarrow p_h} \left[ \left( \int_{O_v(h)}^z \Phi_3^\circ \right) + i\sigma_h \log w_h(z) \right] \\ &\quad - \text{Re} \lim_{z \rightarrow p_{-h}} \left[ \left( \int_{O_v(-h)}^z \Phi_3^\circ \right) + i\sigma_{-h} \log w_{-h}(z) \right] \\ &= \nu_h - \nu_{-h} \end{aligned}$$

where  $\nu_h$  is defined as in Section 2.3. We then have

$$\mathcal{P}^{\text{ver}}(\varepsilon, \phi) = \lambda_h^{\text{ver}}(\varepsilon, \phi) - \gamma_h \ell_h \varepsilon^{-2} + \phi_h + \sigma_h \pi.$$

Since  $\gamma_h = O(t_h)$ , we have at  $\varepsilon = 0$

$$(27) \quad \mathcal{P}_h^{\text{ver}}(0, \phi) = \nu_h - \nu_{-h} + \phi_h + \sigma_h \pi.$$

Now define for  $c$  in the cycle-basis  $\mathbb{C}^*$

$$\mathcal{P}_c^{\text{ver}}(\varepsilon, \phi) = \text{Re} \int_{B_c} \Phi_3 = \sum_{h \in c} \mathcal{P}_h^{\text{hor}}(\varepsilon, \phi).$$

Observe that by Proposition 2.5

$$\sum_{h \in c} (\nu_h - \nu_{-h} + \pi) = \sum_{h \in c} (\nu_h - \nu_{\varsigma(h)} + \pi) = 0 \pmod{2\pi}$$

Hence at  $\varepsilon = 0$ , we have

$$\mathcal{P}_c^{\text{ver}}(0, \phi) = P_c^{\text{ver}}(\phi) \pmod{2\pi}.$$

Regarding the vertical balance equation (11), we define for  $b$  in the cut basis  $\mathbb{B}^*$

$$\mathcal{F}_b^{\text{hor}}(\varepsilon, \phi) = -\text{div}_b(\gamma(\varepsilon, \phi)) = -\sum_{h \in b} \gamma_h(\varepsilon, \phi).$$

Using the Mean Value Inequality and Proposition 6.8, we have for  $\varepsilon \rightarrow 0$

$$\gamma_h(\varepsilon, \phi) = \sigma_h \Upsilon_h \Upsilon_{-h} \text{Im}(t_h(\varepsilon, \phi)) + o(t_h(\varepsilon, \phi)).$$

By Proposition 6.10 and that  $\partial x_h / \partial \varepsilon = 0$ , we have

$$\frac{\partial^2 x_h}{\partial \varepsilon^2}(0) = \exp(i\theta_h^\circ) \left( \frac{\partial^2 \ell_h}{\partial \varepsilon^2}(0) + i\ell_h \frac{\partial^2 \theta_h}{\partial \varepsilon^2}(0) \right) = 2\xi_h,$$

so

$$\frac{\partial^2 \ell_h}{\partial \varepsilon^2}(0) = 2 \text{Re}(\xi_h \exp(-i\theta_h^\circ)).$$

Therefore using Taylor formula

$$\begin{aligned} \operatorname{Im}(t_h(\varepsilon, \phi)) &= -\sigma_h \sin(\phi_h) \exp(-\ell_h(\varepsilon, \phi)\varepsilon^{-2}) \\ &= -\sigma_h \sin(\phi_h) \exp(-\ell_h^\circ \varepsilon^{-2} - \operatorname{Re}(\xi_h \exp(-i\theta_h^\circ)) + O(\varepsilon^2)). \end{aligned}$$

Hence we have

$$\mathcal{F}_b^{\operatorname{ver}}(\varepsilon, \phi) = \sum_{h \in b} \Upsilon_h \Upsilon_{-h} \sin(\phi_h) \exp(-\ell_h^\circ \varepsilon^{-2} - \operatorname{Re}(\xi_h \exp(-i\theta_h^\circ)) + O(\varepsilon^2)).$$

Recall that  $\ell_b^\circ = \min\{\ell_h^\circ \mid h \in b\}$  and define for  $\varepsilon \neq 0$

$$\widetilde{\mathcal{F}}_b^{\operatorname{ver}}(\varepsilon, \phi) = \exp(\ell_b^\circ \varepsilon^{-2}) \mathcal{F}_b^{\operatorname{ver}}(\varepsilon, \phi).$$

Then  $\widetilde{\mathcal{F}}^{\operatorname{ver}}$  extends smoothly at  $\varepsilon = 0$  and in the limit  $\varepsilon \rightarrow 0$  only the shortest edges remain:

$$\widetilde{\mathcal{F}}^{\operatorname{ver}}(0, \phi) = \sum_{h \in m(b)} \Upsilon_h \Upsilon_{-h} \sin(\phi_h) \exp(-\operatorname{Re}(\xi_h \exp(-i\theta_h))) = F^{\operatorname{ver}}(\phi).$$

