

The classification of doubly periodic minimal tori with parallel ends

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ABSTRACT. Let \mathcal{K} be the space of properly embedded minimal tori in quotients of \mathbb{R}^3 by two independent translations, with any fixed (even) number of parallel ends. After an appropriate normalization, we prove that \mathcal{K} is a 3-dimensional real analytic manifold that reduces to the finite coverings of the examples defined by Karcher, Meeks and Rosenberg in [9, 10, 15]. The degenerate limits of surfaces in \mathcal{K} are the catenoid, the helicoid and three 1-parameter families of surfaces: the simply and doubly periodic Scherk minimal surfaces and the Riemann minimal examples.

1 Introduction

In 1988, Karcher [9] defined a 1-parameter family of minimal tori in quotients of \mathbb{R}^3 by two independent translations. Each of these surfaces, called *toroidal halfplane layer*, has four parallel Scherk-type ends in its smallest fundamental domain, is invariant by reflection symmetries in three orthogonal planes and contains four parallel straight lines through the ends that project orthogonally onto the corners of a planar rectangle contained in one of the reflective symmetry planes, and the edges of this rectangle are just the half-period vectors of the surface, see Figure 3 left. Thanks to this richness of symmetries, he gave explicitly the Weierstrass representation of the toroidal halfplane layers in terms of elliptic functions on a 1-parameter family of rectangular tori (these examples will be denoted as $M_{\theta,0,0}$ with $0 < \theta < \frac{\pi}{2}$ in Section 3). Inside a brief remark in his paper and later in another work [10], Karcher exposed two distinct 1-parameter deformations of each $M_{\theta,0,0}$ by losing some of their symmetries (denoted by $M_{\theta,\alpha,0}$, $M_{\theta,0,\beta}$ in Section 3). In

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1989, Meeks and Rosenberg [15] developed a general theory for doubly periodic minimal surfaces with finite topology in the quotient, and used a completely different approach to find again the examples $M_{\theta,0,\beta}$ (although up to now it was not clear that Meeks and Rosenberg's examples were the same as Karcher's). In fact, it is not difficult to produce a 3-parameter family of examples $M_{\theta,\alpha,\beta}$ containing all the above examples, see Section 3.

Hauswirth and Traizet [5] have extended previous ideas by Pérez and Ros [20] and Pérez [18] to study the moduli space of all properly embedded doubly periodic minimal surfaces with a given fixed finite topology in the quotient, proving that in the parallel ends case (resp. nonparallel) and after identifying by translations, homotheties and rotations, this moduli space is a real analytic manifold of dimension 3 (resp. 1) around a *nondegenerate* surface. The nondegeneracy condition concerns the space of infinitesimal deformations of a given surface; we will see in Section 3 that each example $M_{\theta,\alpha,\beta}$ is nondegenerate. All these facts point out to a possible global uniqueness of the four ended surfaces $M_{\theta,\alpha,\beta}$ and their k -sheeted coverings (here k is any positive integer), which in the sequel will be referred to as *standard examples*, among all properly embedded doubly periodic minimal surfaces with genus one and any number of parallel ends in the quotient. In this paper we solve this uniqueness in the affirmative.

Theorem 1 *If M is a properly embedded doubly periodic minimal surface with genus one in the quotient and parallel ends, then M is a standard example.*

Remark 1 *The theorem does not hold if we remove the hypothesis on the ends to be parallel, as demonstrate the 4-ended tori discovered by Hoffman, Karcher and Wei in [6].*

The analysis of the space of standard examples \mathcal{K} with $4k$ parallel ends can be obviously reduced to the case $k = 1$ by taking k -sheeted coverings. In the four ended case, the space $\mathcal{K}_1 = \{M_{\theta,\alpha,\beta}\}_{\theta,\alpha,\beta}$ of standard examples is studied in detail in [23]. \mathcal{K}_1 is a 3-dimensional real analytic manifold and the degenerate limits of sequences in \mathcal{K}_1 are the catenoid, the helicoid, any singly or doubly periodic Scherk minimal surface and any Riemann minimal example (see for instance [14] for a description of these last surfaces). Furthermore \mathcal{K}_1 is self-conjugate, in the sense that the conjugate surface¹ of any element in \mathcal{K}_1 also belongs to \mathcal{K}_1 . Since any standard example admits an orientation reversing involution without

¹Two minimal surfaces $M_1, M_2 \subset \mathbb{R}^3$ are *conjugate* if the coordinate functions of M_2 are harmonic conjugate to the coordinate functions of M_1 .

fixed points, Theorem 1 also gives a classification of all properly embedded minimal Klein bottles with parallel ends in doubly periodic quotients of \mathbb{R}^3 .

Theorem 1 can be viewed as a first step toward the classification of all properly embedded minimal surfaces of genus zero in the complete flat 3-manifold $\mathbb{R}^2 \times \mathbb{S}^1$. Concerning this open problem and except for the flat examples (which reduce to the plane $\mathbb{R}^2 \times \{\theta\}$ and the cylinder $\mathbb{R} \times \mathbb{S}^1$), all the simple ends of any such a surface are simultaneously asymptotic to helicoids or to Scherk minimal surfaces. In the finite topology case, the number of ends is necessarily even and the helicoid is known to be the unique genus zero example with a finite number of helicoidal ends (Pérez and Ros [19]), and it is conjectured that for any integer $k \geq 2$, the only examples with genus zero and $2k$ ends are the $(2k - 3)$ -parameter family of saddle towers having as building blocks the conjugate surface of a Jenkins-Serrin graph over the right triangle which, after symmetrization, generates a convex $2k$ -gon with all edges of the same length (in particular, for $k = 2$ the only examples are the singly-periodic Scherk minimal surfaces). The validity of this conjecture will be proved by Pérez and Traizet in a forthcoming paper [22].

The proof of Theorem 1 is a modified application of the machinery developed by Meeks, Pérez and Ros in their characterization of the Riemann minimal examples [14], hence this reference could be a helpful source to the reader. For $k \in \mathbb{N}$ fixed, one considers the space \mathcal{S} of properly embedded doubly periodic minimal surfaces of genus one in the quotient and $4k$ parallel ends. The goal is to prove that \mathcal{S} reduces to the space \mathcal{K} of standard examples (i.e. k sheeted coverings to the surfaces $M_{\theta, \alpha, \beta} \in \mathcal{K}_1$). The argument is based on modeling \mathcal{S} as an analytic subset in a complex manifold \mathcal{W} of finite dimension (roughly, \mathcal{W} consists of all admissible Weierstrass data for our problem). Then the procedure has three steps:

- *Properness*: Uniform curvature estimates are proven for a sequence of surfaces in \mathcal{S} constrained to certain natural normalizations in terms of the period vector at the ends and of the flux of these surfaces (this flux will be defined in Section 4).
- *Openness*: Any surface in $\mathcal{S} - \mathcal{K}$ can be minimally deformed by moving its period at the ends and its flux. This step depends on the properness part and both together imply, assuming $\mathcal{S} - \mathcal{K} \neq \emptyset$ (the proof of Theorem 1 is by contradiction), that any period at the ends and flux can be achieved by nonstandard examples.
- *Uniqueness around a boundary point of \mathcal{S}* : Only standard examples can occur nearby a certain minimal surface outside \mathcal{S} but obtained as a smooth limit of surfaces in \mathcal{S} .

This property together with the last sentence in the openness point lead to the desired contradiction, thereby proving Theorem 1.

Although the above strategy is adapted from the one in [14], there are several major differences between that work and ours, among which we would like to emphasize two. The main tool in our proof is a detailed study of the map C that associates to each $M \in \mathcal{S}$ two geometric invariants: its period at the ends and its flux along a nontrivial homology class with vanishing period vector. The first main difference between this work and [14] is that C is not proper, in contrast with the properness of the flux map in [14]. Fortunately $C|_{\mathcal{S}-\mathcal{K}}$ becomes proper (recall we assumed $\mathcal{S} - \mathcal{K} \neq \emptyset$). This restricted properness follows from curvature estimates as in the first step of the above procedure, together with a local uniqueness argument similar to the third step, performed around any singly periodic Scherk minimal surface, considered as a point in the boundary $\partial\mathcal{S}$. In fact, to describe the complete list of limits of sequences in \mathcal{S} we need a new characterization of the singly periodic Scherk minimal surfaces among all properly embedded singly periodic genus zero minimal surfaces with Scherk-type ends, provided that all ends are parallel except two of them. This result is a special case of the main Theorem in [22] and it will be used here.

Once we know that $C|_{\mathcal{S}-\mathcal{K}}$ is proper and open, we need a local uniqueness result at a point of $\partial\mathcal{S}$ other than a singly periodic Scherk minimal surface, to conclude the third step in our strategy. This boundary point will be the catenoid, and the local uniqueness follows from the Inverse Function Theorem. The second main discrepancy with [14] is a technical difficulty: nearby the catenoidal limit, the ends of surfaces in \mathcal{S} group in couples, each couple giving rise to a single end of the limit catenoid. Such a collapsing phenomenon makes certain residues to blow-up, and a careful study of speeds of degeneration is needed to rectify the mapping to which the Inverse Function Theorem is applied to.

The paper is organized as follows. In Section 2 we recall the necessary background to tackle our problem. Section 3 is devoted to introduce briefly the 3-parameter family \mathcal{K} of standard examples. The complex manifold of admissible Weierstrass data \mathcal{W} and the natural mappings on it are studied in Section 4. In Section 5 we obtain the curvature estimates needed for the first point of our strategy. Sections 6 and 7 deal with the local uniqueness around the singly periodic Scherk minimal surfaces and the catenoid, respectively. The second point of our above strategy (openness) is the goal of Section 8, and finally Section 9 contains the proof of Theorem 1.

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2 Preliminaries.

Let $\widetilde{M} \subset \mathbb{R}^3$ be a connected orientable² properly embedded minimal surface, invariant by a rank 2 lattice \mathcal{P} generated by two linearly independent translations T_1, T_2 (we will shorten by calling \widetilde{M} a *doubly periodic minimal surface*). \widetilde{M} induces a properly embedded minimal surface $M = \widetilde{M}/\mathcal{P}$ in the complete flat 3-manifold $\mathbb{R}^3/\mathcal{P} = \mathbb{T}^2 \times \mathbb{R}$, where \mathbb{T} is a 2-dimensional torus. Reciprocally, if $M \subset \mathbb{T} \times \mathbb{R}$ is a properly embedded nonflat minimal surface, then its lift $\widetilde{M} \subset \mathbb{R}^3$ is a connected doubly periodic minimal surface by the Strong Halfspace Theorem of Hoffman and Meeks [7]. Existence and classification theorems for doubly periodic minimal surfaces are usually tackled by considering the quotient surfaces in $\mathbb{T} \times \mathbb{R}$. An important result by Meeks and Rosenberg [15] insures that a properly embedded minimal surface $M \subset \mathbb{T} \times \mathbb{R}$ has finite topology if and only if it has finite total curvature, and in this case M has an even number of ends, each one asymptotic to a flat annulus (Scherk-type end). Later, Meeks [13] proved that any properly embedded minimal surface in $\mathbb{T} \times \mathbb{R}$ has a finite number of ends, so the finiteness of its total curvature is equivalent to the finiteness of its genus.

When normalized so that the lattice of periods \mathcal{P} is horizontal, we distinguish two types of ends, depending on whether the well defined third coordinate function on M tends to ∞ (*top* end) or to $-\infty$ (*bottom* end) at the corresponding puncture. By separation properties, there are an even number of top (resp. bottom) ends. Because of embeddedness, top (resp. bottom) ends are always parallel each other. If the top ends are not parallel to the bottom ends, then there exists an algebraic obstruction on the period lattice, which must be *commensurable* as in the classical doubly periodic minimal surfaces defined by Scherk in [25] or in the 4-ended tori found by Hoffman, Karcher and Wei [6]. If the top ends are parallel to the bottom ends, then the cardinals of both families of ends coincide, therefore the total number of ends of M is a multiple of four. For details, see [15].

We will focus on the parallel ends setting, where the simplest possible topology is a finitely punctured torus (properly embedded minimal planar domains in $\mathbb{T} \times \mathbb{R}$ must have nonparallel ends by Theorem 4 in [15]; in fact Lazard-Holly and Meeks [11] proved that the doubly periodic Scherk minimal surfaces are the unique possible examples with genus zero). Our main goal is to give a complete classification of all examples with genus one and parallel ends. To do this, we will normalize appropriately the surfaces under consideration.

²Unless otherwise stated, all surfaces in the paper are supposed to be orientable.

Given a positive integer k , let \mathcal{S} be the space of all properly embedded minimal tori in a quotient $\mathbb{R}^3/\mathcal{P} = \mathbb{T} \times \mathbb{R}$ modulo a rank 2 lattice \mathcal{P} generated by two independent translations (which depend on the surface), one of them being in the direction of the x_2 -axis, with $4k$ horizontal Scherk-type ends. Given $M \in \mathcal{S}$, we denote respectively by $P_\Gamma \in \mathcal{P}$ and F_Γ the period and flux vectors of M along an oriented closed curve $\Gamma \subset M$. By the Divergence Theorem, P_Γ, F_Γ only depend on the homology class of Γ in M . The period and flux vectors H, F at an end of M (defined as the period and flux along a small loop around the puncture with the inward pointing conormal vector respect to the disk that contains the end) are related by the equation $F = H \wedge N_0$, where N_0 is the value of the \mathbb{S}^2 -Gauss map at the puncture. In our normalization, each of the period vectors at the ends of M is of the form $H = \pm(0, \pi a, 0)$ with $a > 0$. The end is called a *left end* if $F = (-\pi a, 0, 0)$, and a *right end* if $F = (\pi a, 0, 0)$. As M is embedded, each family of “sided” ends is naturally ordered by heights; in fact the maximum principle at infinity [16] implies that consecutive left (resp. right) ends are at positive distance. Furthermore, their limit normal vectors are opposite by a trivial separation argument.

We will denote by $\widetilde{M} \subset \mathbb{R}^3$ the (connected) doubly periodic minimal surface obtained by lifting M . Since points in \widetilde{M} homologous by \mathcal{P} have the same normal vector, the stereographically projected Gauss map $g : \widetilde{M} \rightarrow \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ descends to M . As M has finite total curvature, g extends meromorphically to the conformal torus \mathbb{M} obtained after attaching the ends to M , with values $0, \infty$ at the punctures. As the period lattice \mathcal{P} is not horizontal, the third coordinate function x_3 of \widetilde{M} is multivalued on M but the *height differential* $dh = \frac{\partial x_3}{\partial z} dz$ defines a univalent meromorphic differential on M (here z is a holomorphic coordinate). Since M has finite total curvature and horizontal ends, dh extends to a holomorphic differential on \mathbb{M} . The next statement collects some elementary properties of any surface in \mathcal{S} . Given $v \in \mathcal{P} - \{0\}$, \widetilde{M}/v will stand for the singly periodic minimal surface obtained as the quotient of \widetilde{M} by the translation of vector v .

Proposition 1 *Given $M \in \mathcal{S}$, it holds*

1. $g : \mathbb{M} \rightarrow \overline{\mathbb{C}}$ has degree $2k$, total branching number $4k$, does not take vertical directions on M and it is unbranched at the ends.
2. The period vectors at the ends coincide up to sign and we will denote them by $H = \pm(0, \pi a, 0)$, $a > 0$. The period lattice \mathcal{P} of \widetilde{M} is generated by H and a nonhorizontal

vector $T \in \mathbb{R}^3$, this last one being the period vector along a closed curve $\gamma_1 \subset M$ such that $[\gamma_1] \neq 0$ in the homology group $H_1(\mathbb{M}, \mathbb{Z})$.

3. Let \mathcal{E} be the set of Scherk-type ends of \widetilde{M}/H . Then $(\widetilde{M}/H) \cup \mathcal{E}$ is conformally $\mathbb{C}^* = \mathbb{C} - \{0\}$, and the height differential writes as $dh = c \frac{dz}{z}$ in \mathbb{C}^* , with $c \in \mathbb{R}^* = \mathbb{R} - \{0\}$.

Let $\Pi \subset \mathbb{R}^3$ be a horizontal plane.

4. If Π/H is not asymptotic to an end in \mathcal{E} , then $(\widetilde{M} \cap \Pi)/H$ is transversal and connected. The period vector along $(\widetilde{M} \cap \Pi)/H$ either vanishes or equals $\pm H$.
5. We divide \mathcal{E} in right ends and left ends, depending on whether the flux vector at the corresponding end (with the inward pointing conormal vector) is $(a, 0, 0)$ or $(-a, 0, 0)$, respectively. If Π/H is asymptotic to an end in \mathcal{E} , then $(\widetilde{M} \cap \Pi)/H$ consists of one or two properly embedded arcs. If $(\widetilde{M} \cap \Pi)/H$ reduces to one arc Γ , then both ends of Γ diverge to the same end in \mathcal{E} . In the two arcs intersection case, both arcs travel from one left end to one right end in \mathcal{E} .
6. There exists an embedded closed curve $\gamma_2 \subset M$ such that $\{[\gamma_1], [\gamma_2]\}$ is basis of $H_1(\mathbb{M}, \mathbb{Z})$ and $P_{\gamma_2} = 0$. Up to orientation, γ_2 represents the unique nontrivial homology class in $H_1(\mathbb{M}, \mathbb{Z})$ with associated period zero and an embedded representative.
7. Let $[\gamma] \in H_1(M, \mathbb{Z})$ be a homology class with an embedded representative that generates the homology group of $(\widetilde{M}/H) \cup \mathcal{E}$. Then the third component $(F_\gamma)_3$ of the flux of M along any representative $\gamma \in [\gamma]$ neither vanish nor depends on $[\gamma]$ (up to orientation).

Proof. Since dh has no poles on the torus \mathbb{M} , it cannot have zeros, which implies that the only zeros and poles of g are at the ends. As at each puncture one of the meromorphic differentials $g dh$, $\frac{dh}{g}$ has a simple pole (Meeks and Rosenberg [15]), we conclude that g is unbranched at the ends. Since M has $4k$ ends, g must have degree $2k$ and the Riemann-Hurwitz formula implies that its total branching number is $4k$, so statement 1 is proved.

Denote by T_1, T_2 two generators of \mathcal{P} . As M is properly embedded with horizontal ends, we can assume that T_1 is not horizontal and T_2 is horizontal. It follows that the period vector at an end E of M is $H = nT_2$ with $n \in \mathbb{Z} - \{0\}$, and that T_1 is the period vector along a closed curve $\gamma_1 \subset M$ with $[\gamma_1] \neq 0$ in $H_1(\mathbb{M}, \mathbb{Z})$. Part 2 of the Proposition

will be proved if we show that $n = \pm 1$, by taking $T = T_1$. Since E is invariant by T_2 and the period vector of E is nT_2 , it follows that g has branching order $|n| - 1$ at the puncture. As g is unbranched at the ends, we deduce that $n = \pm 1$ as desired.

Since M is a finitely punctured torus, the singly periodic surface \widetilde{M}/H is a cylinder with infinitely many points removed. Note that the third coordinate function x_3 is a well defined harmonic function on \widetilde{M}/H that extends smoothly through \mathcal{E} , giving rise to a proper harmonic function on $(\widetilde{M}/H) \cup \mathcal{E}$. Therefore, $(\widetilde{M}/H) \cup \mathcal{E}$ is conformally $\mathbb{C}^* = \mathbb{C} - \{0\}$ and x_3 writes as $x_3(z) = c \ln |z| + c'$ with $c \in \mathbb{R}^*$, $c' \in \mathbb{R}$, which is statement 3 of the Proposition. This description of x_3 implies that $(\widetilde{M} \cap \Pi)/H$ corresponds in the \mathbb{C}^* -model to a possible punctured circle $C_r = \{|z| = r\}$ for certain $r > 0$. The hypotheses in item 4 correspond to the case that C_r does not contain ends in \mathcal{E} , so the conclusion of 4 is clear. Under the hypotheses of item 5, C_r contains at most two ends of \mathcal{E} (one left end and/or one right end because ends of the same side are separated by heights), and 5 also holds easily.

In order to see item 6, let β be a compact horizontal level section of M . By the description above, β generates the homology of $(\widetilde{M}/H) \cup \mathcal{E} = \mathbb{C}^*$. Since the period vector $T = P_{\gamma_1}$ is not horizontal, we conclude that $\{[\gamma_1], [\beta]\}$ is a basis of $H_1(\mathbb{M}, \mathbb{Z})$. If $P_\beta = 0$, then the first assertion in 6 is proved with $\gamma_2 = \beta$. If $P_\beta \neq 0$, then $P_\beta = \pm H$ by item 4. In this case we choose as γ_2 an embedded closed curve in M homologous to β in \mathbb{C}^* such that $\beta \cup \gamma_2$ bounds just an end with period vector $-P_\beta$. Finally, suppose that $[\Gamma] \in H_1(\mathbb{M}, \mathbb{Z}) - \{0\}$ has an embedded representative Γ (which can be assumed to lie in M) with $P_\Gamma = 0$. Since γ_1, γ_2 and small loops $\alpha_1, \dots, \alpha_{4k}$ around the punctures generate $H_1(M, \mathbb{Z})$, we can write

$$[\Gamma] = a_1[\gamma_1] + a_2[\gamma_2] + \sum_{i=1}^{4k} b_i[\alpha_i] \quad \text{in } H_1(M, \mathbb{Z}) \quad (1)$$

for integers $a_1, a_2, b_1, \dots, b_{4k}$. Taking periods in (1) and having in mind that $P_{\gamma_1} = T$ and that the periods at the ends are $\pm H$, we obtain $0 = P_\Gamma = a_1 T + bH$, where $b \in \mathbb{Z}$. As T, H are linearly independent, it follows $a_1 = 0$. Now (1) implies that $[\Gamma] = a_2[\gamma_2]$ in $H_1(\mathbb{M}, \mathbb{Z})$, so the embeddedness of Γ forces a_2 to be ± 1 . This proves 6.

Finally, recall we have shown that any compact horizontal level section β of M is an embedded closed curve such that $\{[\gamma_1], [\beta]\}$ is a basis for $H_1(\mathbb{M}, \mathbb{Z})$. As the conormal vector to M along β has third coordinate with constant sign, we have $(F_\beta)_3 \neq 0$. Note that if

$\gamma \subset M$ is an embedded closed curve homologous to β in $(\widetilde{M}/H) \cup \mathcal{E}$, then $\gamma \cup \beta$ bounds a finite number of ends in \mathcal{E} , whose fluxes are all horizontal. Thus $(F_\gamma)_3 = (F_\beta)_3$ and the Proposition is proved. \square

Remark 2

- (i) *In general, we cannot expect the curve γ_2 in item 6 above to be a compact horizontal section. For instance, all the horizontal level curves of the standard example $M_{\theta, \frac{\pi}{2}, 0}$ in Section 3 are open arcs when viewed in \mathbb{R}^3 , see Figure 4 right.*
- (ii) *With the notation in Proposition 1, the fact that all the fluxes at the ends of $M \in \mathcal{S}$ point to the x_1 -axis implies that the component $(F_\gamma)_2$ in the direction of the x_2 -axis is also independent (up to orientation) of the homology class $[\gamma] \in H_1(M, \mathbb{Z})$ satisfying the hypotheses of item 7.*

There is a last natural normalization on the surfaces in \mathcal{S} , which we now explain. Given $M \in \mathcal{S}$, Proposition 1 gives a nontrivial homology class in $H_1(M, \mathbb{Z})$ with an embedded representative $\gamma_2 \subset M$ such that $P_{\gamma_2} = 0$ and $(F_{\gamma_2})_3 > 0$. In the sequel, we will always normalize our surfaces so that $(F_{\gamma_2})_3 = 2\pi$, which can be achieved after an homothety. Note that this normalization is independent of the homology class of γ_2 in $H_1(M, \mathbb{Z})$ (up to orientation), see item 7 of Proposition 1.

We label by $\widetilde{\mathcal{S}}$ the set of *marked surfaces* $(M, p_1, \dots, p_{2k}, q_1, \dots, q_{2k}, [\gamma_2])$ where

1. M is a surface in \mathcal{S} whose period lattice is generated by $H, T \in \mathbb{R}^3$, where $H = (0, a, 0)$, $T = (T_1, T_2, T_3)$ and $a, T_3 > 0$;
2. $\{p_1, \dots, p_{2k}\} = g^{-1}(0)$, $\{q_1, \dots, q_{2k}\} = g^{-1}(\infty)$ and the ordered lists $(p_1, q_1, \dots, p_k, q_k)$, $(p_{k+1}, q_{k+1}, \dots, p_{2k}, q_{2k})$ are the two families of “sided” ends of M , both ordered by increasing heights in the quotient;
3. $[\gamma_2] \in H_1(M, \mathbb{Z})$ is the homology class of an embedded closed curve $\gamma_2 \subset M$ satisfying $P_{\gamma_2} = 0$, $(F_{\gamma_2})_3 = 2\pi$. We additionally impose that γ_2 lifts to a curve contained in a fundamental domain of the doubly periodic lifting of M lying between two horizontal planes $\Pi, \Pi + T$.

We will identify in $\widetilde{\mathcal{S}}$ two marked surfaces that differ by a translation that preserves both orientation, the above “sided” ordering of their lists of ends and the associated homology

classes. The same geometric surface in \mathcal{S} can be viewed as different marked surfaces in $\tilde{\mathcal{S}}$. We will simply denote as $M \in \tilde{\mathcal{S}}$ the marked surfaces unless it leads to confusion. $\tilde{\mathcal{S}}$ can be naturally endowed with a topology: a sequence of marked surfaces $\{M_n\}_n \subset \tilde{\mathcal{S}}$ converges to $M \in \tilde{\mathcal{S}}$ if the associated sequence of minimal surfaces $\{M_n\}_n \subset \mathcal{S}$ converges smoothly to $M \in \mathcal{S}$ (in the uniform topology on compact sets), the ordered list of ends associated to M_n converges to the corresponding one for M and the homology classes $[\gamma_{2,n}] \in H_1(M_n, \mathbb{Z})$ in the last component of the marked surfaces M_n have representatives converging uniformly to a representative of the last component of M . With this topology, a geometric minimal surface $M \in \mathcal{S}$ produces a discrete subset in $\tilde{\mathcal{S}}$.

Consider $M \in \tilde{\mathcal{S}}$ with Gauss map g and height differential dh . An elementary calculation gives the periods P_{p_j}, P_{q_j} and fluxes F_{p_j}, F_{q_j} at the ends of M as follows:

$$P_{p_j} + i F_{p_j} = \pi \operatorname{Res}_{p_j} (g^{-1} dh) (i, -1, 0), \quad P_{q_j} + i F_{q_j} = -\pi \operatorname{Res}_{q_j} (g dh) (i, 1, 0), \quad (2)$$

where Res_A denotes the residue of the corresponding meromorphic differential at a point A . The fact that P_{p_j}, P_{q_j} point to the x_2 -axis translates into $\operatorname{Res}_{p_j} (g^{-1} dh), \operatorname{Res}_{q_j} (g dh) \in \mathbb{R}$. By definition of the ordering of the ends of M as a marked surface, we have that

$$\operatorname{Res}_{p_j} (g^{-1} dh) = -\operatorname{Res}_{q_j} (g dh) = \begin{cases} a & (1 \leq j \leq k) \\ -a & (k+1 \leq j \leq 2k), \end{cases} \quad (3)$$

for certain $a \in \mathbb{R}^*$ (the case $a > 0$ corresponds to $p_1, \dots, p_k, q_1, \dots, q_k$ being right ends of M). Recall that $P_{\gamma_2} = 0$ and $(F_{\gamma_2})_3 = 2\pi$. Thus,

$$\overline{\int_{\gamma_2} g^{-1} dh} = \int_{\gamma_2} g dh, \quad \int_{\gamma_2} dh = 2\pi i. \quad (4)$$

3 Standard examples.

We dedicate this Section to introduce briefly the 3-parameter family of standard examples $\mathcal{K} \subset \mathcal{S}$ to which the uniqueness Theorem 1 applies. Some of the properties of \mathcal{K} in this Section are long but straightforward computations that can be found in detail in [23]. As the standard examples with $4k$ -ends are nothing but k -sheeted coverings of 4-ended standard examples in \mathcal{K}_1 , we will concentrate on these last ones. Each $M_{\theta, \alpha, \beta} \in \mathcal{K}_1$ is determined by the 4 branch values of its Gauss map, which consist of two pairs of antipodal points $D, D', D'' = -D, D''' = -D'$ in the sphere \mathbb{S}^2 . Since the Gauss map

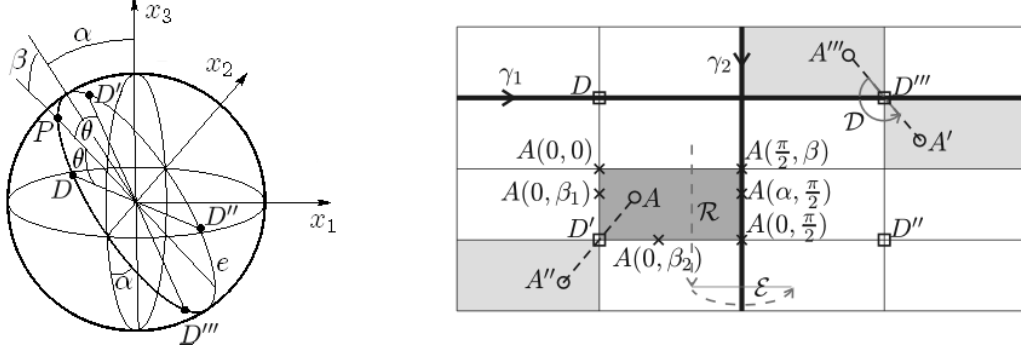


Figure 1: Left: Spherical configuration of $M_{\theta, \alpha, \beta}$. Right: Behavior of the pole $A(\alpha, \beta)$ of g in the dark shaded rectangle \mathcal{R} (here, $0 < \beta_1 < \theta < \beta_2 < \frac{\pi}{2}$, $\alpha \in (0, \frac{\pi}{2})$ and $\beta \in [0, \frac{\pi}{2})$). The remaining ends of $M_{\theta, \alpha, \beta}$ move in the light shaded rectangles as α, β vary.

of any surface in \mathcal{S} is unbranched at the ends, D, D', D'', D''' must be different from the North and South Poles. We also let e be the equator in \mathbb{S}^2 that contains D, D', D'', D''' . Given a point $P \in e$, the branch values of the Gauss map can be determined by giving only one angle $\theta \in (0, \frac{\pi}{2})$, in such a way that the position vectors of D, D' form an angle of 2θ and the position vector of P bisects such an angle. We will call a *spherical configuration* to any set $\{D, D', D'', D'''\}$ as above.

Given $(\theta, \alpha, \beta) \in (0, \frac{\pi}{2}) \times [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ with $(\alpha, \beta) \neq (0, \theta)$, we define the spherical configuration of the potential standard example $M_{\theta, \alpha, \beta}$ by means of the angle θ , the equator e and the point P as follows:

1. Let e_0 be the inverse image of the imaginary axis in \mathbb{C} through the stereographic projection from the North Pole of \mathbb{S}^2 . Then e is the image of e_0 through the rotation by angle α around the x_2 -axis.
2. P is the image of the North Pole through the composition of a rotation by angle β around the x_1 -axis with a rotation by angle α around the x_2 -axis, see Figure 1 left.

Projecting stereographically from the North Pole the branch points D, D', D'', D''' for the spherical configuration associated to $(\theta, 0, 0)$, one just finds the four roots of the polynomial $(z^2 + \lambda^2)(z^2 + \lambda^{-2})$ where $\lambda = \lambda(\theta) = \cot \frac{\theta}{2}$. Therefore, the underlying conformal compactification of the potential surface $M_{\theta, 0, 0}$ is the rectangular torus

$$\Sigma_\theta = \{(z, w) \in \overline{\mathbb{C}}^2 \mid w^2 = (z^2 + \lambda^2)(z^2 + \lambda^{-2})\},$$

and its extended Gauss map is the z -projection $(z, w) \in \Sigma_\theta \mapsto z \in \overline{\mathbb{C}}$ on Σ_θ . Note that the spherical configuration for angles (θ, α, β) differs from the one associated to $(\theta, 0, 0)$ in a Möbius transformation φ . Thus the compactification of any $M_{\theta, \alpha, \beta}$, which is the branched covering of \mathbb{S}^2 through its Gauss map, is Σ_θ . Furthermore, the composition of the Gauss map of $M_{\theta, 0, 0}$ with φ gives the Gauss map of the potential example $M_{\theta, \alpha, \beta}$:

$$g(z, w) = \frac{z \left(i \cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right) \right) + \sin\left(\frac{\alpha-\beta}{2}\right) + i \sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right) + i \cos\left(\frac{\alpha+\beta}{2}\right) - z \left(i \sin\left(\frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha+\beta}{2}\right) \right)}, \quad (z, w) \in \Sigma_\theta,$$

and its ends are $\{A, A', A'', A'''\} = g^{-1}(\{0, \infty\})$. As the height differential dh of $M_{\theta, \alpha, \beta}$ is a holomorphic 1-form on Σ_θ , we have $dh = \mu \frac{dz}{w}$ for certain $\mu = \mu(\theta, \alpha, \beta) \in \mathbb{C}^*$. It will be also useful to have a second representation of Σ_θ as a quotient of the ξ -plane \mathbb{C} by two orthogonal translations. Let $\Omega \subset \Sigma_\theta$ be one of the two connected components of $\{(z, w) \in \Sigma_\theta \mid |z| > 1, -\frac{\pi}{2} < \arg(z) < 0\}$. Ω is topologically a disk and its boundary contains just one branch point $D_1 = (-\lambda i, w(-\lambda i))$ and one pole $A_1 = (\infty, \infty)$ of the z -projection. Let Ω' be an open rectangle of consecutive vertices $A, B, C, D \in \mathbb{C}$ with the segment \overline{AB} being horizontal, such that there exists a biholomorphism $\xi : \{|z| > 1, -\frac{\pi}{2} < \arg(z) < 0\} \rightarrow \Omega'$ with boundary values $\xi(\infty) = A$, $\xi(1) = B$, $\xi(-i) = C$ and $\xi(-\lambda i) = D$. Then the composition of the z -projection with ξ defines a biholomorphism between Ω and Ω' . After symmetric extension of this biholomorphism across the boundary curves of Ω, Ω' we will get a biholomorphism from Σ_θ to the quotient of the ξ -plane modulo the translations given by four times the sides of the rectangle Ω' , see Figure 2. Note that if $\alpha = \beta = 0$, then $g(z, w) = z$ and the equalities $\xi(\infty) = A$, $\xi(-\lambda i) = D$ justify to use the same symbols A, D previously defined for two of the corners of the rectangle Ω' .

The above identification allows us to see the ends A, A', A'', A''' as functions of $(\alpha, \beta) \in [0, \frac{\pi}{2}]^2 - \{(0, \theta)\}$ valued in the ξ -plane model of Σ_θ . $A(\alpha, \beta)$ moves on the dark shaded rectangle \mathcal{R} in Figure 1 right. For $(\alpha, \beta) = (0, 0)$, we choose $A(0, 0)$ as the up-left corner of \mathcal{R} . As α increases in $[0, \frac{\pi}{2}]$, the end $A(\alpha, 0)$ of $M_{\theta, \alpha, 0}$ moves continuously and horizontally to the right, until reaching for $\alpha = \frac{\pi}{2}$ the up-right corner of \mathcal{R} . The curve $\beta \in [0, \theta] \mapsto A(0, \beta)$ parametrizes downwards the left vertical boundary edge of \mathcal{R} , until reaching the branch point D' . The curve $\beta \in (\theta, \frac{\pi}{2}] \mapsto A(0, \beta)$ parametrizes the lower horizontal edge of $\partial\mathcal{R}$ from D' until the down-right corner of \mathcal{R} . The curve $\alpha \in [0, \frac{\pi}{2}] \mapsto A(\alpha, \frac{\pi}{2})$ parametrizes upwards the right vertical edge of $\partial\mathcal{R}$. Finally, the curve $\beta \in [0, \frac{\pi}{2}] \mapsto A(\frac{\pi}{2}, \beta)$ is constant at the up-right corner of \mathcal{R} . These boundary values of $A(\alpha, \beta)$ extend continuously and

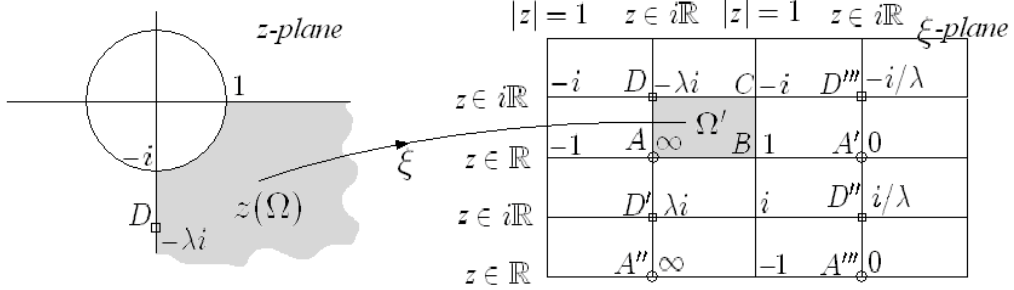


Figure 2: The biholomorphism ξ between the shaded regions. We have included some values of the z -projection in the ξ -plane model of Σ_θ .

bijectionally from $(0, \frac{\pi}{2})^2$ to the interior of \mathcal{R} . The behavior of the remaining three ends of $M_{\theta, \alpha, \beta}$ on the ξ -plane model can be deduced from $A(\alpha, \beta)$ by using the isometry group $\text{Iso}(\theta, \alpha, \beta)$ of the induced metric $ds^2 = \frac{1}{4}(|g| + |g|^{-1})^2 |dh|^2$, which we now investigate.