We want to solve the system

$$\begin{cases} \mathcal{P}_c^{\operatorname{ver}}(\varepsilon, \phi) = 0, & c \in \mathbf{F}^*, \\ \mathcal{P}_{c_i}^{\operatorname{ver}}(\varepsilon, \phi) = \Psi_i, & i = 1, 2, \\ \widetilde{\mathcal{F}}_b^{\operatorname{ver}}(\varepsilon, \phi) = 0, & b \in \mathbf{B}^*. \end{cases}$$

If the phase function  $\phi^\circ$  is balanced, the system is solved at  $\varepsilon = 0$  by  $\phi = \phi^\circ$ . If  $\phi^\circ$  is rigid and the cut basis  $\mathbf{B}^*$  is  $\mathbf{B}_m^*$  as given by Proposition 3.8, then by the Implicit Function Theorem, the system has a unique solution  $\phi(\varepsilon)$ , depending smoothly on  $\varepsilon$  for  $\varepsilon$  in a neighborhood of 0, such that  $\phi(0) = \phi^\circ$ .  $\square$

**6.7. Geometry and Embeddedness.** We have constructed a 1-parameter family of Weierstrass data  $(\Phi_{1,\varepsilon}, \Phi_{2,\varepsilon}, \Phi_{3,\varepsilon})$  depending on  $\varepsilon > 0$  that solve the Conformality and Period Problems. All parameters are now smooth functions of  $\varepsilon$  and are denoted accordingly  $t_h(\varepsilon)$ ,  $\theta_h(\varepsilon)$ , etc... We denote

$$f_\varepsilon = (f_\varepsilon^{\operatorname{hor}}, f_\varepsilon^{\operatorname{ver}}) : \Sigma_{t(\varepsilon)} \rightarrow \mathbb{T}_\varepsilon^3$$

the immersion given by Weierstrass Representation formula, decomposed into its horizontal and vertical components. Note that  $\Sigma_{t(\varepsilon)}$  and  $f_\varepsilon$  are independent of the positive number  $\delta$  used to define  $\Sigma_t$  (although the smaller  $\delta$ , the smaller  $\varepsilon$  must be in the definition of  $\Sigma_t$ ). We first prove that the immersion is regular, that is:

**Proposition 6.12.** *we have*

$$(28) \quad |\Phi_{1,\varepsilon}|^2 + |\Phi_{2,\varepsilon}|^2 + |\Phi_{3,\varepsilon}|^2 > 0$$

on  $\Sigma_{t(\varepsilon)}$  for sufficiently small  $\varepsilon$ .

*Proof.* The statement holds in  $U_\delta$  because the Saddle towers are regular embeddings, and the inequality (28) remains for sufficiently small  $\varepsilon$ . It suffices to prove (28) on the annuli that we identified to form necks. This is the case if  $\Phi_{3,\varepsilon}$  has no zero in the necks.

Recall that  $\Phi_3^\circ$  has  $\deg(v)$  poles hence  $\deg(v) - 2$  zeros in  $\hat{\mathbb{C}}_v$ . By choosing  $\delta$  sufficiently small, we may assume that all  $2|E| - 2|V| = 2|F| = 2g - 2$  zeros lie in  $U_\delta$ , which remains true for sufficiently small  $\varepsilon$ . But for  $\varepsilon \neq 0$ ,  $\Phi_{3,\varepsilon}$ , being a holomorphic function on a Riemann surface of genus  $g$  (without nodes), has exactly  $2g - 2$  zeros, therefore no other zeros. The statement then follows readily.  $\square$



It remains to prove that the image of  $f_\varepsilon$  has the desired geometry, as stated in Points (1), (2), (3) of Theorem 4.16, and is embedded. Define

$$X_v(\varepsilon) = f_\varepsilon^{\text{hor}}(O_v).$$

Then  $f_\varepsilon - X_v(\varepsilon)$  converges smoothly on  $U_{v,\delta}$  to  $f_{v,0}$  which parametrize a Saddle tower which we denote  $\mathcal{S}_v$ . Hence  $f_\varepsilon(U_{v,\delta})$  is embedded for  $\varepsilon > 0$  small enough. Let  $\mathcal{S}_v^\circ$  be the reference saddle tower, namely translated so that the image of  $O_v$  is the origin. We denote  $\varphi(\mathcal{S})$  the phase of a saddle tower. Using Equation (27) and Proposition 2.5, we have for  $h \in \mathbb{H}$

$$\begin{aligned} \varphi(\mathcal{S}_{v(-h)}) - \varphi(\mathcal{S}_{v(h)}) &= \varphi(\mathcal{S}_{v(-h)}^\circ) - \varphi(\mathcal{S}_{v(h)}^\circ) + \lim_{\varepsilon \rightarrow 0} \mathcal{P}_h^{\text{ver}}(\varepsilon) \\ &= \varphi(\mathcal{S}_{v(-h)}^\circ) - \varphi(\mathcal{S}_{v(h)}^\circ) + \phi_h^\circ + \nu_h - \nu_{-h} + \sigma_h \pi \\ &= \phi_h^\circ \pmod{2\pi}. \end{aligned}$$

This proves Point (3). Using Equation (26), we have for  $h \in \mathbb{H}$

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 (X_{v(-h)}(\varepsilon) - X_{v(h)}(\varepsilon)) = \lim_{\varepsilon \rightarrow 0} \mathcal{P}_h^{\text{hor}}(\varepsilon) = x_h^\circ.$$

This proves Point (2). Moreover, because the vertices are represented by distinct points, the images  $f_\varepsilon(U_{v,\delta})$  for  $v \in \mathbb{V}$  are disjoint for  $\varepsilon > 0$  small enough.

For  $\delta$  and  $\varepsilon$  small enough, the image of the circle  $|w_h| = \delta$  is close to a vertical segment  $s_h(\varepsilon)$  lying in  $f_\varepsilon(U_{v,\delta})$ . On the boundary of the annulus  $|t_h|/\delta < |w_h| < \delta$ , the gauss map is close to a horizontal constant (from the convergence to saddle towers), so by the maximum principle for holomorphic functions, it is close to this constant on the whole annulus. Given the boundary behavior, the image of this annulus is a graph over a vertical plane, hence embedded. By the convex hull property of minimal surfaces, the image of the annulus  $|t_h|/\delta < |w_h| < \delta$  stays close to the vertical plane  $B_{e(h)}(\varepsilon)$  bounded by  $s_h(\varepsilon)$  and  $s_{-h}(\varepsilon)$ . After a scaling by  $\varepsilon^2$ , Point (1) follows.

If there are no parallel edges, by taking  $\delta > 0$  small enough, we may ensure that the planes  $B_e(\varepsilon)$  for  $e \in \mathbb{E}$  are disjoint and conclude that the image of  $f_\varepsilon$  is embedded because the graph is embedded. The situation is more subtle in the case of parallel edges. Assume that the edges  $e(h_1), \dots, e(h_k)$  are parallel, with  $h_1, \dots, h_k \in v$  and  $\zeta(h_i) = h_{i+1}$  for  $1 \leq i \leq k-1$ . Because the saddle tower  $\mathcal{S}_v$  is embedded, the corresponding ends are asymptotic to parallel but distinct vertical planes. Hence, by our choice of ordering of the ends, the abscissa of the vertical segments  $s_{h_i}(\varepsilon)$  are decreasing and separated by a distance greater than some uniform  $r > 0$ . On the other side, the abscissa of the vertical segments  $s_{-h_i}(\varepsilon)$  are also decreasing and uniformly separated. Hence for  $\varepsilon$  small enough, the planes  $B_{e(h_i)}$  for  $1 \leq i \leq k$  are disjoint, and we conclude again that the image of  $f_\varepsilon$  is embedded. This concludes the proof of Theorem 4.16.

## APPENDIX A. RIGIDITY OF SADDLE TOWERS

This section is dedicated to a proof of Theorem 2.8. Let  $\dot{p} \in \ker D\Lambda(p^\circ)$ . We want to prove that, if  $\dot{p}_h = 0$  for three  $h \in \mathbb{H}$ , then  $\dot{p} = 0$ .

Define  $p(t) = p^\circ + \varepsilon \dot{p}$ . Let  $(\Phi_1(\varepsilon), \Phi_2(\varepsilon), \Phi_3(\varepsilon))$  be the Weierstrass data given by Equation (1) and  $X(\varepsilon)$  the corresponding immersion given by Equation (2). Note that  $X(\varepsilon)$  is harmonic but not minimal. Let  $H(\varepsilon)$  be the mean curvature and  $N(\varepsilon)$  be the normal of  $X(\varepsilon)$ . To ease computation, the dependence on  $\varepsilon$  will not be written. Instead, we use an exponent  $\circ$  to denote the value at  $\varepsilon = 0$  and a dot to denote the partial derivative with respect to  $\varepsilon$  at  $\varepsilon = 0$ .