First note that the identity in \mathbb{S}^2 lifts via g to two different isometries of ds^2 , namely the identity in Σ_θ and the deck transformation $\mathcal{D}(z, w) = (z, -w)$, both restricted to $\Sigma_\theta - g^{-1}(\{0, \infty\})$. \mathcal{D} corresponds in the ξ -plane to the 180° -rotation about any of the branch points of the z -projection. The antipodal map \aleph on \mathbb{S}^2 also leaves invariant both the spherical configuration of $M_{\theta, \alpha, \beta}$ and the set $\{(0, 0, \pm 1)\}$, and the equality $[w(\frac{-1}{z})]^2 = \frac{[w(z)]^2}{z^4}$ for any $(z, w) \in \Sigma_\theta$ implies that \aleph lifts through g to two isometries of ds^2 , which we call \mathcal{E} and $\mathcal{F} = \mathcal{D} \circ \mathcal{E}$. Both \mathcal{E}, \mathcal{F} are antiholomorphic involutions of Σ_θ without fixed points. This property implies that one of these involutions, say \mathcal{E} , corresponds in the ξ -plane model to the composition of the translation by the vector $\overline{D}, \overline{D'}$ with the symmetry with respect to the right vertical edge of $\partial\mathcal{R}$, see Figure 1 right. The remaining ends of $M_{\theta, \beta, \alpha}$ in terms of $A = A(\alpha, \beta)$ are (up to relabeling)

$$A'' = \mathcal{D}(A), \quad A''' = \mathcal{E}(A), \quad A' = \mathcal{D}(A'''). \quad (5)$$

We now study the period problem for $M_{\theta, \alpha, \beta}$. Recall that $dh = \mu \frac{dz}{w}$ on Σ_θ with $\mu \in \mathbb{C}^*$. From now on, take $\mu \in \mathbb{R}^*$. The period P_A and flux F_A of $M_{\theta, \alpha, \beta}$ at the end A with $g(A) = \infty$ are given by

$$P_A = \pi\mu(i E(\theta, \alpha, \beta), 0), \quad F_A = \pi\mu(E(\theta, \alpha, \beta), 0), \quad (6)$$

where we have used the identification $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$ by $(a, b, c) \equiv (a + ib, c)$, and $E(\theta, \alpha, \beta) = [\cos^2 \alpha + \csc^2 \theta (\sin \alpha \cos \beta - i \sin \beta)^2]^{-1/2}$ (we have chosen a branch of w for computing

(6), which only affects the result up to sign). The periods and fluxes at A', A'', A''' can be easily obtained using (5), (6) and the pullbacks by \mathcal{D}, \mathcal{E} of the Weierstrass form $\Phi = (\frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1)dh$ of $M_{\theta, \alpha, \beta}$:

$$\mathcal{D}^* \Phi = -\Phi, \quad \mathcal{E}^* \Phi = -\bar{\Phi}. \quad (7)$$

Concerning the period problem in homology, let γ_1, γ_2 be the simple closed curves in Σ_θ obtained respectively as quotients of the horizontal and vertical lines in the ξ -plane passing through D, D''' and through the right vertical edge of $\partial\mathcal{R}$ (see Figure 1 right). Clearly $\{[\gamma_1], [\gamma_2]\}$ is a basis of $H_1(\Sigma_\theta, \mathbb{Z})$. Both γ_1, γ_2 miss the ends A, A', A'', A''' except for certain extreme values of (α, β) (for instance, $A(\pi/2, 0)$ lies in γ_2), but one can think of the homology classes $[\gamma_1], [\gamma_2] \in H_1(M_{\theta, \alpha, \beta}, \mathbb{Z})$ as being independent of (α, β) . Furthermore,

$$\mathcal{E}_*[\gamma_1] = -[\gamma_1] - [\gamma_A] - [\gamma_{A'}], \quad \mathcal{E}_*[\gamma_2] = [\gamma_2] \quad \text{in } H_1(M_{\theta, \alpha, \beta}, \mathbb{Z}), \quad (8)$$

where γ_X denotes a small loop around a point $X \in \Sigma_\theta$. Equations (5), (7) and the first equality in (8) imply $-\overline{\int_{\gamma_1} \Phi} = -\int_{\gamma_1} \Phi - \int_{\gamma_A} \Phi + \overline{\int_{\gamma_{A'}} \Phi}$. Taking imaginary parts, we find

$$F_{\gamma_1} = -F_{\gamma_A} = -F_A. \quad (9)$$

Similarly, the second equalities in (7), (8) insure that $\int_{\gamma_2} \Phi = -\overline{\int_{\gamma_2} \Phi}$, which real part gives

$$P_{\gamma_2} = 0. \quad (10)$$

Equation (10) implies that we can take γ_2 as the embedded closed curve appearing in item 6 of Proposition 1 (except for the extreme values of (α, β) mentioned above, in which we deform γ_2 keeping $[\gamma_2] \in H_1(M_{\theta, \alpha, \beta}, \mathbb{Z})$ constant in α, β). As dh is holomorphic and nontrivial on Σ_θ , (10) also implies that $\int_{\gamma_2} dh \in i\mathbb{R}^*$, so we must rescale to have $\int_{\gamma_2} dh = 2\pi i$ as part of the normalization of our surfaces in \mathcal{S} . This rescaling gives μ :

$$\mu = \mu(\theta) = \frac{\pi}{2 \int_{-\pi/2}^0 \sqrt{2 \cos(2t) + \lambda^2(\theta) + \lambda^{-2}(\theta)} dt} = \frac{\pi}{\sin \theta \text{EllipticK}(\sin^2 \theta)}, \quad (11)$$

where $\text{EllipticK}(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 u} du$ is the complete elliptic integral of the first kind. Since $\Re \int_{\gamma_2} dh = 0$ and dh is holomorphic on Σ_θ , we deduce that P_{γ_1} has nonvanishing third component. In particular, P_A and P_{γ_1} are linearly independent. All these facts imply

that $M_{\theta,\alpha,\beta}$ is a complete immersed doubly periodic minimal torus with four horizontal embedded Scherk-type ends and period lattice generated by P_A, P_{γ_1} .

We claim that $M_{\theta,\alpha,\beta}$ is embedded for any (θ, α, β) . First note that $M_{\theta,0,0}$ is the toroidal halfplane layer defined by Karcher in [9], who proved that $M_{\theta,0,0}$ decomposes in 16 congruent disjoint pieces, each one being the conjugate surface of certain Jenkins-Serrin graph. In particular, $M_{\theta,0,0}$ is embedded for each $\theta \in (0, \frac{\pi}{2})$. Since for θ fixed the heights of the ends of $M_{\theta,\alpha,\beta}$ depend continuously on (α, β) in the connected set $[0, \frac{\pi}{2}]^2 - \{(0, \theta)\}$ (this is clear in the ξ -plane model), a standard application of the maximum principle insures that $M_{\theta,\alpha,\beta}$ is embedded for all values of θ, α, β .

Remark 3 *Since P_A does not necessarily point to the x_2 -axis, we must possibly rotate $M_{\theta,\alpha,\beta}$ by a suitable angle around the x_3 -axis in order to see it in \mathcal{S} .*

Next we study the conjugate surface $M_{\theta,\alpha,\beta}^*$ of $M_{\theta,\alpha,\beta}$. Since the flux (resp. the period) of the conjugate surface along a given curve in the parameter domain equals the period (resp. the opposite of the flux) of the original surface along the same curve, we deduce from (6), (9) and (10) that the period of $M_{\theta,\alpha,\beta}^*$ at its ends is $\pm\pi\mu(E(\theta, \alpha, \beta), 0) \in \mathbb{C} \times \mathbb{R}$, its period along γ_1 is $\pi\mu(E(\theta, \alpha, \beta), 0)$ and the third component of the period of $M_{\theta,\alpha,\beta}^*$ along γ_2 is -2π . Therefore, $M_{\theta,\alpha,\beta}^*$ is a complete immersed doubly periodic torus with four horizontal embedded Scherk-type ends, whose period lattice is generated by $\pi\mu(E(\theta, \alpha, 0), 0)$ and $(F(\gamma_2), 2\pi)$, where $F(\gamma_2) \in \mathbb{C}$ denotes the horizontal part of the flux of $M_{\theta,\alpha,\beta}$ along γ_2 . Again any $M_{\theta,\alpha,\beta}^*$ is embedded by the maximum principle and the embeddedness of $M_{\theta,0,0}^*$ (which decomposes in 16 congruent Jenkins-Serrin graphs). Also note that in order to see $M_{\theta,\alpha,\beta}^*$ inside the normalized space \mathcal{S} , we must rotate this surface around the x_3 -axis and rescale it suitably. After this identification, the curve with period zero in the sense of item 6 of Proposition 1 can be taken as $\gamma_2^* = \gamma_1 + \gamma_A \subset M_{\theta,\alpha,\beta}^*$.

We have defined two families of examples $\mathcal{A} = \{M_{\theta,\alpha,\beta}\}$, $\mathcal{A}^* = \{M_{\theta,\alpha,\beta}^*\}$ inside \mathcal{S} , with (θ, α, β) varying in $\mathcal{I} = \{(\theta, \alpha, \beta) \in (0, \frac{\pi}{2}) \times [0, \frac{\pi}{2}]^2 \mid (\alpha, \beta) \neq (0, \theta)\}$. Clearly this definition can be extended to larger ranges in (θ, α, β) , but such an extension only produces symmetric images of these surfaces with respect to certain planes orthogonal to the x_1, x_2 or x_3 -axes. Nevertheless, some of these geometrically equivalent surfaces are considered as distinct points in the space $\tilde{\mathcal{S}}$ defined in Section 2. Another interesting property is that $\mathcal{A}^* = \mathcal{A}$ (modulo the aforementioned identifications), as the following Lemma shows. In this sense, we can assure that the space of standard examples is *self-conjugate*.

Lemma 1 Given $(\theta, \alpha, \beta) \in \mathcal{I}$, the surface $M_{\theta, \alpha, \beta}^*$ coincides with $M_{\frac{\pi}{2}-\theta, \alpha, \frac{\pi}{2}-\beta}$ up to a symmetry in a plane orthogonal to the x_2 -axis.

Proof. Fix $(\theta, \alpha, \beta) \in \mathcal{I}$. By direct calculation, $\Sigma_{\frac{\pi}{2}-\theta} = \{(\tilde{z}, \tilde{w}) \mid \tilde{w}^2 = (\tilde{z}^2 - 1)^2 + 4\tilde{z}^2 \sec^2 \theta\}$. Since the Möbius transformation $\varphi(z) = \frac{1-iz}{-i+z}$ applies the set of branch points of the z -projection of Σ_θ bijectively on the set of branch points of the \tilde{z} -projection of $\Sigma_{\frac{\pi}{2}-\theta}$, it follows that $\Delta(z, w) = (\varphi(z), \tilde{w}(\varphi(z)))$ is a biholomorphism between Σ_θ and $\Sigma_{\frac{\pi}{2}-\theta}$. A simple thought of the role of the angle β gives that if we extend the definition of the standard examples to $\beta' \in [-\frac{\pi}{2}, 0)$, then $M_{\theta, \alpha, \beta'}$ is nothing but the reflected image of $M_{\theta, \alpha, -\beta'}$ with respect to the (x_1, x_3) -plane (up to a translation). Furthermore, it is straightforward to check that $g_{\theta, \alpha, -\beta} = g_{\frac{\pi}{2}-\theta, \alpha, \frac{\pi}{2}-\beta} \circ \Delta$, where the subindex for g means the parameter angles of the standard example whose Gauss map is g . Denoting by dh_θ its height differential (which only depends on θ), a direct computation gives $\Delta^* dh_{\frac{\pi}{2}-\theta} = \mu(\frac{\pi}{2} - \theta) \frac{\varphi'(z) dz}{\tilde{w}(\varphi(z))} = i \frac{\mu(\frac{\pi}{2}-\theta)}{\tan \theta} \frac{dz}{w} = i \frac{\mu(\frac{\pi}{2}-\theta)}{\mu(\theta) \tan \theta} dh_\theta$. Now the Lemma is proved. \square

One could ask for which angles (θ, α, β) the surface $M_{\theta, \alpha, \beta}$ is self-conjugate (i.e. when it is congruent to its conjugate surface). First note that for any $\theta \in (0, \frac{\pi}{2})$, $M_{\theta, \frac{\pi}{2}, \beta}$ does not depend on $\beta \in [0, \frac{\pi}{2}]$ since its spherical configuration is just the rotation image of the one of $M_{\theta, \frac{\pi}{2}, 0}$ around the x_3 -axis by angle β . Having this in mind and using Lemma 1 we have that the range of angles for which $M_{\theta, \alpha, \beta}$ is self-conjugate is $(\{\frac{\pi}{4}\} \times [0, \frac{\pi}{2}] \times \{\frac{\pi}{4}\}) \cup (\{\frac{\pi}{4}\} \times \{\frac{\pi}{2}\} \times [0, \frac{\pi}{2}])$.

As the branch values of the Gauss map of any $M_{\theta, \alpha, \beta}$ lie on a spherical equator, a result by Montiel and Ros [17] insures that the space of bounded Jacobi functions on $M_{\theta, \alpha, \beta}$ is 3-dimensional (they reduce to the linear functions of the Gauss map), a condition usually referred in literature as the *nondegeneracy* of $M_{\theta, \alpha, \beta}$. This nondegeneracy can be interpreted by means of an Implicit Function Theorem argument to obtain that around $M_{\theta, \alpha, \beta}$, the space \mathcal{S} is a 3-dimensional real analytic manifold (Hauswirth and Traizet [5]); in particular, the only elements in \mathcal{S} around a standard example are themselves standard. This local uniqueness result will be extended in the large by Theorem 1 in this paper.

We finish this section by summarizing some additional properties of the standard examples, that can be checked using their Weierstrass representation. For details, see [23].

Proposition 2

1. For any $\theta \in (0, \frac{\pi}{2})$, $M_{\theta, 0, 0}$ admits 3 reflection symmetries S_1, S_2, S_3 in orthogonal planes

and contains a straight line parallel to the x_1 -axis, that induces a 180° -rotation symmetry R_D . The isometry group $\text{Iso}(\theta, 0, 0)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$, with generators S_1, S_2, S_3, R_D (see Figure 3 left).

2. For any $(\theta, \alpha) \in (0, \frac{\pi}{2})^2$, $M_{\theta, \alpha, 0}$ is invariant by a reflection S_2 in a plane orthogonal to the x_2 -axis and by a 180° -rotation R_2 around a line parallel to the x_2 -axis that cuts the surface orthogonally (Figure 4 left). $\text{Iso}(\theta, \alpha, 0)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, with generators S_2, R_2, \mathcal{D} . Furthermore, $\text{Iso}(\theta, \frac{\pi}{2}, 0) = \text{Iso}(\theta, 0, 0)$ (Figure 4 right).
3. For any $(\theta, \beta) \in (0, \frac{\pi}{2})^2 - \{(\theta, \theta)\}$, $M_{\theta, 0, \beta}$ is invariant by a reflection S_1 in a plane orthogonal to the x_1 -axis, by a 180° -rotation symmetry R_D around a straight line parallel to the x_1 -axis contained in the surface, and by a 180° -rotation R_1 around another line parallel to the x_1 -axis that cuts the surface orthogonally. $\text{Iso}(\theta, 0, \beta)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, with generators S_1, R_D, R_1 (Figure 3 right). Furthermore, $\text{Iso}(\theta, 0, \frac{\pi}{2}) = \text{Iso}(\theta, 0, 0)$.
4. For any $(\theta, \alpha, \beta) \in (0, \frac{\pi}{2})^3$, $\text{Iso}(\theta, \alpha, \beta)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ with generators \mathcal{D}, \mathcal{E} .
5. For any $(\theta, \alpha) \in (0, \frac{\pi}{2})^2$, $M_{\theta, \alpha, \frac{\pi}{2}}$ is invariant by a reflection S_3 in a plane orthogonal to the x_2 -axis and by a 180° -rotation R_3 around a line parallel to the x_2 -axis that intersects $M_{\theta, \alpha, \frac{\pi}{2}}$ orthogonally. $\text{Iso}(\theta, \alpha, \frac{\pi}{2})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, with generators S_3, R_3, \mathcal{D} .
6. When $(\theta, \alpha, \beta) \rightarrow (0, 0, 0)$, $M_{\theta, \alpha, \beta}$ converges smoothly to two vertical catenoids, both with flux $(0, 0, 2\pi)$.
7. Let $\theta_0 \in (0, \frac{\pi}{2})$. When $(\theta, \alpha, \beta) \rightarrow (\theta_0, 0, \theta_0)$, $M_{\theta, \alpha, \beta}$ converges to a Riemann minimal example with two horizontal ends, vertical part of its flux 2π and branch values of its Gauss map at $0, \infty, i \tan \theta_0, -i \cot \theta_0 \in \overline{\mathbb{C}}$.
8. When $(\theta, \alpha, \beta) \rightarrow (\frac{\pi}{2}, 0, \frac{\pi}{2})$, $M_{\theta, \alpha, \beta}$ converges (after blowing up) to two vertical helicoids.
9. Let $(\alpha_0, \beta_0) \in [\theta, \frac{\pi}{2}]^2 - \{(0, 0)\}$. When $(\theta, \alpha, \beta) \rightarrow (0, \alpha_0, \beta_0)$, $M_{\theta, \alpha, 0}$ converges to two singly periodic Scherk minimal surfaces, each one with two horizontal ends and two ends forming angle $\arccos(\cos \alpha_0 \cos \beta_0)$ with the horizontal.

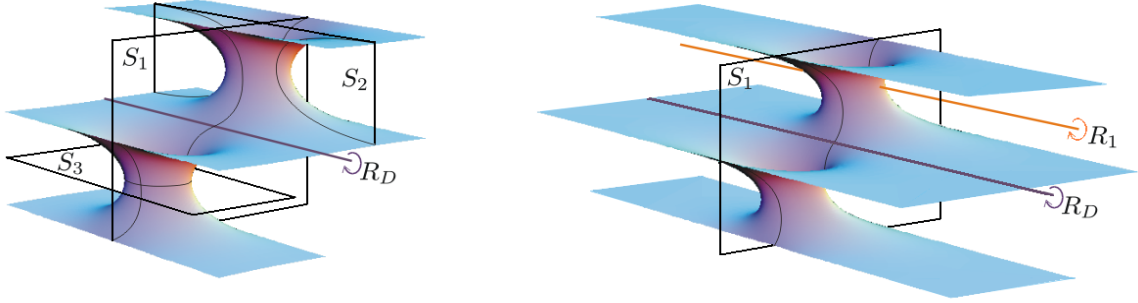


Figure 3: Left: The toroidal halfplane layer $M_{\pi/4,0,0}$. Right: The surface $M_{\pi/4,0,\pi/8}$.

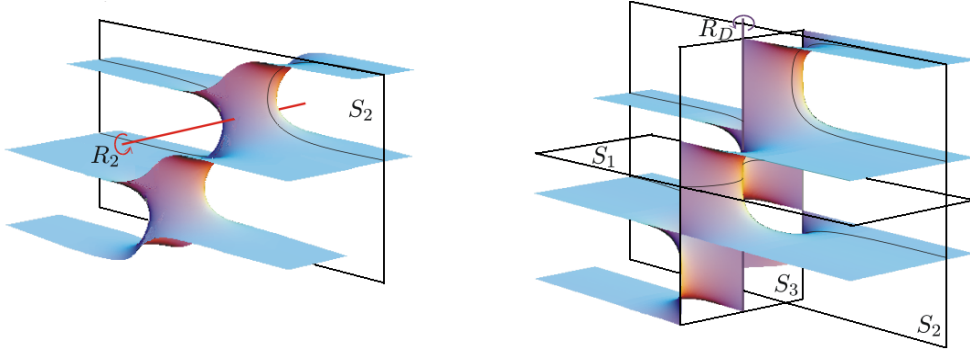


Figure 4: Left: The surface $M_{\pi/4,\pi/4,0}$. Right: $M_{\pi/4,\pi/2,0}$, all whose horizontal level curves are open arcs in \mathbb{R}^3 .

10. Let $(\alpha_0, \beta_0) \in [\theta, \frac{\pi}{2}]^2 - \{(0, \frac{\pi}{2})\}$. When $(\theta, \alpha, \beta) \rightarrow (\frac{\pi}{2}, \alpha_0, \beta_0)$, $M_{\theta,\alpha,0}$ converges (after blowing up) to two doubly periodic Scherk minimal surfaces, each one with two horizontal ends and two ends forming angle $\arccos(\cos \alpha_0 \sin \beta_0)$ with the horizontal.

4 The moduli space \mathcal{W} of Weierstrass representations.

Any surface in the hypotheses of Theorem 1 can be seen as an element of a finite dimensional complex manifold by means of the *Hurwitz schemes*, a process which endows families of meromorphic functions of prescribed degree with structures of finite dimensional complex manifolds, see for instance [3]. This general construction is simpler in our setting, as we now explain.

We will call \mathcal{W} to the space of lists $(\mathbb{M}, g, p_1, \dots, p_{2k}, q_1, \dots, q_{2k}, [\gamma])$, where $g : \mathbb{M} \rightarrow \overline{\mathbb{C}}$ is a meromorphic degree $2k$ function defined on a torus \mathbb{M} , which is unbranched at its zeros $\{p_1, \dots, p_{2k}\} = g^{-1}(0)$ and poles $\{q_1, \dots, q_{2k}\} = g^{-1}(\infty)$, and $[\gamma]$ is a homology class in $\mathbb{M} - \{p_1, \dots, p_{2k}, q_1, \dots, q_{2k}\}$ with $[\gamma] \neq 0$ in $H_1(\mathbb{M}, \mathbb{Z})$. Note that the same map g can be viewed as an infinite subset of \mathcal{W} by considering different orderings on the points p_j, q_j and different homology classes $[\gamma]$. This infinite subset associated to the same g will be discrete with the topology to be defined below. We will simply denote by g the elements of \mathcal{W} , which will be referred to as *marked meromorphic maps*.

Next we endow \mathcal{W} with a topology. Given $g \in \mathcal{W}$, let $b_1, \dots, b_l \in \mathbb{C}^*$, $l \leq 4k$, be the distinct branch values of g (by the Riemann-Hurwitz formula, g has $4k$ branch points counting multiplicity). Take l small pairwise disjoint disks D_i in \mathbb{C}^* centered at the points b_i and denote by $\Omega = \overline{\mathbb{C}} - \cup_{i=1}^l D_i$. By taking the radii of these disks small enough, we can assume that both Ω and $g^{-1}(\Omega)$ are connected and each component of $g^{-1}(D_i)$ is a disk ($1 \leq i \leq l$). Therefore, $g : g^{-1}(\Omega) \rightarrow \Omega$ is an unbranched $2k$ -sheeted covering with connected total space, and each component of $g^{-1}(D_i)$ contains at most one branch point of g , possibly of high multiplicity. With these data g, D_1, \dots, D_l we define a neighborhood $U(g)$ of g in \mathcal{W} as the set of marked meromorphic maps $f \in \mathcal{W}$ such that:

- (1) $f|_{f^{-1}(\Omega)}$ is unbranched and the restrictions $f : f^{-1}(\Omega) \rightarrow \Omega$, $g : g^{-1}(\Omega) \rightarrow \Omega$ are isomorphic as covering maps. This allows us to identify conformally $f^{-1}(\Omega)$ and $g^{-1}(\Omega)$, so that zeros (resp. poles) of g identify with zeros (resp. poles) of f .
- (2) Each component of $f^{-1}(D_i)$ is a disk in the torus $f^{-1}(\overline{\mathbb{C}})$. These disks are in bijective correspondence with those of g by means of the identification in (1), and the total branching order of g, f at corresponding disks necessarily coincides.
- (3) The identification in (1) gives a bijection between the zeros (resp. poles) of g and those of f . We impose that the ordering in the set of zeros and poles of the marked meromorphic map f coincides with the corresponding one for g through this bijection.
- (4) The homology class $[\gamma] \in H_1(g^{-1}(\mathbb{C}^*), \mathbb{Z})$ can be represented by an embedded cycle $\gamma \in g^{-1}(\Omega)$, which by (1) can be also seen as an embedded cycle in $f^{-1}(\Omega)$. Such a cycle induces an homology class in $f^{-1}(\mathbb{C}^*)$, which we denote $[\gamma]$ as well. It is easy to show that $[\gamma] \neq 0$ in $H_1(f^{-1}(\mathbb{C}^*), \mathbb{Z})$. We impose that the homology class associated to the marked meromorphic map f coincides with $[\gamma]$.

Next we endow \mathcal{W} with a structure of $4k$ -dimensional complex manifold. The topology defined above implies that if $g \in \mathcal{W}$ has $4k$ distinct branch values, then every $f \in \mathcal{W}$ close enough to g has also $4k$ distinct branch values. In this case, the map that applies this f to its list of branch values (choosing an ordering) is a local chart for \mathcal{W} around g . Around a generic $g \in \mathcal{W}$, a local chart can be obtained by exchanging the branch values of g by the list $(\sigma_1(g), \dots, \sigma_{4k}(g))$, where $\sigma_i(g)$ is the value of the symmetric elementary polynomial of degree i on the unordered list of $4k$ (not necessarily distinct) branch values of g , $1 \leq i \leq 4k$. These symmetric elementary polynomials can be considered as globally defined holomorphic functions $\sigma_i : \mathcal{W} \rightarrow \mathbb{C}$, $1 \leq i \leq 4k$. Also, the map $(\sigma_1, \dots, \sigma_{4k}) : \mathcal{W} \rightarrow \mathbb{C}^{4k}$ is a local diffeomorphism, hence \mathcal{W} can be seen as an open submanifold of \mathbb{C}^{4k} .

The following result deals with compact analytic subvarieties of \mathcal{W} . A subset V of a complex manifold N is said to be an *analytic subvariety* if for any $p \in N$ there exists a neighborhood U of p in N and a finite number of holomorphic functions f_1, \dots, f_r on U such that $U \cap V = \{q \in U \mid f_i(q) = 0, 1 \leq i \leq r\}$. The proof of the following lemma can be found in [14], Lemma 4 (although the definition of \mathcal{W} in [14] is different, the proof still works in our case).

Lemma 2 *The only compact analytic subvarieties of \mathcal{W} are finite subsets.*

4.1 The height differential associated to a marked meromorphic map.

Consider a marked meromorphic map $g = (\mathbb{M}, g, p_1, \dots, p_{2k}, q_1, \dots, q_{2k}, [\gamma]) \in \mathcal{W}$. Since the complex space of holomorphic differentials on \mathbb{M} is 1-dimensional, there exists a unique holomorphic 1-form $\phi = \phi(g)$ on \mathbb{M} such that

$$\int_{\gamma} \phi = 2\pi i. \tag{12}$$

The pair (g, ϕ) must be seen as the Weierstrass data of a potential surface in the setting of Theorem 1, defined on $M = g^{-1}(\mathbb{C}^*)$, with Gauss map g and height differential ϕ . Equation (12) means that the period of (g, ϕ) along γ is horizontal and its flux along γ has third coordinate 2π . We will say that $g \in \mathcal{W}$ *closes periods* when there exists $a \in \mathbb{R}^*$ such that (3) and the first equation in (4) hold with $dh = \phi$ and $\gamma_2 = \gamma$ (note that the second equation in (4) holds by definition of ϕ).

Lemma 3 *If $g \in \mathcal{W}$ closes periods, then (g, ϕ) is the Weierstrass pair of a properly immersed minimal surface $M \subset \mathbb{T} \times \mathbb{R}$ for a certain flat torus \mathbb{T} , with total curvature $8k\pi$ and $4k$ horizontal Scherk-type ends. Furthermore, the fluxes at the ends $p_1, q_1, \dots, p_k, q_k$ are equal to $(\pi a, 0, 0)$ and opposite to the fluxes at $p_{k+1}, q_{k+1}, \dots, p_{2k}, q_{2k}$ (here $a \in \mathbb{R}^*$ comes from equation (3)).*

Proof. Since g closes periods, it follows that the period vectors of (g, ϕ) at the ends are $\pm(0, \pi a, 0)$. Hence it suffices to check that if $\alpha \subset g^{-1}(\mathbb{C}^*)$ is a closed curve with $\{[\alpha], [\gamma]\}$ forming a basis of $H_1(g^{-1}(\overline{\mathbb{C}}), \mathbb{Z})$, then the period vector of (g, ϕ) along α is linearly independent of $(0, \pi a, 0)$. This holds because ϕ is a nonzero holomorphic differential on the torus $g^{-1}(\overline{\mathbb{C}})$ with purely imaginary integral along γ , hence $\Re \int_{\alpha} \phi \neq 0$. \square

Equation (3) gives $4k$ residue equations on a marked meromorphic map $g \in \mathcal{W}$ which are necessary for g to close periods with $dh = \phi$. The fact that the sum of the residues of a meromorphic differential on a compact Riemann surface equals zero implies that it suffices to impose (3) for $1 \leq j \leq 2k - 1$, so we end up with $4k - 2$ equations. Provided that the first equation in (4) also holds for g with $\gamma_2 = \gamma$, the horizontal component of the flux of the corresponding immersed minimal surface M in Lemma 3 is given by

$$F(\gamma) = i \int_{\gamma} g\phi \in \mathbb{C} \equiv \mathbb{R}^2. \quad (13)$$

4.2 The ligature map.

We define the *ligature map* $L : \mathcal{W} \rightarrow \mathbb{C}^{4k}$ as the map that associates to each marked meromorphic map $g \in \mathcal{W}$ the $4k$ -tuple

$$L(g) = \left(\text{Res}_{p_1}(g^{-1}\phi), \dots, \text{Res}_{p_{2k-1}}(g^{-1}\phi), \text{Res}_{q_1}(g\phi), \dots, \text{Res}_{q_{2k-1}}(g\phi), \int_{\gamma} g^{-1}\phi, \int_{\gamma} g\phi \right).$$

As ϕ depends holomorphically on g and all the components of L can be computed as integrals along curves contained in $g^{-1}(\Omega)$ (see the definition of the topology of \mathcal{W}), we conclude that L is holomorphic. We consider the subset of marked meromorphic maps that close periods, $\mathcal{M} = \{g \in \mathcal{W} \mid L(g) = L_{(a,b)}, \text{ with } a \in \mathbb{R}^*, b \in \mathbb{C}\}$, where

$$L_{(a,b)} = \left(\underbrace{a, \dots, a}_{1 \leq j \leq k}, \underbrace{-a, \dots, -a}_{k+1 \leq j \leq 2k-1}, \underbrace{-a, \dots, -a}_{2k \leq j \leq 3k-1}, \underbrace{a, \dots, a}_{3k \leq j \leq 4k-2}, b, \bar{b} \right) \in \mathbb{C}^{4k}.$$

The following map J defines a canonical injection from the topological space $\tilde{\mathcal{S}}$ of marked surfaces into \mathcal{M} ,

$$(M, p_1, \dots, p_{2k}, q_1, \dots, q_{2k}, [\gamma_2]) \mapsto J(M) = (g^{-1}(\bar{\mathbb{C}}), g, p_1, \dots, p_{2k}, q_1, \dots, q_{2k}, [\gamma_2]),$$

with g being the Gauss map of M .

Lemma 4 $J : \tilde{\mathcal{S}} \rightarrow \mathcal{M}$ is an embedding, where \mathcal{M} has the restricted topology from the one of \mathcal{W} . Moreover if we identify $\tilde{\mathcal{S}}$ with $J(\tilde{\mathcal{S}})$, then $\tilde{\mathcal{S}}$ is open and closed in \mathcal{M} .

Proof. First consider a sequence $\{M_n\}_n \subset \tilde{\mathcal{S}}$ converging to a marked surface $M \in \tilde{\mathcal{S}}$. It is a standard fact that the Gauss map g_n of M_n converges uniformly as $n \rightarrow \infty$ to the Gauss map g of M on the compactified torus. From here it is not difficult to check that $J(M_n)$ lies in an arbitrarily small neighborhood of $J(M)$ in the topology of \mathcal{W} for n large enough, so J is continuous. Both the continuity of the inverse map J^{-1} and the openness and closeness of $\tilde{\mathcal{S}}$ in \mathcal{M} can be deduced from an argument similar to the one in the proof of Lemma 6 of [14] (in fact, our setting is easier than [14] because the Gauss map of any surface in \mathcal{S} is unbranched at the ends). \square

In the sequel, we will identify J with the inclusion map and see $\tilde{\mathcal{S}}$ as a subset of \mathcal{M} .

Proposition 3 Let $H \in \mathbb{R}^3 - \{0\}$ be a vector parallel to the x_2 -axis and $F \in \mathbb{C}$. Then, the set of marked surfaces $M \in \tilde{\mathcal{S}}$ whose periods at the ends are $\pm H$ and whose flux vector along the homology class in the last component of M equals $(F, 2\pi) \in \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ is an analytic subvariety of \mathcal{W} .

Proof. Fix $a \in \mathbb{R}^*$ and $b \in \mathbb{C}$. It follows from (2) and (13) that the set $\mathcal{M}(a, b) = \{g \in \mathcal{W} \mid L(g) = L_{(a,b)}\}$ coincides with the set of immersed minimal surfaces $M \in \mathcal{M}$ whose periods and fluxes at the ends are $P_{p_j} = -P_{p_{k+j}} = -P_{q_j} = P_{q_{k+j}} = -(0, \pi a, 0)$, $F_{p_j} = -F_{p_{k+j}} = F_{q_j} = -F_{q_{k+j}} = (\pi a, 0, 0)$, for $j = 1, \dots, k$, and whose flux along the last component $[\gamma]$ of the marked meromorphic map g is $F_\gamma = (i\bar{b}, 2\pi)$. As L is holomorphic, $\mathcal{M}(a, b)$ is an analytic subvariety of \mathcal{W} . As a simultaneously open and closed subset of an analytic subvariety is also an analytic subvariety, Lemma 4 implies that $\tilde{\mathcal{S}} \cap \mathcal{M}(a, b)$ is an analytic subvariety of \mathcal{W} , which proves the Proposition. \square

Definition 1 The value of the ligature map L at a marked surface $M \in \tilde{\mathcal{S}}$ is determined by two numbers $a \in \mathbb{R}^*$, $b \in \mathbb{C}$ so that $\text{Res}_{p_1}(g^{-1}dh) = a$ and $F_{\gamma_2} = (i\bar{b}, 2\pi)$. We define the *classifying map* $C : \tilde{\mathcal{S}} \rightarrow \mathbb{R}^* \times \mathbb{C}$ by $C(M) = (a, b)$.

Remark 4 Let $M \in \mathcal{S}$ be a geometric surface, seen as two marked surfaces $M_1, M_2 \in \tilde{\mathcal{S}}$ with associated homology classes $[\gamma_2(M_1)], [\gamma_2(M_2)] \in H_1(M, \mathbb{Z})$ such that $[\gamma_2(M_1)] = [\gamma_2(M_2)]$ in $H_1(\mathbb{M}, \mathbb{Z})$ (here \mathbb{M} is the compactification of M). Then, $\gamma_2(M_1) \cup \gamma_2(M_2)$ bounds an even number of ends whose periods add up to zero, and the components of C at M_1, M_2 satisfy $a(M_1) = \pm a(M_2) \in \mathbb{R}^*$, $b(M_1) = b(M_2) + m\pi a(M_1)$ with $m \in \mathbb{Z}$ even.

5 Properness.

In the sequel, we will denote by K_Σ the Gaussian curvature function of any surface Σ .

Lemma 5 Let \mathcal{P} be a rank 2 lattice in \mathbb{R}^3 and $\Sigma \subset \mathbb{R}^3/\mathcal{P}$ a properly embedded nonflat (orientable) minimal surface with finite topology and $4k$ horizontal Scherk-type ends. Denote by H, T two generators of \mathcal{P} and assume that H points to the x_2 -axis. Suppose also that $|K_\Sigma| \leq c$ where $c > 0$. Then,

- (i) Both $\|H\|$ and the vertical distance between consecutive left (resp. right) ends of Σ are not less than $2/\sqrt{c}$, and the third coordinate of T satisfies $|T_3| \geq 4k/\sqrt{c}$.
- (ii) The injectivity radius of \mathbb{R}^3/\mathcal{P} is bounded by below by $1/\sqrt{c}$ and Σ admits a regular open neighborhood of radius $1/\sqrt{c}$ in \mathbb{R}^3/\mathcal{P} .

Proof. Let $\tilde{\Sigma} \subset \mathbb{R}^3$ be the connected properly embedded doubly periodic minimal surface obtained by lifting Σ . As $|K_{\tilde{\Sigma}}| \leq c$ and the ends of $\tilde{\Sigma}$ are asymptotic to horizontal halfplanes, a standard application of the maximum principle shows that $\tilde{\Sigma}$ has a regular neighborhood $\tilde{\Sigma}(\varepsilon)$ of positive radius $\varepsilon = 1/\sqrt{c}$ (see [16, 24] for similar arguments). In particular, the vertical separation between two consecutive left ends of $\tilde{\Sigma}$ in the same fundamental domain is greater than or equal to 2ε . As Σ has $2k$ left ends, it follows that $|T_3| \geq 4k\varepsilon$. Now take a point $p \in \tilde{\Sigma}$ where the normal vector to $\tilde{\Sigma}$ points to $(0, 1, 0)$. $\tilde{\Sigma}$ can be expressed locally around p as a graph \mathcal{G} over a disk in the tangent space at p and the same holds around the homologous point $p + H$. Reasoning as above with \mathcal{G} , $\mathcal{G} + H$ instead of consecutive left ends of $\tilde{\Sigma}$, we have that $\|H\| \geq 2\varepsilon$. This proves (i).