**Lemma A.1.** *We have  $\dot{H} = 0$  on the Riemann sphere minus the points  $p_h^\circ$  for  $h \in \mathbb{H}$ .*

*Proof.* We have for all  $\epsilon$

$$Q = \sum_{i=1}^{n-2} \lambda_i \frac{\Phi_1 dz}{z - \zeta_i} \quad \text{with} \quad \lambda_i = \text{Res} \left( \frac{Q}{\Phi_1}, \zeta_i \right).$$

We have  $\lambda_i^\circ = 0$  and  $\dot{\lambda}_i = D\Lambda_i(p^\circ) \cdot \dot{p} = 0$  so  $\dot{Q} = 0$ . Let  $w = x_1 + ix_2$  be a local complex coordinate on the Riemann sphere. Recall the standard formula for the mean curvature

$$(29) \quad H = \frac{g_{22}b_{11} + g_{11}b_{22} - 2g_{12}b_{12}}{2(g_{11}g_{22} - g_{12}^2)}$$

where  $g$  and  $b$  are respectively the matrices of the first and second fundamental forms in the coordinate system  $(x_1, x_2)$ . We have for all  $\epsilon$

$$Q = \left( \left\| \frac{\partial X}{\partial x_1} \right\|^2 - \left\| \frac{\partial X}{\partial x_2} \right\|^2 - 2i \left\langle \frac{\partial X}{\partial x_1}, \frac{\partial X}{\partial x_2} \right\rangle \right) dw^2 = (g_{11} - g_{22} - 2ig_{12})dw^2.$$

Hence from  $Q^\circ = 0$  and  $\dot{Q} = 0$  we obtain

$$(30) \quad g_{11}^\circ = g_{22}^\circ, \quad g_{12}^\circ = 0, \quad \dot{g}_{11} = \dot{g}_{22} \quad \text{and} \quad \dot{g}_{12} = 0.$$

Since  $X(t)$  is harmonic for all  $\epsilon$ , we have  $b_{11} + b_{22} = 0$  for all  $\epsilon$ , so

$$b_{11}^\circ + b_{22}^\circ = 0 \quad \text{and} \quad \dot{b}_{11} + \dot{b}_{22} = 0.$$

Taking the derivative of (29), we obtain  $\dot{H} = 0$ . □

By Lemma A.1,  $u = \langle \dot{X}, N^\circ \rangle$  is a Jacobi field. In a neighborhood of  $p_h^\circ$ , we have

$$(31) \quad \dot{X} = \text{Re} \left[ (\cos(\theta_h), \sin(\theta_h), i\sigma_h) \frac{\dot{p}_h}{z - p_h^\circ} \right] + \text{bounded terms}$$

$$N^\circ = \sigma_h(-\sin(\theta_h), \cos(\theta_h), 0) + O(z - p_h^\circ)$$

so  $u$  is a bounded Jacobi field. Since all saddle towers are rigid, there exists  $c \in \mathbb{R}^3$  such that  $u = \langle N^\circ, c \rangle$ . Replacing  $X(\epsilon)$  by  $Y(\epsilon) = X(\epsilon) - \epsilon c$ , we obtain  $u = 0$ . Hence  $\dot{Y}$  is a tangent vector. Using the local complex coordinate  $w = x_1 + ix_2$ , we decompose  $\dot{Y}$  in the tangent space as

$$\dot{Y} = \xi_1 \frac{\partial X^\circ}{\partial x_1} + \xi_2 \frac{\partial X^\circ}{\partial x_2} = \xi \frac{\partial X^\circ}{\partial w} + \bar{\xi} \frac{\partial X^\circ}{\partial \bar{w}} \quad \text{with} \quad \xi = \xi_1 + i\xi_2.$$

**Lemma A.2.**  $\xi \frac{d}{dw}$  defines a holomorphic vector field on the Riemann sphere. Moreover,  $\xi(p_h^\circ) = -\dot{p}_h$  for  $h \in \mathbb{H}$ .

*Proof.* Since  $X^\circ$  is conformal and harmonic, we have

$$\left\langle \frac{\partial X^\circ}{\partial w}, \frac{\partial X^\circ}{\partial w} \right\rangle = 0, \quad \left\langle \frac{\partial X^\circ}{\partial w}, \frac{\partial^2 X^\circ}{\partial w^2} \right\rangle = 0, \quad \text{and} \quad \frac{\partial^2 X^\circ}{\partial w \partial \bar{w}} = 0.$$

For all  $\epsilon$

$$\left\langle \frac{\partial Y}{\partial w}, \frac{\partial Y}{\partial w} \right\rangle = \frac{1}{4}(g_{11} - g_{22} - 2ig_{12}).$$