Let r be the injectivity radius of \mathbb{R}^3/\mathcal{P} . Since \mathbb{R}^3/\mathcal{P} is flat, there are no conjugate points on geodesics in \mathbb{R}^3/\mathcal{P} and thus r is half of the minimum length of a closed geodesic in \mathbb{R}^3/\mathcal{P} . Therefore,

$$2r = \min_{v \in \mathcal{P} - \{0\}} \|v\| \geq \min(\|H\|, |T_3|) \geq \min\left(\frac{2}{\sqrt{c}}, \frac{4k}{\sqrt{c}}\right) = \frac{2}{\sqrt{c}}.$$

Finally, any two distinct points $p, q \in \mathbb{R}^3$ homologous by \mathcal{P} are separated by a distance greater than or equal to $2r$. It follows that $\tilde{\Sigma}(\varepsilon)/\mathcal{P}$ is a regular neighborhood of Σ of radius $1/\sqrt{c}$. \square

Next we describe all possible limits of sequences $\{M_n\}_n$ of surfaces under the hypotheses of Theorem 1, with the additional assumption of having uniform curvature bounds. For such a surface M_n , Proposition 1 insures that its period lattice \mathcal{P}_n is generated by the horizontal period vector $H_n = (0, \pi a_n, 0)$ at the ends of M_n and by a nonhorizontal vector $T_n = P_{\gamma_1(n)} \neq 0$, where $a_n > 0$, $\gamma_1(n) \subset M_n$ is a closed curve so that $[\gamma_1(n)] \neq 0$ in $H_1(\mathbb{M}_n, \mathbb{Z})$ and \mathbb{M}_n is the compactification of M_n . We will use this notation throughout the proof of the following Proposition.

Proposition 4 *Let $\{\tilde{M}_n\}_n$ be a sequence of properly embedded doubly periodic minimal surfaces, each \tilde{M}_n invariant by a rank 2 lattice \mathcal{P}_n as above. Suppose that for all n , $M_n = \tilde{M}_n/\mathcal{P}_n$ has genus 1 and $4k$ horizontal Scherk-type ends, \tilde{M}_n passes through the origin of \mathbb{R}^3 and $|K_{\tilde{M}_n}(0)| = 1$ is a maximum value of $|K_{\tilde{M}_n}|$. Then (after passing to a subsequence), \tilde{M}_n converges uniformly on compact subsets of \mathbb{R}^3 with multiplicity 1 to a properly embedded minimal surface $\tilde{M}_\infty \subset \mathbb{R}^3$ which lies in one of the following cases:*

- (i) \tilde{M}_∞ is a vertical catenoid with flux $(0, 0, 2\pi)$. In this case, both $\{H_n\}_n, \{T_n\}_n$ are unbounded for any choice of T_n as above.
- (ii) \tilde{M}_∞ is a vertical helicoid with period vector $(0, 0, 2\pi m)$ for some $m \in \mathbb{N}$. Now $\{H_n\}_n$ is unbounded and there exists a choice of T_n for which $\{T_n\}_n \rightarrow (0, 0, 2\pi m)$ as $n \rightarrow \infty$.
- (iii) \tilde{M}_∞ is a Riemann minimal example with horizontal ends. Moreover, $\{H_n\}_n$ is unbounded and certain choice of $\{T_n\}_n$ converges to the period vector of \tilde{M}_∞ .
- (iv) \tilde{M}_∞ is a singly periodic Scherk minimal surface, two of whose ends are horizontal. Furthermore, any choice of $\{T_n\}_n$ is unbounded, $\{H_n\}_n$ converges to the period vector $H_\infty = (0, a, 0)$ of \tilde{M}_∞ (with $a > 0$), and $\tilde{M}_\infty/H_\infty$ has genus zero.

- (v) \widetilde{M}_∞ is a doubly periodic Scherk minimal surface. In this case, $\{H_n\}_n, \{T_n\}_n$ converge respectively to period vectors H_∞, T_∞ of \widetilde{M}_∞ , and $\widetilde{M}_\infty/\{H_\infty, T_\infty\}$ has genus zero with at least two horizontal ends and exactly two nonhorizontal ends.
- (vi) \widetilde{M}_∞ is a doubly periodic minimal surface invariant by a rank 2 lattice \mathcal{P}_∞ , $M_\infty = \widetilde{M}_\infty/\mathcal{P}_\infty$ has genus one and $4k$ horizontal Scherk-type ends, and $\{H_n\}_n \rightarrow H_\infty, \{T_n\}_n \rightarrow T_\infty$, where H_∞, T_∞ are defined by Proposition 1 applied to M_∞ .

Proof. By Lemma 5, $\{\widetilde{M}_n\}_n$ has local uniform area bounds. As $\{|K_{\widetilde{M}_n}|\}_n$ is uniformly bounded and the origin is an accumulation point of $\{\widetilde{M}_n\}_n$, after choosing a subsequence $\{\widetilde{M}_n\}_n$ converges uniformly on compact subsets of \mathbb{R}^3 to a properly embedded minimal surface \widetilde{M}_∞ with $0 \in \widetilde{M}_\infty$ and $|K_{\widetilde{M}_\infty}(0)| = 1$. Since \widetilde{M}_∞ is complete, orientable and not flat, it is not stable (see e.g. do Carmo and Peng [1] or Fischer-Colbrie and Schoen [2]). This implies that the multiplicity of the limit $\{\widetilde{M}_n\}_n \rightarrow \widetilde{M}_\infty$ is one, see for instance [21]. We now discuss on the boundedness of the sequences $\{H_n\}_n, \{T_n\}_n$.

Suppose that both $\{H_n\}_n$ and any choice of $\{T_n\}_n$ are unbounded. After passing to a subsequence, we can assume that the minimum length of a nonzero vector in \mathcal{P}_n diverges to ∞ as $n \rightarrow \infty$. Thus \widetilde{M}_∞ can be viewed as limit of surfaces with boundary $\widetilde{\Sigma}_n$ contained in fundamental domains of \widetilde{M}_n , so \widetilde{M}_∞ has finite total curvature. As each $\widetilde{\Sigma}_n$ has genus zero, the same holds for \widetilde{M}_∞ . Therefore \widetilde{M}_∞ is a catenoid (López and Ros [12]). As the Gauss map of \widetilde{M}_n omits the vertical directions for all n , the same holds for \widetilde{M}_∞ by the Open Mapping Theorem. Hence \widetilde{M}_∞ is a vertical catenoid. Since $|K_{\widetilde{M}_\infty}|$ reaches a maximum at the origin with value 1, the flux of \widetilde{M}_∞ must be $(0, 0, 2\pi)$.

Suppose that $\{H_n\}_n$ is unbounded and there is a bounded choice of $\{T_n\}_n$. We do not loss generality assuming that $\{H_n\}_n \rightarrow \infty$ and $\{T_n\}_n \rightarrow T_\infty \in \mathbb{R}^3$. By Lemma 5, $|(T_n)_3| \geq 4k$ for all n , thus $|(T_\infty)_3| \geq 4k$. In particular $T_\infty \neq 0$ and \widetilde{M}_∞ is singly periodic, invariant by the translation by T_∞ . As the Gauss map of $\widetilde{M}_\infty/T_\infty$ has degree at most $2k$ and this surface is properly embedded in \mathbb{R}^3/T_∞ , it must have finite topology, and all its ends are simultaneously asymptotic to ends of planes, helicoids or Scherk minimal surfaces. Note that $\widetilde{M}_\infty/T_\infty$ cannot have Scherk-type ends, because otherwise $\widetilde{M}_\infty/T_\infty$ would have at least two horizontal Scherk-type ends (the Gauss map of $\widetilde{M}_\infty/T_\infty$ omits the vertical directions), so T_∞ would be horizontal, a contradiction.

Next we prove that if the ends of $\widetilde{M}_\infty/T_\infty$ are helicoidal, then \widetilde{M}_∞ is a vertical helicoid. As the Gauss map g_∞ of $\widetilde{M}_\infty/T_\infty$ misses $0, \infty$, we deduce that the height differential dh_∞

of $\widetilde{M}_\infty/T_\infty$ has no zeros and all the ends of $\widetilde{M}_\infty/T_\infty$ are asymptotic to vertical helicoids, where dh_∞ has simple poles. Calling P to the number of ends of $\widetilde{M}_\infty/T_\infty$ and G to the genus of its compactification, it follows that $P + 2(G - 1) = 0$ from where $P = 2$ and $G = 0$. In this situation, it is well known that \widetilde{M}_∞ is a vertical helicoid (Toubiana [26] or Pérez and Ros [19]). Furthermore, its period vector is of the form $(0, 0, 2\pi m)$ with $m \in \mathbb{N}$ because the maximum absolute curvature of $\widetilde{M}_\infty/T_\infty$ is one. Thus we have proved (ii).

Now assume that the ends of $\widetilde{M}_\infty/T_\infty$ are planar. A direct lifting argument and the maximum principle insure that $\widetilde{M}_\infty/T_\infty$ cannot have genus zero, thus it is a properly embedded minimal torus in \mathbb{R}^3/T_∞ with a finite number of planar ends. By a Theorem of Meeks, Pérez and Ros [14], $\widetilde{M}_\infty/T_\infty$ is a Riemann minimal example. As its Gauss map misses the vertical directions, the ends of $\widetilde{M}_\infty/T_\infty$ are horizontal and we have (iii).

Suppose that $\{H_n\}_n$ is bounded and any choice of $\{T_n\}_n$ is unbounded. After extracting a subsequence, we can assume $\{H_n\}_n \rightarrow H_\infty = (0, a, 0)$ with $a \in [0, \infty)$ and $\{T_n\}_n \rightarrow \infty$. By Lemma 5, $a \geq 2$ and \widetilde{M}_∞ is singly periodic, invariant by translation by H_∞ . The well defined Gauss map g_∞ of $\widetilde{M}_\infty/H_\infty$ has degree less than or equal to $2k$, hence $\widetilde{M}_\infty/H_\infty$ has finite total curvature. As it is embedded and proper, all ends of $\widetilde{M}_\infty/H_\infty$ are again simultaneously planar, helicoidal or of Scherk-type. Moreover, $\widetilde{M}_\infty/H_\infty$ has at least two ends with vertical limit normal vector because g_∞ omits $0, \infty$. As H_∞ is horizontal, properness of $\widetilde{M}_\infty/H_\infty$ prevents its ends to be planar. Also, its ends are not helicoidal because in such case the period vectors at the ends would be vertical. Hence $\widetilde{M}_\infty/H_\infty$ has Scherk-type ends (at least two of them horizontal) and genus zero or one.

Assertion 1 *In the above situation, the well defined height differential dh_∞ of $\widetilde{M}_\infty/H_\infty$ cannot have a zero at any horizontal end (equivalently, g_∞ is unbranched at the horizontal ends of $\widetilde{M}_\infty/H_\infty$).*

To see the Assertion, suppose on the contrary that g_∞ is branched at a puncture p_∞ corresponding to a horizontal Scherk-type end of $\widetilde{M}_\infty/H_\infty$, say with value zero. Since the Gauss map g_n of \widetilde{M}_n/H_n is unbranched at its zeros, there exists an integer $l \geq 2$ such that l distinct zeros $p_1(n), \dots, p_l(n)$ of g_n converge to p_∞ (we can think of pieces of the compactifications of $\widetilde{M}_\infty/T_\infty, \widetilde{M}_n/T_n$ defined on a common disk D centered at p_∞ so that $p_1(n), \dots, p_l(n) \in D$ for n large). Moreover, the heights of the ends of \widetilde{M}_n/H_n corresponding to $p_1(n), \dots, p_l(n)$ converge to the height of the end p_∞ of $\widetilde{M}_\infty/H_\infty$. By Lemma 5, we deduce that $l = 2$ and $p_1(n), p_2(n)$ are a left end and a right end, and so

their fluxes are opposite and the same holds for their periods. Therefore both the period and flux of $\widetilde{M}_\infty/H_\infty$ at p_∞ vanish, a contradiction. This proves Assertion 1.

Note that dh_∞ has a pole at each nonhorizontal Scherk-type end of $\widetilde{M}_\infty/H_\infty$ (and there are at least two of these ends by the maximum principle). Using Assertion 1 and the fact that g_∞ misses $0, \infty$ on $\widetilde{M}_\infty/H_\infty$, we conclude that dh_∞ has no zeros at the compactification of $\widetilde{M}_\infty/H_\infty$. Thus $\widetilde{M}_\infty/H_\infty$ has genus zero and dh_∞ has exactly two poles or equivalently, $\widetilde{M}_\infty/H_\infty$ has exactly two nonhorizontal Scherk-type ends. In this situation, Pérez and Traizet [22] have proved that $\widetilde{M}_\infty/H_\infty$ is a singly periodic Scherk minimal surface of genus zero. This shows (iv).

Suppose that $\{H_n\}_n$ is bounded and there exists a bounded choice of $\{T_n\}_n$. As before, it can be supposed that $\{H_n\}_n \rightarrow H_\infty = (0, a, 0)$ with $a \geq 2$ and $\{T_n\}_n \rightarrow T_\infty \in \mathbb{R}^3$ with $|(T_\infty)_3| \geq 4k$. Thus, \widetilde{M}_∞ is doubly periodic, invariant by the rank 2 lattice \mathcal{P}_∞ generated by H_∞, T_∞ and $\widetilde{M}_\infty/\mathcal{P}_\infty$ has genus zero or one with a finite number of Scherk-type ends.

Since the Gauss map g_∞ of $\widetilde{M}_\infty/\mathcal{P}_\infty$ misses $0, \infty$, this surface has at least two horizontal ends. A suitable modification of Assertion 1 gives that the height differential dh_∞ of $\widetilde{M}_\infty/\mathcal{P}_\infty$ (which is also well defined) does not vanish at any of the horizontal ends of this surface. Therefore either $\widetilde{M}_\infty/\mathcal{P}_\infty$ has genus zero with exactly two nonhorizontal Scherk-type ends, or it has genus one and all its ends are horizontal. In the first case, a theorem by Wei [28] (see also Lazard-Holly and Meeks [11]) insures that \widetilde{M}_∞ is a doubly periodic Scherk minimal surface, so (v) is proved. Finally, assume that $\widetilde{M}_\infty/\mathcal{P}_\infty$ has genus one. Since the total curvature of M_n is $8k\pi$, $\widetilde{M}_\infty/\mathcal{P}_\infty$ must have total curvature at most $8k\pi$. Using the Meeks-Rosenberg formula [15] and the fact that $\widetilde{M}_\infty/\mathcal{P}_\infty$ is properly embedded with genus one and Scherk-type ends, we deduce that $\widetilde{M}_\infty/\mathcal{P}_\infty$ has at most $4k$ ends. On the other hand, the spacing between left (resp. right) ends of M_n is bounded away from zero by Lemma 5, which implies that the $2k$ heights corresponding to the left (resp. right) ends of M_n converge to $2k$ distinct heights in \mathbb{R}^3/H_∞ . Since noncompact horizontal level sets of the \widetilde{M}_n/H_n tend to noncompact level sets of $\widetilde{M}_\infty/H_\infty$, we conclude that $\widetilde{M}_\infty/\mathcal{P}_\infty$ has exactly $4k$ horizontal ends. Now the proof is complete. \square

The following result is a crucial curvature estimate in terms of the classifying map C .

Proposition 5 *Let $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ be a sequence of marked surfaces. Suppose that $C(M_n) = (a_n, b_n) \in \mathbb{R}^* \times \mathbb{C}$ satisfies*

(i) $\{a_n\}_n$ is bounded away from zero.

(ii) $\{|b_n|\}_n$ is bounded by above.

Then, the sequence of Gaussian curvatures $\{K_{M_n}\}_n$ is uniformly bounded.

Proof. The proof is based on the one of Theorem 4 in [14], so we will only go into the details of what is new in this setting. By contradiction, assume that $\lambda_n := \max_{M_n} \sqrt{|K_{M_n}|} \rightarrow \infty$ as $n \rightarrow \infty$. Let $\Sigma_n = \lambda_n M_n \subset \mathbb{R}^3 / \lambda_n \mathcal{P}_n$, where $\mathcal{P}_n = \text{Span}\{H_n, T_n\}$ is the rank 2 lattice associated to M_n and H_n is the period vector at the ends of M_n (up to sign). Let us also call $\widehat{\Sigma}_n, \widetilde{\Sigma}_n$ to the respective liftings of Σ_n to $\mathbb{R}^3 / \lambda_n T_n$ and to \mathbb{R}^3 .

After translation of $\widetilde{\Sigma}_n$ to have maximum absolute Gauss curvature one at the origin, Proposition 4 implies that (after passing to a subsequence) $\{\widetilde{\Sigma}_n\}_n$ converges smoothly to a properly embedded minimal surface $\mathcal{H}_1 \subset \mathbb{R}^3$, which must lie in one of the six possibilities in Proposition 4. Since a_n is bounded away from zero and $\lambda_n \rightarrow \infty$, the period vectors $\lambda_n H_n$ at the ends of Σ_n diverge to ∞ . Therefore \mathcal{H}_1 must be a vertical catenoid, a vertical helicoid or a Riemann minimal example with horizontal ends. If \mathcal{H}_1 were a catenoid or a Riemann minimal example, then the vertical part of the flux of $\widetilde{\Sigma}_n$ along a compact horizontal section, which is $2\pi\lambda_n$ by item 7 of Proposition 1, would converge to the vertical part of the flux of \mathcal{H}_1 , which is finite. This contradicts that $\lambda_n \rightarrow \infty$ and so, \mathcal{H}_1 is a vertical helicoid with period vector $T = (0, 0, 2\pi m)$ for certain $m \in \mathbb{N}$ (furthermore, we can choose the period vector T_n of M_n with $\{\lambda_n T_n\}_n \rightarrow T$ as $n \rightarrow \infty$, see Proposition 4).

Let $\pi_n : \mathbb{R}^3 / \lambda_n \mathcal{P}_n \rightarrow \{x_3 = 0\} / \lambda_n \mathcal{P}_n$ be the linear projection in the direction of T_n , $\Pi_n^H : \mathbb{R}^3 / \lambda_n T_n \rightarrow \mathbb{R}^3 / \lambda_n \mathcal{P}_n$ the quotient projection modulo $\lambda_n H_n$ and \mathbb{D} the unit disk centered at the origin in $\{x_3 = 0\}$. From now on we will only consider n large so that $\mathcal{H}_1(n) = \Sigma_n \cap \pi_n^{-1}(\mathbb{D} / \lambda_n \mathcal{P}_n)$ is connected and extremely close to a piece of \mathcal{H}_1 containing its axis, $\lambda_n T_n$ being close to T . Let $\overline{\Sigma}_n$ be the torus obtained by attaching to Σ_n its $4k$ ends. Note that $\mathcal{H}_1(n)$ does not separate $\overline{\Sigma}_n$, hence $\mathcal{F}_n = \overline{\Sigma}_n - \mathcal{H}_1(n)$ is a compact annulus that contains the $4k$ ends of Σ_n . Let N_n be the Gauss map of Σ_n .