Taking the derivative with respect to  $\epsilon$  and using Equation (30), we obtain

$$2 \left\langle \frac{\partial X^\circ}{\partial w}, \frac{\partial}{\partial w} \left( \xi \frac{\partial X^\circ}{\partial w} + \bar{\xi} \frac{\partial X^\circ}{\partial \bar{w}} \right) \right\rangle = 2 \frac{\partial \bar{\xi}}{\partial w} \left\langle \frac{\partial X^\circ}{\partial w}, \frac{\partial X^\circ}{\partial \bar{w}} \right\rangle = 0.$$

Hence  $\frac{\partial \bar{\xi}}{\partial w} = 0$  and  $\xi$  is holomorphic. If  $w'$  is another local complex coordinate, we can write

$$\dot{Y} = \xi' \frac{\partial X^\circ}{\partial w'} + \bar{\xi}' \frac{\partial X^\circ}{\partial \bar{w}'} = \xi' \frac{\partial X^\circ}{\partial w} \frac{dw}{dw'} + \bar{\xi}' \frac{\partial X^\circ}{\partial \bar{w}} \overline{\left( \frac{dw}{dw'} \right)}$$

This gives

$$\xi = \xi' \frac{dw}{dw'}$$

so  $\xi$  transforms as a holomorphic vector field under change of coordinate. Using the complex coordinate  $w = z - p_h^\circ$  in a neighborhood of  $p_h^\circ$ , we have

$$\frac{\partial X^\circ}{\partial w} = \frac{1}{2w}(-\cos(\theta_h), -\sin(\theta_h), -i\sigma_h) + O(1).$$

Hence by Equation (31),

$$\xi(p_h^\circ) = -\dot{p}_h.$$

□

Recall that we have fixed the position of three points  $p_h$ . Hence  $\xi$  has at least three zeros. Now a non-zero holomorphic vector field on the Riemann sphere has two zeros, so  $\xi = 0$ . This implies that  $\dot{p} = 0$  and concludes the proof of Theorem 2.8.

## APPENDIX B. HORIZONTAL RIGIDITY OF TRIANGULATED GRAPHS

This section is dedicated to a proof of Theorem 4.21. Consider, for  $x \in \mathcal{H}$ , the total length

$$\mathcal{L}(x) = \frac{1}{2} \sum_{h \in \mathbf{H}} \|x_h\|.$$

If  $P^{\text{hor}}(\chi) = 0$ , so that  $\chi = \text{grad } f$ , we have

$$\begin{aligned} D\mathcal{L}(x^\circ) \cdot \chi &= \frac{1}{2} \sum_{h \in \mathbf{H}} \frac{1}{\|x_h^\circ\|} \langle x_h^\circ, f_{v(-h)} - f_{v(h)} \rangle \\ &= - \sum_{h \in \mathbf{H}} \frac{1}{\|x_h^\circ\|} \langle x_h^\circ, f_{v(h)} \rangle \quad (\text{using } h \rightarrow -h \text{ for } f_{v(-h)}) \\ &= - \sum_{v \in \mathbf{V}} \sum_{h \in b(v)} \frac{1}{\|x_h^\circ\|} \langle x_h^\circ, f_v \rangle \\ &= - \sum_{v \in \mathbf{V}} \langle F_{b(v)}^{\text{hor}}, f_v \rangle. \end{aligned}$$

We have proved that

**Proposition B.1.** *The graph is balanced if and only if  $x^\circ$  is a critical point of  $\mathcal{L}$  restricted to those  $x \in \mathcal{A}$  such that  $P^{\text{hor}}(x) = P^{\text{hor}}(x^\circ)$ .*

Assume that  $F^{\text{hor}}(x^\circ) = 0$  and let  $\chi = \text{grad}(f)$  be in the kernel of  $(DF^{\text{hor}}(x^\circ), P^{\text{hor}})$ . Differentiating the above equation

$$D^2\mathcal{L}(x^\circ) \cdot (\chi, \chi) = - \sum_{v \in \mathbf{V}} \langle DF_{b(v)}^{\text{hor}}(x^\circ) \cdot \chi, f_v \rangle = 0.$$

On the other hand, a direct computation gives

$$D^2\mathcal{L}(x^\circ) \cdot (\chi, \chi) = \frac{1}{2} \sum_{h \in \mathbf{H}} \frac{1}{\|x_h^\circ\|^3} (\|x_h^\circ\|^2 \|\chi_h\|^2 - \langle x_h^\circ, \chi_h \rangle^2).$$

The summands are all non-negative, hence must be all zero, which means that  $\chi_h$  is parallel to  $x_h^\circ$  for all  $h \in \mathbf{H}$ .

If all faces have 2 or 3 edges, this implies that  $\chi_h = \lambda x_h^\circ$  for some  $\lambda \in \mathbb{R}$ . Then  $P_{c_1}^{\text{hor}}(\chi) = \lambda T_1 = 0$  implies  $\lambda = 0$  and  $\chi = 0$ , which proves Theorem 4.21.

## APPENDIX C. CLASSIFICATION OF BALANCED CONFIGURATIONS OF GENUS 3

As promised in Section 5.1, we prove the following Classification Theorem.

**Theorem C.1.** *The Meeks, aG, aH, and aI configurations in Figure 4 are the only balanced configurations whose graphs are orientable with two faces. Hence they are the only possible configurations that give rise to TPMSs of genus 3.*

*Proof.* If a configuration gives rise to a TPMS of genus three, its graph must have two faces. By Euler's formula, the average degree of the graph is

$$2|E|/|V| = 2(|V| + |F|)/|V| = 2 + 4/|V|.$$

By Assumption 3.6, the average degree is at least 4, hence the graph has at most 2 vertices. We discuss two cases

$|V| = 1, |E| = 3$ : Then all edges are loops. Since edges are represented by straight segments, no loop is null-homologous. If two loops are homologous, they must be parallel, and represented by the same segment that cuts the torus into an annulus. To form a 2-cell embedding, the remaining edge must be in a different homology class. But then, it is impossible to orient the half-edges alternately incoming and outgoing around the vertex, contradicting the orientability.

So we have three pairwise non-homologous simple loops. They only intersect at the vertex, hence any two of them form a homology basis, and the remaining loop must be homologous to their concatenation. So the graph must be homeomorphic to that of the aH.

Any configuration with this graph is trivially balanced because all edges are loops. It is also trivially rigid as there is only one vertex, so the cut space is trivial.

$|V| = 2, |E| = 4$ : We first prove that such a graph has no loop. The graph is connected, hence the edges can not be all loops. If exactly one or three edges are loops, the degree of a vertex will be smaller than 4, contradicting Assumption 3.6. If exactly two edges are loops, they must be adjacent to different vertices. So they divide the torus into two annuli. Since the graph is represented as a limit of 2-cell embeddings, the remaining edges must lie in different annuli. But then, it is impossible to orient the half-edges alternately incoming and outgoing around each vertex, contradicting the orientability. This proves that none of the edges is a loop.

So we have four edges between two vertices. Any two of the edges form a cycle. If some of these cycles are null-homologous, we will have parallel edges. If at most one edge is simple, it is not possible to form a 2-cell embedding. If exactly two edges are simple, the only 2-cell embedding does not have a proper orientation.

So we have four simple edges between two vertices. Then the graph must be as shown in Figure 9. This graph is balanced if and only if the half-edges form two collinear pairs around each vertex. So if one vertex is at 0, the other vertex must lie at a 2-division point.

By the period condition (6), the phase function must be of the form

$$c - \Psi_2/2, \quad c + \Psi_2/2, \quad c + \Psi_1 + \Psi_2/2, \quad c + \Psi_1 - \Psi_2/2$$

on the half-edges around a vertex, where  $\Psi_1$  and  $\Psi_2$  are the fundamental shifts. See Figure 9.

If  $\arg(T_2/T_1) \neq \pi/2$ , the shortest edges of the graph form a collinear pair. Assuming  $\arg(T_2/T_1) < \pi/2$  as in Figure 9, then the phase function

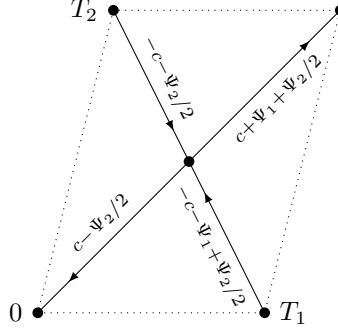


FIGURE 9

is balanced if and only if

$$\sin(c + \Psi_1/2) = 0 \quad \text{or} \quad \cos \frac{\Psi_2 - \Psi_1}{2} = 0.$$

The solution  $\Psi_1 = -2c$  gives Meeks' configuration which is generically rigid. The solution  $\Psi_2 - \Psi_1 = \pi$  gives the aI configurations, which is not rigid because  $c$  remains a free variable.

If  $\arg(T_2/T_1) = \pi/2$ , the phase function is balanced if and only if

$$\sin c + \sin(c + \Psi_1) = 0 \quad \text{or} \quad \cos(\Psi_2/2) = 0.$$

The solution  $\Psi_1 = -2c$  gives again Meeks' configurations. The solutions  $\Psi_1 = \pi$  and  $\Psi_2 = \pi$  give the aG configurations, which is not rigid because  $c$  remains a free variable.

□

#### APPENDIX D. INTEGRAL OF A HOLOMORPHIC 1-FORM THROUGH A NECK

To prove Lemma 6.3, we first prove a general result, Theorem D.1 below. This was in fact proved in [Tra02b, Lemma 1] and has been used in several papers, but was not clearly stated as an independent result and the value at  $t = 0$  was not explicitly given.