We claim that for all n large, N_n takes horizontal values on \mathcal{F}_n . To see this, consider the compact intersection Γ_n of Σ_n with a totally geodesic horizontal cylinder in $\mathbb{R}^3 / \lambda_n \mathcal{P}_n$ not asymptotic to the ends of Σ_n . Viewed in $\overline{\Sigma}_n$, Γ_n is a non nulhomotopic embedded closed curve whose intersections with $\mathcal{H}_1(n), \mathcal{F}_n$ are two open arcs with common end points A_n, B_n . As $\mathcal{H}_1(n)$ is very close to a piece of a vertical helicoid containing its axis,

we deduce that $N_n(A_n), N_n(B_n)$ are in different hemispheres of \mathbb{S}^2 with respect to the vertical direction. Now our claim follows by continuity.

Let $\mathbb{S}^1 \subset \mathbb{S}^2$ be the horizontal equator in the sphere. We next show that for all $\theta \in \mathbb{S}^1$ and $n \in \mathbb{N}$ large enough, $\mathcal{F}_n \cap N_n^{-1}(\theta)$ is not empty. Since $\mathcal{H}_1(n)$ is almost a piece of a vertical helicoid containing its axis, $(N_n|_{\mathcal{H}_1(n)})^{-1}(\mathbb{S}^1)$ must be a simple closed curve in Σ_n (viewed in \mathbb{R}^3 , $(N_n|_{\mathcal{H}_1(n)})^{-1}(\mathbb{S}^1)$ is an open embedded arc whose end points differ in $\lambda_n T_n$) that covers \mathbb{S}^1 with finite multiplicity through N_n . As N_n is horizontal somewhere in \mathcal{F}_n , we deduce that the number of sheets of this last covering is less than $2k$, which implies that for all $\theta \in \mathbb{S}^1$, $\mathcal{F}_n \cap N_n^{-1}(\theta)$ is not empty as desired.

Next we produce a second helicoid as a limit of different translations of the $\tilde{\Sigma}_n$. Since the total branching number of N_n is $4k$, there exists a spherical disk $D_{\mathbb{S}^2}(\theta_n, \varepsilon) \subset \mathbb{S}^2$ centered at some $\theta_n \in \mathbb{S}^1$ with uniform radius $\varepsilon > 0$ which is free of branch values of N_n . By the last paragraph, for n large we can find points $p_n \in \tilde{\Sigma}_n$ with $N_n(p_n) = \theta_n$ and whose projections into $\mathbb{R}^3/\lambda_n \mathcal{P}_n$ lie inside \mathcal{F}_n . The sequence of translated surfaces $\{\tilde{\Sigma}_n - p_n\}_n$ has uniform curvature and area bounds and all the $\tilde{\Sigma}_n - p_n$ have normal vector θ_n at the origin. After passing to a subsequence, $\tilde{\Sigma}_n - p_n$ converges uniformly on compact subsets of \mathbb{R}^3 to a properly embedded minimal surface \mathcal{H}_2 having vertical tangent plane at the origin. \mathcal{H}_2 is not flat, because otherwise one could construct an arc $\tilde{\alpha}_n$ contained in $\{x \in \tilde{\Sigma}_n \mid d_{\tilde{\Sigma}_n}(x, p_n) < \frac{3}{2}\|T\|\}$, $\tilde{\alpha}_n$ giving rise to the period vector $\lambda_n T_n$. By construction, $N_n(\tilde{\alpha}_n)$ would lie in $D_{\mathbb{S}^2}(\theta_n, \varepsilon)$ for n large enough, but this last disk is free of branch values of N_n hence $N_n^{-1}(D_{\mathbb{S}^2}(\theta_n, \varepsilon))$ consists of disjoint disks inside Σ_n , one of which would contain $\alpha_n = \tilde{\alpha}_n/\lambda_n \mathcal{P}_n$ in contradiction with the fact that α_n is homotopically nontrivial on Σ_n . Hence \mathcal{H}_2 is not flat. Applying Proposition 4 (suitably modified so that the absolute Gaussian curvature of the surfaces in the sequence is not 1 at the origin, but it is uniformly bounded by above and that the limit surface is not a plane) and our previous arguments to eliminate all limits other than a helicoid, we conclude that \mathcal{H}_2 is another vertical helicoid. Since both $\mathcal{H}_1, \mathcal{H}_2$ are limits of translations of the $\tilde{\Sigma}_n$, the period vector of \mathcal{H}_2 is again $T = \lim_n \lambda_n T_n$.

Consider two disjoint round disks $\mathcal{D}_1(n), \mathcal{D}_2(n)$ in the totally geodesic cylinder $\{x_3 = 0\}/\lambda_n \mathcal{P}_n \subset \mathbb{R}^3/\lambda_n \mathcal{P}_n$, with common radius r_n such that the annular component $\mathcal{H}_i(n) = \Sigma_n \cap \pi_n^{-1}(\mathcal{D}_i(n))$ is arbitrarily close to a translated copy of the forming helicoid \mathcal{H}_i/T minus neighborhoods of its ends, $i = 1, 2$. After passing to a subsequence, we can also choose r_n so that

1. $r_n \rightarrow \infty$ as $n \rightarrow \infty$.
2. $\frac{r_n}{\lambda_n} \rightarrow 0$ as $n \rightarrow \infty$.
3. The normal direction to Σ_n along the helix-type curves in the boundary of $\mathcal{H}_i(n)$ makes an angle less than $1/n$ with the vertical, $i = 1, 2$.

(Note that we have exchanged the former $\mathcal{H}_1(n) = \Sigma_n \cap \pi_n^{-1}(\mathbb{D})$ by a bigger one $\mathcal{H}_1(n) = \Sigma_n \cap \pi_n^{-1}(D_1(n))$ but all the preceding arguments remain valid now). We claim that the extended Gauss map N_n applies $\bar{\Sigma}_n - (\mathcal{H}_1(n) \cup \mathcal{H}_2(n))$ in the spherical disks centered at the North and South Poles of \mathbb{S}^2 with radius $1/n$. Since N_n is an open map, the condition 3 above shows that it suffices to check that $\Sigma_n - (\mathcal{H}_1(n) \cup \mathcal{H}_2(n))$ has no points with horizontal normal vector. If horizontal normal vectors occurred in $\Sigma_n - (\mathcal{H}_1(n) \cup \mathcal{H}_2(n))$, then our former arguments would give further vertical helicoids $\mathcal{H}_3, \dots, \mathcal{H}_s$ as limits of translations of subsequences of the $\tilde{\Sigma}_n$ (finitely many limits because each one consumes at least -4π of total curvature, and the total curvature of the Σ_n is fixed $-8k\pi$). Therefore we have s pairwise disjoint annuli $\mathcal{H}_j(n) = \Sigma_n \cap \pi_n^{-1}(D_j(n))$, each one arbitrarily close to the corresponding vertical helicoid \mathcal{H}_j minus a neighborhood of its two ends, where each $D_j(n) \subset \{x_3 = 0\}/\lambda_n \mathcal{P}_n$ is a round disk as before, $j = 1, \dots, s$. In this setting, the complement of $\mathcal{H}_1(n) \cup \dots \cup \mathcal{H}_s(n)$ in $\bar{\Sigma}_n$ would consist of s closed annuli $\mathcal{F}_1(n), \dots, \mathcal{F}_s(n)$ labelled so that $\mathcal{H}_i(n)$ is consecutive to $\mathcal{F}_i(n)$ and all $\mathcal{H}_i(n), \mathcal{F}_i(n)$ are disposed cyclically for $i = 1, \dots, s$. Moreover, $N_n(\mathcal{F}_1(n) \cup \dots \cup \mathcal{F}_s(n))$ omits \mathbb{S}^1 hence $N_n(\mathcal{F}_1(n) \cup \dots \cup \mathcal{F}_s(n))$ is contained in a small neighborhood of the vertical directions in \mathbb{S}^2 . When restricted to the surface $\mathcal{F}_i(n)$, the projection π_n extends smoothly through the ends producing a map

$$f_n := \pi_n|_{\mathcal{F}_i(n)} : \mathcal{F}_i(n) \rightarrow \frac{\{x_3 = 0\} \cup \{\infty\}}{\lambda_n \mathcal{P}_n} - \cup_{j=1}^s D_j(n). \quad (14)$$

f_n is a proper local diffeomorphism, hence a finite sheeted covering map. Since $\partial\mathcal{F}_i(n)$ has two components it follows that $s = 2$, a contradiction. Thus $N_n[\bar{\Sigma}_n - (\mathcal{H}_1(n) \cup \mathcal{H}_2(n))] \subset D_{\mathbb{S}^2}(*, 1/n)$ where $* = 0, \infty$.

So far, we have proved that the two components $\mathcal{F}_1(n), \mathcal{F}_2(n)$ of $\bar{\Sigma}_n - (\mathcal{H}_1(n) \cup \mathcal{H}_2(n))$ are closed annuli, each one is noncompact when viewed in Σ_n . The number of ends of Σ_n in $\mathcal{F}_i(n)$ is twice the number of sheets $\#_i$ of the covering map f_n in (14). Moreover, the boundary components $\alpha_{i,1}(n), \alpha_{i,2}(n)$ of $\mathcal{F}_i(n)$ apply by f_n respectively on the circumferences $\partial\mathcal{D}_1(n), \partial\mathcal{D}_2(n)$, both with multiplicity $\#_i$. Since $\alpha_{1,j}(n), \alpha_{2,j}(n)$ are the helix-type

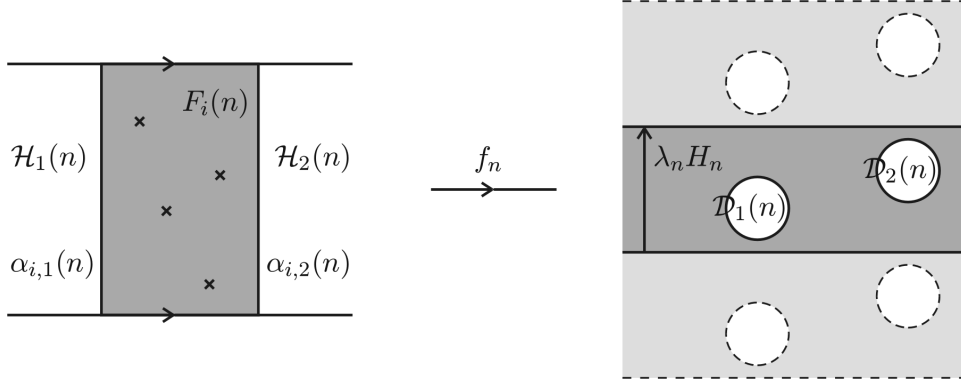


Figure 5: In this case, the covering map f_n has two sheets.

boundary curves of the forming helicoid $\mathcal{H}_j(n)$, we deduce that $\#_1 = \#_2$. Since there are the same number of left and right ends of Σ_n in $\mathcal{F}_i(n)$, it follows that $\#_1 = k$, see Figure 5.

In this setting and coming back to the original scale, one can follow the argument in pages 117-118 of [14] to construct a closed embedded curve $\Gamma_n \subset M_n$ formed by four consecutive arcs $L_1(n)^{-1} * \beta_1(n) * L_2(n) * \beta_2(n)$, where $L_1(n), L_2(n)$ are liftings of the distance minimizing horizontal segment $L(n)$ from $\frac{1}{\lambda_n} \partial \mathcal{D}_1(n)$ to $\frac{1}{\lambda_n} \partial \mathcal{D}_2(n)$, lying in consecutive sheets by the covering

$$f_n : \frac{1}{\lambda_n} [\mathcal{F}_1(n) \cup \mathcal{F}_2(n)] \rightarrow \frac{\{x_3=0\} \cup \{\infty\}}{\mathcal{P}_n} - \frac{1}{\lambda_n} [\mathcal{D}_1(n) \cup \mathcal{D}_2(n)]$$

and $\beta_1(n), \beta_2(n)$ are small arcs contained in $\frac{1}{\lambda_n} \mathcal{H}_1(n), \frac{1}{\lambda_n} \mathcal{H}_2(n)$ respectively (we abuse of notation keeping the label f_n in the original scale of M_n , although it has been defined in the scale of $\Sigma_n = \lambda_n M_n$). For later uses, we will describe $\beta_1(n), \beta_2(n)$ more precisely. $\beta_1(n)$ consists of the union of three consecutive arcs l_1, l_2, l_3 , where l_1, l_3 are at almost constant height and $l_2 \subset N_n^{-1}(\mathbb{S}^1)$, and $\beta_2(n)$ is similarly defined, see Figure 6.

Next we study the relation between the homology classes $[\Gamma_n], [\gamma_2(n)] \in H_1(M_n, \mathbb{Z})$, where $[\gamma_2(n)]$ is the last component of the marked surface $M_n \in \tilde{\mathcal{S}}$ (recall that $C(M_n) = (a_n, b_n)$ where $F_{\gamma_2(n)} = (F(\gamma_2(n)), 2\pi) = (i\overline{b_n}, 2\pi)$ is the flux vector of M_n along $\gamma_2(n)$). Let g_n be the complex Gauss map of M_n . Since Γ_n is embedded, not trivial in $H_1(g_n^{-1}(\overline{\mathbb{C}}), \mathbb{Z})$ and has period zero, Proposition 1 implies that Γ_n can be oriented so that $[\Gamma_n] = [\gamma_2(n)]$ in $H_1(g_n^{-1}(\overline{\mathbb{C}}), \mathbb{Z})$. Viewed in $H_1(M_n, \mathbb{Z})$, the classes $[\Gamma_n], [\gamma_2(n)]$ differ in a finite sum of

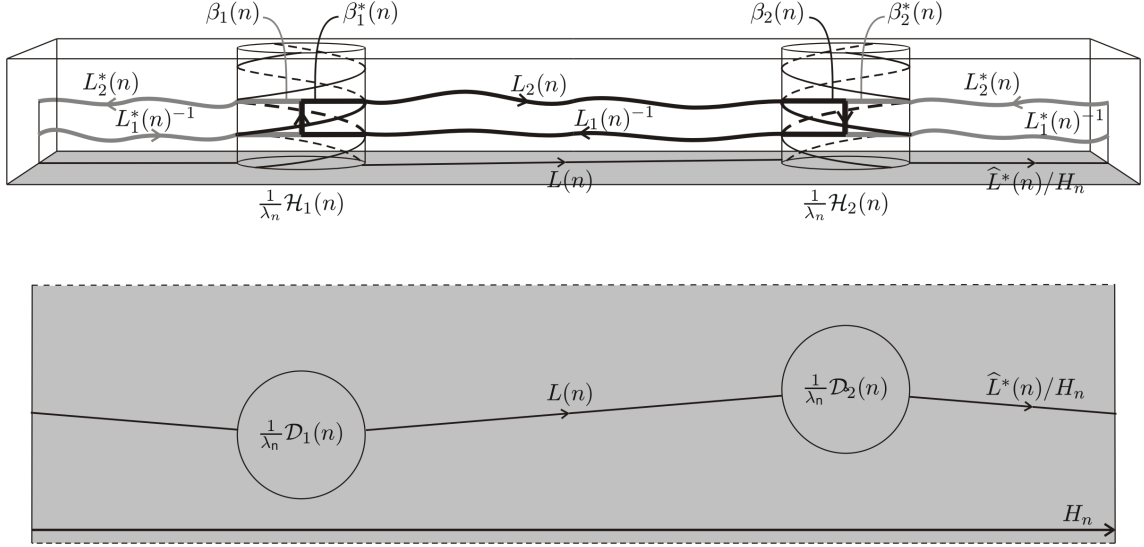


Figure 6: Front and top views of two helicoids forming with the direction of $L(n)$ tending to the direction of the x_2 -axis. In the front view, the sided ends are placed ahead and behind the box that contains both helicoids.

loops around ends, from where

$$F(\Gamma_n) = i\overline{b_n} + t(n)\pi a_n, \quad (15)$$

where $t(n) \in \mathbb{Z}$. Since both $\Gamma_n, \gamma_2(n)$ can be chosen in the same fundamental domain of the doubly periodic lifting \widetilde{M}_n of M_n lying between two horizontal planes $\Pi, \Pi + T_n$, the embeddedness of both curves insures that $\{t(n)\}_n$ is bounded. Passing to a subsequence, we can suppose that $t = t(n)$ does not depend on n . Also note that t is even since the periods of M_n along both $\Gamma_n, \gamma_2(n)$ are zero. By item 7 of Proposition 1, $(F_{\Gamma_n})_3 = (F_{\gamma_2(n)})_3 = 2\pi$. This property implies that the lengths of $L_1(n), L_2(n)$ diverge to ∞ as $n \rightarrow \infty$, see [14] page 118. Those arguments in [14] also show that the limit as $n \rightarrow \infty$ of the horizontal flux $F(\Gamma_n)$ can be computed as the limit of the horizontal fluxes along $L_1(n) \cup L_2(n)$, and that the horizontal flux along $L_1(n) \cup L_2(n)$ divided by the length of $L(n)$ converges to a complex number of modulus 2 as $n \rightarrow \infty$, so $F(\Gamma_n) \rightarrow \infty$. Dividing equation (15) by $|F(\Gamma_n)|$ and using that b_n is bounded, we have (after extracting a subsequence) that both $|F(\Gamma_n)|^{-1}F(\Gamma_n), t|F(\Gamma_n)|^{-1}\pi a_n$ converge to the same limit $e^{i\theta}$, $\theta \in [0, 2\pi)$, from where $t \neq 0$ and $a_n \rightarrow \infty$. Since $a_n \in \mathbb{R}^*$, we also have $\theta = 0$ or π . In particular, the

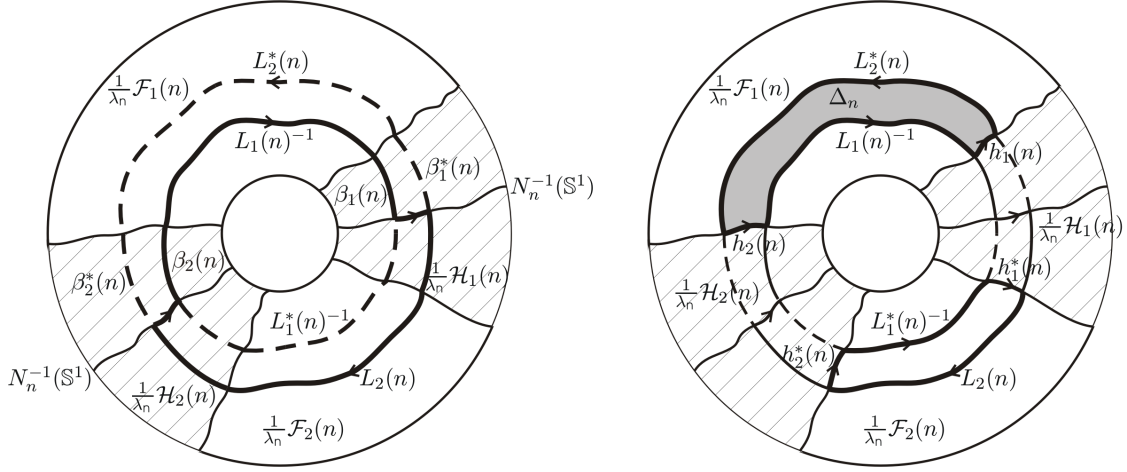


Figure 7: Left: The curves Γ_n, Γ_n^* . Right: The curve ζ_n .

direction of the segment $L(n)$ tends to the direction of the x_2 -axis as $n \rightarrow \infty$.