Fix some numbers  $0 < \epsilon_1 < \epsilon$ . For  $t \in \mathbb{C}$  such that  $0 < |t| < \epsilon^2$ , let  $\mathcal{A}_t \subset \mathbb{C}$  be the annulus  $|t|/\epsilon < |z| < \epsilon$  and  $\psi_t : \mathcal{A}_t \rightarrow \mathcal{A}_t$  be the involution defined by  $\psi_t(z) = t/z$ . Let  $\beta_t$  be the curve from  $\epsilon_1$  to  $t/\epsilon_1$  parameterized for  $s \in [0, 1]$  by

$$\beta_t(s) = \epsilon_1^{1-2s} t^s.$$

Note that  $\beta_t$  depends on the choice of the argument of  $t$ . Let  $\gamma$  be the circle parameterized by  $\gamma(s) = \epsilon_1 e^{2\pi i s}$ .

**Theorem D.1.** *Let  $\omega_t$  be a family of holomorphic 1-forms on  $\mathcal{A}_t$ , depending holomorphically on  $t \in D^*(0, \epsilon^2)$ , and let  $\tilde{\omega}_t = \psi_t^* \omega_t$ . Define*

$$\alpha_t = \frac{1}{2\pi i} \int_{\gamma} \omega_t = -\frac{1}{2\pi i} \int_{\gamma} \tilde{\omega}_t.$$

Assume that

$$\lim_{t \rightarrow 0} \omega_t = \omega_0 \quad \text{and} \quad \lim_{t \rightarrow 0} \tilde{\omega}_t = \tilde{\omega}_0$$

where  $\omega_0$  and  $\tilde{\omega}_0$  are holomorphic in  $D^*(0, \epsilon)$  with at most simple poles at  $z = 0$ , and the limit is uniform on compact subsets of  $D^*(0, \epsilon)$ .

Then  $\int_{\beta_t} \omega_t - \alpha_t \log t$  is a well-defined holomorphic function of  $t \neq 0$  which extends holomorphically at  $t = 0$ . Moreover, its value at  $t = 0$  is

$$\lim_{z \rightarrow 0} \left[ \left( \int_{\epsilon_1}^z \omega_0 \right) - \alpha_0 \log z \right] - \lim_{z \rightarrow 0} \left[ \left( \int_{\epsilon_1}^z \tilde{\omega}_0 \right) + \alpha_0 \log z \right].$$

*Proof.* If  $\arg(t)$  is increased by  $2\pi$ , the homotopy class of  $\beta_t$  is multiplied on the left by  $\gamma$ , so  $\int_{\beta_t} \omega_t$  is increased by  $2\pi i \alpha_t$ . On the other hand,  $\log t$  is increased by  $2\pi i$ , so the difference  $\int_{\beta_t} \omega_t - \alpha_t \log t$  is a well-defined holomorphic function of  $t$  in  $D^*(0, \epsilon^2)$ . Using the change of variable rule, we write

$$(32) \quad \left( \int_{\beta_t} \omega_t \right) - \alpha_t \log t = \left[ \left( \int_{\epsilon_1}^{\sqrt{t}} \omega_t \right) - \alpha_t \log \sqrt{t} \right] - \left[ \left( \int_{\epsilon_1}^{\sqrt{t}} \tilde{\omega}_t \right) + \alpha_t \log \sqrt{t} \right].$$

To estimate the first term, we fix  $\epsilon_1 < \epsilon_2 < \epsilon$  and expand  $\omega_t$  in Laurent series in the annulus  $\mathcal{A}_t$  as

$$\omega_t = \sum_{n \in \mathbb{Z}} a_n(t) z^{n-1} dz$$

with  $a_0(t) = \alpha_t$  and

$$a_n(t) = \frac{1}{2\pi i} \int_{|z|=\epsilon_2} z^{-n} \omega_t = \frac{-1}{2\pi i} \int_{|z|=\epsilon_2} \psi_t^*(z^{-n} \omega_t) = \frac{-1}{2\pi i} \int_{|z|=\epsilon_2} t^{-n} z^n \tilde{\omega}_t.$$

Since  $\omega_t$  and  $\tilde{\omega}_t$  are uniformly bounded on the circle  $|z| = \epsilon_2$ , this gives the estimates, for  $n > 0$  and a uniform constant  $C$ ,

$$(33) \quad |a_n(t)| \leq \frac{C}{(\epsilon_2)^n} \quad \text{and} \quad |a_{-n}(t)| \leq \frac{C|t|^n}{(\epsilon_2)^n}.$$