Now consider respective liftings $\widehat{L}(n), \widehat{\mathcal{D}}_1(n), \widehat{\mathcal{D}}_2(n)$ of $L(n), \mathcal{D}_1(n), \mathcal{D}_2(n)$ such that $\widehat{L}(n) \cup \frac{1}{\lambda_n} [\widehat{\mathcal{D}}_1(n), \widehat{\mathcal{D}}_2(n)]$ lies in the same fundamental domain of \widetilde{M}_n/T_n . Let $\widehat{L}^*(n)$ be the length minimizing horizontal segment from $\frac{1}{\lambda_n} \partial \widehat{\mathcal{D}}_2(n)$ to $\frac{1}{\lambda_n} \partial \widehat{\mathcal{D}}_1(n) + H_n$. Let $\Gamma_n^* \subset M_n$ be another embedded closed curve constructed in a similar way as Γ_n , i.e. $\Gamma_n^* = L_1^*(n)^{-1} * \beta_1^*(n) * L_2^*(n) * \beta_2^*(n)$ where $L_1^*(n), L_2^*(n)$ are liftings in consecutive sheets of $\widehat{L}^*(n)/H_n$ and $\beta_1^*(n), \beta_2^*(n)$ are small arcs inside $\frac{1}{\lambda_n} \mathcal{H}_1(n), \frac{1}{\lambda_n} \mathcal{H}_2(n)$ respectively. As before, each of the $\beta_i^*(n)$ consists of the consecutive union of three arcs, two at almost constant height joined by a central one which we choose as $\beta_i(n) \cap N_n^{-1}(\mathbb{S}^1)$, $i = 1, 2$. We orient Γ_n^* in such a way that Γ_n, Γ_n^* share their orientations along the arcs $\beta_i(n) \cap N_n^{-1}(\mathbb{S}^1)$, $i = 1, 2$. Viewed in $H_1(g_n^{-1}(\overline{\mathbb{C}}), \mathbb{Z})$, it holds $[\Gamma_n] = -[\Gamma_n^*]$, see Figure 7 left. As above, we have that after passing to a subsequence, $F(\Gamma_n^*) = -i\overline{b}_n + t^* \pi a_n$ for certain nonzero even integer t^* .

Let $\zeta_n \subset M_n$ be the embedded closed curve defined by $\zeta_n = L_1(n)^{-1} * h_1(n) * L_2^*(n) * h_2(n)$, where $h_1(n) \subset \frac{1}{\lambda_n} \partial \mathcal{H}_1(n)$ is a helix-type curve joining the end point of $L_1(n)^{-1}$ with the starting point of $L_2^*(n)$ (so the covering map f_n restricts to $h_1(n)$ as a diffeomorphism onto an arc in $\frac{1}{\lambda_n} \partial \mathcal{D}_1(n)$ arbitrarily close to a halfcircumference), and $h_2(n) \subset \frac{1}{\lambda_n} \partial \mathcal{H}_2(n)$ is similarly defined. Note that we can endow ζ_n with a natural orientation by restricting the orientation of either Γ_n or Γ_n^* to their common arcs with ζ_n (both restrictions give

rise to the same orientation). ζ_n is the boundary of a disk Δ_n inside $\frac{1}{\lambda_n}[\mathcal{F}_1(n) \cup \mathcal{F}_2(n)]$. We do not lose generality by supposing $\Delta_n \subset \frac{1}{\lambda_n}\mathcal{F}_1(n)$. Since f_n restricts to ζ_n as a diffeomorphism onto a curve that represents the period vector H_n , we deduce that Δ_n has just one end of M_n . In particular $F(\zeta_n) = \varepsilon\pi a_n$, with $\varepsilon = \pm 1$. Now consider for $i = 1, 2$ the helix-type curve $h_i^*(n) \subset \frac{1}{\lambda_n}\partial\mathcal{H}_i(n)$ that joins the corresponding extreme points of $L_1^*(n)^{-1}$ and of $L_2(n)$. $h_i^*(n)$ is almost opposite to $h_i(n)$ with respect to the almost axis of $\frac{1}{\lambda_n}\partial\mathcal{H}_i(n)$, and let ζ_n^* be the closed curve $L_1^*(n)^{-1} * h_1^*(n) * L_2(n) * h_2^*(n)$ endowed with the orientation induced by the ones of either Γ_n or Γ_n^* along their common arcs with ζ_n^* . The same argument as before gives $F(\zeta_n^*) = \pm\pi a_n$. Since the unit conormal vector along $L_1(n)^{-1}$ projects horizontally on the same side as the unit conormal vector along $L_2(n)$, we conclude that $F(\zeta_n^*) = F(\zeta_n)$. A similar argument with the unit conormal vector implies that $\frac{F(\Gamma_n)}{F(\Gamma_n^*)}$ converges to a positive real number, from where $\frac{t}{t^*} > 0$.

As we are assuming $\frac{r_n}{\lambda_n} \rightarrow 0$, then the lengths of $\beta_i(n), \beta_i^*(n), h_i(n), h_i^*(n)$ go to 0 as $n \rightarrow \infty, i = 1, 2$. Thus the limit of $F(\Gamma_n) + F(\Gamma_n^*)$ coincide with the limit of $F(\zeta_n) + F(\zeta_n^*) = 2\varepsilon\pi a_n$ as $n \rightarrow \infty$, and

$$2 = \lim_{n \rightarrow \infty} \frac{(F(\Gamma_n) + F(\Gamma_n^*))}{\varepsilon\pi a_n} = \lim_{n \rightarrow \infty} \frac{-2i\overline{b_n} + (t + t^*)\pi a_n}{\varepsilon\pi a_n} = \varepsilon(t + t^*),$$

which contradicts that both t, t^* are nonzero even integers with the same sign, thereby proving Proposition 5. \square

Remark 5 *If we remove the hypothesis (ii) in Proposition 5, then it is possible to find two highly sheeted vertical helicoids forming inside surfaces in \mathcal{S} , which makes their curvatures to blow-up. For instance, the standard examples $M_{\theta,0,\pi/2}$ with $\theta \nearrow \pi/2$ contain two helicoids forming with axes joined horizontally by a line parallel to the period vector at the ends. The surfaces $M_{\theta,0,\beta}$ with $\beta < \theta < \pi/2$ and $\beta \nearrow \pi/2$ have also two forming helicoids, but now their axes join by a horizontal line orthogonal to the period vector at the ends.*

6 Uniqueness of examples around the singly periodic Scherk surfaces.

In this Section we will prove that if $\{M_n\}_n \subset \tilde{\mathcal{S}}$ degenerates in a singly periodic Scherk minimal surface (case (iv) of Proposition 4), then $L(M_n)$ tends to a tuple in \mathbb{C}^{4k} . In

particular, the classifying map $C : \tilde{\mathcal{S}} \rightarrow \mathbb{R}^* \times \mathbb{C}$ cannot be proper. In order to overcome this lack of properness, we will prove that only standard examples can occur in $\tilde{\mathcal{S}}$ nearby the singly periodic Scherk limit. This will be essential when proving that the restriction of C to the space of nonstandard examples is proper (Theorem 5).

Proposition 6 *Let $\{M_n\}_n \subset \tilde{\mathcal{S}}$ be a sequence of marked surfaces with $\{H_n\}_n \rightarrow H_\infty = (0, \pi a, 0)$ ($a \in \mathbb{R}^*$), $\{T_n\}_n \rightarrow \infty$ (for any choice of T_n as in Proposition 4) and $\{C(M_n)\}_n \rightarrow (a, b)$ where $b \in \mathbb{C}$. Then for n large, the geometric surface M_n is close to $2k$ translated images of arbitrarily large compact regions of a singly periodic Scherk minimal surface of genus zero with two horizontal ends, together with $2k$ annular regions $C_n(1), \dots, C_n(2k)$ each of which has two distinct simple branch points of the Gauss map $g_n : M_n \rightarrow \overline{\mathbb{C}}$. Moreover, there exists a nonhorizontal plane $\Pi \subset \mathbb{R}^3$ such that any annulus $C_n(j)$ is a graph over the intersection of Π/H_n with a certain horizontal slab, $j = 1, \dots, 2k$.*

Proof. Note that it suffices to show that there exists a subsequence of $\{M_n\}_n$ verifying the conclusions of the Proposition. Since the total branching number of the Gauss map N_n of M_n is fixed $4k$, we can find a small $\varepsilon > 0$ such that for every n , there exists a $\theta_n \in \mathbb{S}^2 \cap \{x_2 = 0\}$ with the disk $D_{\mathbb{S}^2}(\theta_n, \varepsilon)$ of radius ε in \mathbb{S}^2 around θ_n disjoint from the branch locus of N_n and from the North and South Poles. In particular, $N_n^{-1}(\theta_n)$ consists of $2k$ distinct points $p_n(1), \dots, p_n(2k) \in M_n$. We label by $\tilde{p}_n(i)$ the lift of $p_n(i)$ to a fundamental domain of the doubly periodic surface $\tilde{M}_n \subset \mathbb{R}^3$ whose quotient is M_n , $i = 1, \dots, 2k$. Since $\{C(M_n)\}_n$ converges in $\mathbb{R}^* \times \mathbb{C}$, Proposition 5 and Lemma 5 imply that the sequence $\{\tilde{M}_n - \tilde{p}_n(1)\}_n$ has uniform curvature and local area bounds. After passing to a subsequence, $\tilde{M}_n - \tilde{p}_n(1)$ will converge to a (not necessarily connected) properly embedded minimal surface $\tilde{M}_\infty(1) \subset \mathbb{R}^3$ uniformly on compact subsets of \mathbb{R}^3 . Since $\theta_n \in \mathbb{S}^2 \cap \{x_2 = 0\}$, any plane orthogonal to the limit of $\{\theta_n\}_n$ contains the vector H_∞ . Using this fact, minor modifications in the arguments inside the proof of Proposition 5 (when we proved that \mathcal{H}_2 is not flat) show that $\tilde{M}_\infty(1)$ is not flat. Applying Proposition 4 with the modified hypothesis of the limit surface not being flat instead of the normalization of the absolute curvature to have a maximum at the origin, and using that H_n converges while any choice of T_n diverges, we deduce that $\tilde{M}_\infty(1)$ is a singly periodic Scherk minimal surface of genus zero, two of whose ends are horizontal.

Consider an infinite closed horizontal cylinder $\mathcal{C} \subset \mathbb{R}^3$ around the x_2 -axis with radius large enough such that if we define $\Omega(1) = (\tilde{M}_\infty(1) \cap \mathcal{C})/H_\infty$, then $(\tilde{M}_\infty(1)/H_\infty) - \Omega(1)$

consists of four extremely flat annular Scherk-type ends. Since the region $\tilde{\Omega}_n(1) = [\tilde{M}_n - \tilde{p}_n(1)] \cap \mathcal{C}$ satisfies that $\tilde{\Omega}_n(1)/H_n$ converges uniformly to $\Omega(1)$ as $n \rightarrow \infty$ and $\Omega(1)$ has injective Gauss map, we deduce that $\tilde{\Omega}_n(1)/H_n$ has also injective Gauss map for n large. This implies that the points $\tilde{p}_n(i) - \tilde{p}_n(1)$, $2 \leq i \leq 2k$, are outside $\tilde{\Omega}_n(1)$. Using the same arguments we conclude that each of the sequences $\{\tilde{M}_n - \tilde{p}_n(2)\}_n, \dots, \{\tilde{M}_n - \tilde{p}_n(2k)\}_n$ (after passing to a common subsequence) converges to respective singly periodic Scherk minimal surfaces $\tilde{M}_\infty(2), \dots, \tilde{M}_\infty(2k)$, each one with two horizontal ends. Note that these singly periodic Scherk limits have the same period vector H_∞ . Taking the radius of \mathcal{C} large enough, we can assume that $(\tilde{M}_\infty(i)/H_\infty) - \Omega(i)$ consists of four extremely flat annular Scherk-type ends, where $\Omega(i) = (\tilde{M}_\infty(i) \cap \mathcal{C})/H_\infty$ for each $i = 1, \dots, 2k$. As before, if we define $\tilde{\Omega}_n(i) = [\tilde{M}_n - \tilde{p}_n(i)] \cap \mathcal{C}$ then $\tilde{\Omega}_n(i)/H_n$ converges uniformly to $\Omega(i)$ as $n \rightarrow \infty$. Furthermore, $[(\tilde{\Omega}_n(1) + \tilde{p}_n(1)) \cup \dots \cup (\tilde{\Omega}_n(2k) + \tilde{p}_n(2k))]/H_n$ embeds into M_n . Since any compact horizontal level section of M_n is connected and any compact horizontal level section of $\tilde{\Omega}_n(i)/H_n$ is a closed curve, we conclude that the minimum closed horizontal slab $S_n(i) \subset \mathbb{R}^3/H_n$ containing to $[\tilde{\Omega}_n(i) + \tilde{p}_n(i)]/H_n$ satisfies $S_n(i) \cap S_n(j) = \emptyset$ whenever $i \neq j$. Since the radius of the cylinder \mathcal{C} can be made arbitrarily large, we also conclude that both the width of the slabs $S_n(i)$ and $(T_n)_3$ diverge to ∞ as n increases.

We claim that all the limits $\tilde{M}_\infty(1), \dots, \tilde{M}_\infty(2k)$ are in fact the same singly periodic Scherk minimal surface. Clearly it suffices to check that the angle between the nonhorizontal ends of these surfaces and the horizontal does not depend on $i = 1, \dots, 2k$. Without loss of generality, we can assume that $\tilde{M}_\infty(1), \dots, \tilde{M}_\infty(2k)$ are ordered increasingly in heights, in the sense that for all n , $S_n(i+1)$ lies above $S_n(i)$, $i = 1, \dots, 2k-1$. Since the absolute total curvature of M_n is $8k\pi$, the one of a Scherk minimal surface with genus zero is 4π and we dispose of $2k$ of these limit surfaces, we conclude that the Gauss map N_n restricted to the complement of $M_n \cap (\cup_{i=1}^{2k} (S_n(i)/T_n))$ covers a set in \mathbb{S}^2 of arbitrarily small area. Let $C_n(j)$ be the component of $M_n - (\cup_{i=1}^{2k} (S_n(i)/T_n))$ that glues to $[\tilde{\Omega}_n(j) + \tilde{p}_n(j)]/\mathcal{P}_n, [\tilde{\Omega}_n(j+1) + \tilde{p}_n(j+1)]/\mathcal{P}_n$. Since $C_n(j)$ is a compact annulus in M_n , the Divergence Theorem shows that the flux vectors of $C_n(j)$ along its boundary curves are opposite; but such flux vectors converge as $n \rightarrow \infty$ to the fluxes of the limit Scherk surfaces $\tilde{M}_\infty(j)/H_\infty, \tilde{M}_\infty(j+1)/H_\infty$ around their respective upward and downward pointing nonhorizontal ends. Thus, $\tilde{M}_\infty(1) = \dots = \tilde{M}_\infty(2k)$ (and the Gauss map of $\tilde{M}_\infty(j)$ is opposite to the one of $\tilde{M}_\infty(j+1)$).

Since the Gauss map of a singly periodic Scherk surface of genus zero is unbranched, the $4k$ branch values (counting multiplicity) of the Gauss map g_n of M_n are located in $C_n(1) \cup \dots \cup C_n(2k)$. Given $j = 1, \dots, 2k$, g_n restricts to each of the two boundary components of the annulus $C_n(j)$ as a bijection onto the boundary of a small spherical disk centered at a point $N(j)$, this one being the limit normal vector of $\widetilde{M}_\infty(j)$ at its upward pointing nonhorizontal Scherk-type end or equivalently, the limit normal vector of $\widetilde{M}_\infty(j+1)$ at its downward pointing nonhorizontal Scherk-type end. By gluing two suitable disks D_1, D_2 to $C_n(j)$ along its boundary components, one can construct a meromorphic degree 2 map $G : C_n(j) \cup D_1 \cup D_2 \rightarrow \overline{\mathbb{C}}$. Since $C_n(j) \cup D_1 \cup D_2$ is a sphere, Riemann-Hurwitz formula gives that G has total branching number 2 and so, G has exactly two distinct simple branch points which lie necessarily in $C_n(j)$. Finally, let $\Pi \subset \mathbb{R}^3$ be a plane parallel to the nonhorizontal ends of $\widetilde{M}_\infty(1)$. Since for n large $g_n|_{C_n(j)}$ is contained in an arbitrarily small spherical disk centered in $N(j)$, we conclude that $C_n(j)$ can be expressed as the graph of a function $u_n(j) : (\Pi/H_n) \cap S'_n(j) \rightarrow \mathbb{R}$, where $S'_n(j)$ is the horizontal slab (quotiented by H_n) between $S_n(j)$ and $S_n(j+1)$. This finishes the proof. \square

Let S_ρ be the singly periodic Scherk minimal surface that appears as a limit in Proposition 6. Since the period vector H_∞ of S_ρ points to the x_2 -axis, the normal vectors at the ends of S_ρ (stereographically projected) can be written as $0, \infty, \rho, -1/\rho$ with $\rho > 0$, so we can parametrize S_ρ by the Weierstrass data

$$g(z) = z, \quad dh = c \frac{dz}{(z - \rho)(\rho z + 1)}, \quad z \in \overline{\mathbb{C}} - \{0, \infty, \rho, -1/\rho\}, \quad (16)$$

where $c \in \mathbb{R}^*$. To determine c , note that the intersection of S_ρ with the quotient by H_∞ of a horizontal plane at large positive height consists of a compact embedded curve Γ with period H_∞ (up to sign). Γ is the uniform limit of compact horizontal level sections Γ_n of the surfaces M_n (suitably translated and with the notation of Proposition 6). Since the vertical parts of the period and flux vectors of M_n along Γ_n are respectively 0, 2π for all n , it follows that

$$2\pi i = c \int_\Gamma \frac{dz}{(z - \rho)(\rho z + 1)} = 2\pi i c \operatorname{Res}_\rho \frac{dz}{(z - \rho)(\rho z + 1)} = \frac{2\pi i c}{\rho^2 + 1},$$

thus $c = \rho^2 + 1$ (we have assumed that the limit normal vector of S_ρ at its upward pointing nonhorizontal end is ρ and oriented Γ so that the second equality above holds; these choices determine the sign of c).