Then we have

$$\left( \int_{\epsilon_1}^{\sqrt{t}} \omega_t \right) - \alpha_t \log \sqrt{t} = -\alpha_t \log \epsilon_1 + \sum_{n=1}^{\infty} \left[ \frac{a_n}{n} (t^{n/2} - (\epsilon_1)^n) - \frac{a_{-n}}{n} (t^{-n/2} - (\epsilon_1)^{-n}) \right].$$

Using the estimates (33) and  $\epsilon_1 < \epsilon_2$ , it is straightforward to check that the sum is uniformly bounded with respect to  $t$  and that we have

$$\lim_{t \rightarrow 0} \left[ \left( \int_{\epsilon_1}^{\sqrt{t}} \omega_t \right) - \alpha_t \log \sqrt{t} \right] = -\alpha_0 \log \epsilon_1 - \sum_{n=1}^{\infty} \frac{a_n(0)}{n} \epsilon_1^n.$$

On the other hand, we have

$$\int_{\epsilon_1}^z \omega_0 = \alpha_0 \log \left( \frac{z}{\epsilon_1} \right) + \sum_{n=1}^{\infty} \frac{a_n(0)}{n} (z^n - (\epsilon_1)^n)$$

so

$$(34) \quad \lim_{t \rightarrow 0} \left[ \left( \int_{\epsilon_1}^{\sqrt{t}} \omega_t \right) - \alpha_t \log \sqrt{t} \right] = \lim_{z \rightarrow 0} \left[ \left( \int_{\epsilon_1}^z \omega_0 \right) - \alpha_0 \log z \right].$$

The second term in (32) is estimated in the exact same way, leading to

$$(35) \quad \lim_{t \rightarrow 0} \left[ \left( \int_{\epsilon_1}^{\sqrt{t}} \tilde{\omega}_t \right) + \alpha_t \log \sqrt{t} \right] = \lim_{z \rightarrow 0} \left[ \left( \int_{\epsilon_1}^z \tilde{\omega}_0 \right) + \alpha_0 \log z \right].$$

The function  $\int_{\beta_t} \omega_t - \alpha_t \log t$  is bounded so extends holomorphically at  $t = 0$  by Riemann Extension Theorem. The last point of Theorem D.1 follows from Equations (32), (34) and (35).  $\square$

Proof of Lemma 6.3. Recall the definition of the path  $B_h$  just before Lemma 6.3. On path number (1),  $\omega_t$  depends holomorphically on  $t_h$  in a neighborhood of 0 so

$$\int_{O_v(h)}^{w_h=\delta} \omega_t$$

is a holomorphic function of  $t_h$  in a neighborhood of 0. Same for path number (3). Regarding path number (2) we write

$$(36) \quad \left( \int_{w_h=\delta}^{w_h=t_h/\delta} \omega_t \right) - \alpha_h \log t_h = \left( \int_{\delta}^{t_h/\delta} (w_h^{-1})^* \omega_t \right) - \alpha_h \log t_h$$

and we apply Theorem D.1 to the 1-form  $(w_h^{-1})^* \omega_t$  with  $\epsilon_1 = \delta$ ,  $t = t_h$ , observing that

$$\psi_{t_h}^* (w_h^{-1})^* \omega_t = ((\psi_{t_h} \circ w_h)^{-1})^* \omega_t = (w_{-h}^{-1})^* \omega_t$$

so the hypotheses of Theorem D.1 are satisfied. Hence (36) extends holomorphically at  $t_h = 0$  and its value there is

$$\begin{aligned} & \lim_{z \rightarrow 0} \left[ \left( \int_{\delta}^z (w_h^{-1})^* \omega_0 \right) - \alpha_h \log z \right] - \lim_{z \rightarrow 0} \left[ \left( \int_{\delta}^z (w_{-h}^{-1})^* \omega_0 \right) - \alpha_{-h} \log z \right] \\ &= \lim_{z \rightarrow p_h} \left[ \left( \int_{w_h=\delta}^z \omega_0 \right) - \alpha_h \log w_h(z) \right] - \lim_{z \rightarrow p_{-h}} \left[ \left( \int_{w_{-h}=\delta}^z \omega_0 \right) - \alpha_{-h} \log w_{-h}(z) \right]. \end{aligned}$$

Adding the three terms gives Lemma 6.3.

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(Chen) GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN, INSTITUT FÜR NUMERISCHE UND ANGEWANDTE MATHEMATIK

*Email address:* [h.chen@math.uni-goettingen.de](mailto:h.chen@math.uni-goettingen.de)

(Traizet) INSTITUT DENIS POISSON, CNRS UMR 7350, FACULTÉ DES SCIENCES ET TECHNIQUES, UNIVERSITÉ DE TOURS

*Email address:* [martin.traizet@univ-tours.fr](mailto:martin.traizet@univ-tours.fr)