6.1 Weierstrass data.

Next we give a local chart for \mathcal{W} around the boundary point described in Proposition 6. Fix $\rho > 0$ and let $D(*, \varepsilon) \subset \mathbb{C}$ be a small disk of radius $\varepsilon > 0$ centered at $* = \rho, -1/\rho$. Given k unordered couples of points $a_{2i-1}, b_{2i-1} \in D(\rho, \varepsilon)$ with $a_{2i-1} \neq b_{2i-1}$ and another k couples $a_{2i}, b_{2i} \in D(-1/\rho, \varepsilon)$ with $a_{2i} \neq b_{2i}$, $1 \leq i \leq k$, we can construct a marked meromorphic map $g \in \mathcal{W}$ associated to these couples as follows. Consider $2k$ copies $\overline{\mathbb{C}}_1, \overline{\mathbb{C}}_2, \dots, \overline{\mathbb{C}}_{2k}$ of $\overline{\mathbb{C}}$. Cut $\overline{\mathbb{C}}_1$ along small disjoint arcs β_1, β_2 so that β_1 joins a_1 with b_1 and β_2 joins a_2 with b_2 . Cut $\overline{\mathbb{C}}_2$ along a copy of β_2 and also along an arc β_3 joining a_3 with b_3 , and glue $\overline{\mathbb{C}}_1$ with $\overline{\mathbb{C}}_2$ along the common cut β_2 in the usual way. Repeat the process so that $\overline{\mathbb{C}}_{2k}$ glues with $\overline{\mathbb{C}}_1$ along the common cut β_1 . This surgery produces a torus \mathbb{M} and the natural z -map on each copy of $\overline{\mathbb{C}}$ gives a well defined degree $2k$ meromorphic map $g : \mathbb{M} \rightarrow \overline{\mathbb{C}}$ with branch values $\{a_1, \dots, b_{2k}\}$. The ordered list of zeros and poles of the marked meromorphic map to be defined will be $(0_1, \dots, 0_{2k}, \infty_1, \dots, \infty_{2k})$, where the subindexes refer to the copy of $\overline{\mathbb{C}}$ which the zero or pole of z belongs to (we do not lose generality by assuming that $0, \infty$ do not lie in none of the β_j -curves). Finally, the nontrivial homology class $[\gamma] \in H_1(\mathbb{M} - \{\text{zeros, poles}\}, \mathbb{Z})$ is defined to be the class of $\{|z| = 1\}$ in $\overline{\mathbb{C}}_1$ with the anticlockwise orientation.

Remark 6 *For any $\rho \in (0, \infty) - \{1\}$, the circle $\{|z| = 1\}$ is a closed embedded curve in S_ρ with period zero (when $\rho = 1$ we must perturb slightly $\{|z| = 1\}$ to keep the vanishing period condition true). This fact justifies the above choice of the homology class $[\gamma]$.*

With the above procedure, the map $(a_1, b_1, \dots, a_{2k}, b_{2k}) \mapsto g \in \mathcal{W}$ is not injective, as one can exchange a_i by b_i obtaining the same g . Rather than parametrizing these marked meromorphic maps by the lists of its branch values, one can use the symmetric elementary polynomials in two variables for each couple a_i, b_i . Note that when a_i, b_i are close to ρ (resp. $-1/\rho$) then $(a_i + b_i, a_i b_i)$ is close to $(2\rho, \rho^2)$ (resp. $(-2/\rho, 1/\rho^2)$). The computations that follow simplify if we use the arithmetic and geometric means instead of the elementary symmetric polynomials, which is possible since the map $(u, v) \mapsto (u/2, \sqrt{v})$ is a local diffeomorphism outside $(0, 0)$. We define

$$x_i = \frac{1}{2}(a_i + b_i), \quad y_i = \sqrt{a_i b_i},$$

so (x_i, y_i) lies in a neighborhood of (ρ, ρ) or $(-1/\rho, 1/\rho)$. Given $\varepsilon' > 0$, we label $\mathcal{U}(\varepsilon') = \left[D(\rho, \varepsilon') \times D(\rho, \varepsilon') \times D(\frac{-1}{\rho}, \varepsilon') \times D(\frac{1}{\rho}, \varepsilon') \right]^k$. As $x_i^2 - y_i^2 = \frac{1}{4}(a_i - b_i)^2$, the condition $a_i \neq b_i$

necessary for the above construction is equivalent to $x_i^2 \neq y_i^2$. Let $\mathcal{A} = \{(x_1, y_1, \dots, x_{2k}, y_{2k}) \mid x_i^2 = y_i^2 \text{ for some } i = 1 \dots, 2k\}$. Clearly \mathcal{A} is an analytic subvariety of \mathbb{C}^{4k} . It can be shown that for $\varepsilon' > 0$ small, the correspondence $\mathbf{z} = (x_1, y_1, \dots, x_{2k}, y_{2k}) \in \mathcal{U}(\varepsilon') - \mathcal{A} \mapsto \aleph(\mathbf{z}) = g \in \mathcal{W}$ defines a local chart for \mathcal{W} .

Remark 7

- (i) *If a marked meromorphic map $g = \aleph(\mathbf{z})$ produces a marked surface M , then the ordered list $(0_1, \dots, 0_{2k}, \infty_1, \dots, \infty_{2k})$ does not necessarily coincide with the ordering on the ends of $M \in \tilde{\mathcal{S}}$. This is only a matter of notation and will not affect to the arguments that follow.*
- (ii) *Consider a sequence $\{M_n\}_n \subset \tilde{\mathcal{S}}$ with $\{C(M_n) = (a_n, b_n)\}_n \rightarrow (a, b) \in \mathbb{R}^* \times \mathbb{C}$ and $\{T_n\}_n \rightarrow \infty$ as $n \rightarrow \infty$ (for any choice of T_n). By Proposition 6, the sequence of geometric surfaces $\{M_n\}_n$ converges uniformly to $2k$ -copies of a singly periodic Scherk minimal surface S_ρ parametrized as in (16), for certain $\rho > 0$. Let Γ be the curve $\{|z| = 1\}$ viewed in one of the copies of S_ρ . Clearly, Γ is the uniform limit as $n \rightarrow \infty$ of a sequence of closed curves $\Gamma_n \subset M_n$ with $P_{\Gamma_n} = 0$. After exchanging the homology class of the marked surface M_n by $[\Gamma_n] \in H_1(M_n, \mathbb{Z})$, we can see the same geometric surface M_n as a new marked surface M'_n inside the domain of the chart \aleph for n large enough (also note that the second component of $C(M'_n)$ differs from b_n in a fixed even multiple of $i\pi a_n$).*

6.2 Holomorphic extension.

When $\mathbf{z} \in \mathcal{A}$, the continuous extension of the above cut-and-paste process gives a Riemann surface with nodes (see [8] page 245 for the definition of a Riemann surface with nodes), each node occurring between copies $\overline{\mathbb{C}}_{j-1}, \overline{\mathbb{C}}_j$ such that $a_j = b_j$. The corresponding differential ϕ also extends through \mathbf{z} , in the following manner.

Proposition 7 *Each $\mathbf{z} \in \mathcal{A}$ produces l spheres S_1, \dots, S_l joined by l node points $P_i, Q_i \in S_i$ (here $Q_i = P_{i+1}$ and the subindexes are cyclic), g degenerates in l nonconstant meromorphic maps $g(i) : S_i \rightarrow \overline{\mathbb{C}}$ with the degrees of $g(1), \dots, g(l)$ adding up to $2k$ and $g(i)$ takes the values ρ and/or $-1/\rho$ at P_i, Q_i . ϕ degenerates in the l unique meromorphic differentials $\phi(i)$ on S_i , such that $\phi(i)$ has exactly two simple poles at P_i, Q_i with residues 1*

at P_i and -1 at Q_i (these residues are determined by the equation $\int_{|z|=1} \phi = 2\pi i$). Finally, both g and ϕ depend holomorphically on all parameters (including at points of \mathcal{A}).

Proof. See Lemma 8 of [14] for a similar situation as ours (tori degenerating in spheres); see also Section 3.4 of [27] for arbitrary genus. \square

Lemma 6 *The ligature map L extends holomorphically to $\mathcal{U}(\varepsilon')$.*

Proof. Since $\mathcal{U}(\varepsilon') \cap \mathcal{A}$ is an analytic subvariety of $\mathcal{U}(\varepsilon')$, it suffices to check that L is bounded by the Riemann extension Theorem for several variables (see for instance [4] page 9). Let $g_n = \aleph(\mathbf{z}(n))$ be a sequence with $\mathbf{z}(n) \in \mathcal{U}(\varepsilon') - \mathcal{A}$ converging to $\mathbf{z} \in \mathcal{U}(\varepsilon') \cap \mathcal{A}$. Note that each component of $L(g_n)$ can be written as an integral along a curve α independent of n which lies in one of the twice punctured spheres $S_j - \{P_j, Q_j\}$ of Proposition 7, of a holomorphic differential $\varphi(n)$, and that $\{\varphi(n)\}_n$ converge uniformly on α as $n \rightarrow \infty$ to some holomorphic differential φ on $S_j - \{P_j, Q_j\}$. From here we directly deduce that $L(g_n)$ is bounded as desired. \square

6.3 Partial derivatives and Inverse Function Theorem.

Theorem 2 *There exists $\varepsilon' > 0$ small such that $L|_{\mathcal{U}(\varepsilon')}$ is a biholomorphism.*

Proof. Given a list $\mathbf{z} = (x_1, y_1, \dots, x_{2k}, y_{2k}) \in \mathcal{U}(\varepsilon') - \mathcal{A}$, we will denote by $g = \aleph(\mathbf{z})$ the associated marked meromorphic map. Recall that $0_j, \infty_j$ belong to the copy $\overline{\mathbb{C}}_j$ of $\overline{\mathbb{C}}$. For $j = 1, \dots, k$, we let Γ_{2j-1} denote the closed curve in $g^{-1}(\overline{\mathbb{C}})$ that corresponds to the loop $\{|z| = 1\}$ on $\overline{\mathbb{C}}_{2j-1}$ (if $\rho = 1$, then we modify slightly $\{|z| = 1\}$ as mentioned in Remark 6). We orient Γ_1 to coincide with the last component of g and the remaining Γ_{2j-1} to be homologous to Γ_1 in $g^{-1}(\overline{\mathbb{C}})$. Hence $\text{Res}_{0_{2j}}(\frac{\phi}{g}) + \text{Res}_{\infty_{2j+1}}(\frac{\phi}{g}) = \frac{1}{2\pi i} (\int_{\Gamma_{2j+1}} \frac{\phi}{g} - \int_{\Gamma_{2j-1}} \frac{\phi}{g})$ and a similar formula holds for poles of g . Thus, the composition of L with a certain regular linear transformation in \mathbb{C}^{4k} writes as $\widehat{L} : \mathcal{U}(\varepsilon') \rightarrow \mathbb{C}^{4k}$ where

$$\widehat{L}(\mathbf{z}) = \left(\underbrace{\text{Res}_{0_{2j-1}}\left(\frac{\phi}{g}\right), \text{Res}_{\infty_{2j-1}}(g\phi)}_{1 \leq j \leq k}, \underbrace{\int_{\Gamma_{2j-1}} \frac{\phi}{g}, \int_{\Gamma_{2j-1}} g\phi}_{1 \leq j \leq k} \right).$$

By the Inverse Function Theorem, it suffices to prove that the differential of \widehat{L} at the point $(\rho, \rho, \frac{-1}{\rho}, \frac{1}{\rho})^k \in \mathbb{C}^{4k}$ corresponding to $(2k$ copies of) the singly periodic Scherk minimal

surface S_ρ , is an automorphism of \mathbb{C}^{4k} . The computations that follow are similar to those in the proof of Lemma 9 in [14], so we will only explain them briefly.

Fix $j = 1, \dots, k$. To compute $\frac{\partial \widehat{L}}{\partial x_{2j-1}}(S_\rho)$, we differentiate at $x = \rho$ the composition of \widehat{L} with the curve $x \in D(\rho, \varepsilon') \mapsto (\rho, \rho, \frac{-1}{\rho}, \frac{1}{\rho}, \dots, \frac{1}{\rho}, x, \rho, \frac{-1}{\rho}, \frac{1}{\rho}, \dots, \rho, \rho, \frac{-1}{\rho}, \frac{1}{\rho}) \in \mathcal{U}(\varepsilon')$, where x is placed at the $(4j-3)$ -th component of the $4k$ -tuple. This curve produces $2k-1$ spheres $S_1, S_2, \dots, S_{2j-2}, S_{2j}, \dots, S_{2k}$ and meromorphic maps $g_1, g_2, \dots, g_{2j-2}, g_{2j}, \dots, g_{2k}$ on them such that

1. For each $m = 1, \dots, 2k$ with $m \neq 2j-2, 2j-1$, $g_m : S_m \rightarrow \overline{\mathbb{C}}$ is a biholomorphism. Thus we can parametrize S_m by $\overline{\mathbb{C}}$ with $g_m(z) = z$. With this parameter, the node points correspond to $\rho, \frac{-1}{\rho}$, and $\phi = c \frac{dz}{(z-\rho)(z+\rho^{-1})}$ where $c \in \mathbb{C}^*$ is determined by the equation $\int_{|z|=1} \phi = 2\pi i$ (in particular, c does not depend on x).
2. $g_{2j-2} : S_{2j-2} \rightarrow \overline{\mathbb{C}}$ has degree two. We can parametrize S_{2j-2} by $\{(z, w) \in \overline{\mathbb{C}}^2 \mid w^2 = z^2 - 2xz + \rho^2\}$, hence $w = \sqrt{z^2 - 2xz + \rho^2}$ is well defined on S_{2j-2} (we fix the sign of the square root so that $w \sim z - \rho$ in $\overline{\mathbb{C}}_{2j-1}$ and $w \sim -(z - \rho)$ in $\overline{\mathbb{C}}_{2j-2}$). Let Q denote the point $z = -1/\rho$ in $\overline{\mathbb{C}}_{2j-1}$. Then $g(z, w) = z$ and $\phi = c(x) \frac{dz}{(z+\rho^{-1})w}$, where

$$-1 = \text{Res}_Q \phi = \text{Res}_Q \frac{c(x) dz}{(z + \rho^{-1})w} = \frac{c(x)}{w(Q)}, \quad (17)$$

where we have assumed $\rho < 1$ (the case $\rho > 1$ is similar). Since the components of \widehat{L} computed on S_m with $m \neq 2j-2, 2j-1$ do not depend on x , the corresponding derivative with respect to x vanishes. From (17) one has $c(\rho) = -w(Q)|_{x=\rho} = \frac{\rho^2+1}{\rho}$, $c'(\rho) = -\frac{d}{dx}\Big|_{x=\rho} w(Q) = \frac{1}{\rho^2+1}$. An elementary calculation gives

$$\begin{aligned} \frac{d}{dx}\Big|_{x=\rho} \text{Res}_{0_{2j-1}} \left(\frac{\phi}{g} \right) &= \frac{d}{dx}\Big|_{\rho} \text{Res}_{(0, -\rho)} \frac{c(x) dz}{z(z + \rho^{-1})w} = -c'(\rho), & \frac{d}{dx}\Big|_{\rho} \text{Res}_{\infty_{2j-1}} (g\phi) &= -c'(\rho), \\ \frac{d}{dx}\Big|_{\rho} \int_{\Gamma_{2j-1}} \frac{\phi}{g} &= \frac{d}{dx}\Big|_{\rho} \int_{|z|=1} \frac{c(x) dz}{z(z + \rho^{-1})w} = 0, & \frac{d}{dx}\Big|_{\rho} \int_{\Gamma_{2j-1}} g\phi &= \frac{2\pi i}{\rho^2 + 1}, \end{aligned}$$

where $\Gamma_{2j-1} \subset S_{2j-2}$ is the connected lift of $\{|z|=1\}$ to $\overline{\mathbb{C}}_{2j-1}$ through the z -map. Thus,

$$\frac{\partial \widehat{L}}{\partial x_{2j-1}} = \frac{1}{\rho^2+1} \left(0, \dots, 0, \overset{(2j-1)}{-1}, \overset{(2j)}{-1}, 0, \dots, 0, \overset{(2k+2j-1)}{0}, \overset{(2k+2j)}{2\pi i}, 0, \dots, 0 \right).$$

Analogous derivation of the composition of \widehat{L} with similar curves gives

$$\begin{aligned}\frac{\partial \widehat{L}}{\partial y_{2j-1}} &= \frac{\rho^2}{\rho^2+1} \left(0, \dots, 0, 1 - \frac{(2j-1)(\rho^2+1)^2}{\rho^4}, \quad \begin{matrix} (2j) \\ -1 \end{matrix}, 0, \dots, 0, \quad \begin{matrix} (2k+2j-1) \\ 0 \end{matrix}, \quad \begin{matrix} (2k+2j) \\ 2\pi i \end{matrix}, 0, \dots, 0 \right); \\ \frac{\partial \widehat{L}}{\partial x_{2j}} &= \frac{\rho^2}{\rho^2+1} \left(0, \dots, 0, \quad \begin{matrix} 1 \\ 1 \end{matrix}, \quad \begin{matrix} 1 \\ 1 \end{matrix}, 0, \dots, 0, \quad \begin{matrix} 2\pi i \\ 0 \end{matrix}, \quad \begin{matrix} 0 \\ 0 \end{matrix}, 0, \dots, 0 \right); \\ \frac{\partial \widehat{L}}{\partial y_{2j}} &= \frac{(\rho^2+1)^2-1}{\rho^2+1} \left(0, \dots, 0, \quad \begin{matrix} 1 \\ \frac{1}{1-(\rho^2+1)^2} \end{matrix}, 0, \dots, 0, \quad \begin{matrix} 2\pi i \\ 0 \end{matrix}, \quad \begin{matrix} 0 \\ 0 \end{matrix}, 0, \dots, 0 \right).\end{aligned}$$

The absolute Jacobian of \widehat{L} at the point S_ρ is the absolute value of the determinant of the above files, which turns out to be $(2\pi)^{2k}$, and the Theorem follows. \square

7 Uniqueness of examples around the catenoid.

When a sequence $\{M_n\}_n \in \widetilde{\mathcal{S}}$ degenerates in a vertical catenoid (case (i) of Proposition 4), the residues in the ligature map L diverge to ∞ . In this Section we will modify L to have a well defined locally invertible extension through this boundary point of \mathcal{W} .

Proposition 8 *Let $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ be a sequence with $\{C(M_n) = (a_n, b_n)\}_n \rightarrow (\infty, 0)$. Then for n large, the geometric surface M_n is close to $2k$ translated images of arbitrarily large compact regions of a catenoid with flux $(0, 0, 2\pi)$, together with $2k$ regions $C_n(1), \dots, C_n(2k)$. Each $C_n(j)$ is a twice punctured annulus with one left end, one right end of M_n and two distinct simple branch points of its Gauss map. Furthermore, $C_n(j)$ is a graph over its horizontal projection on $\{x_3 = 0\}/H_n$.*

Proof. As in Proposition 6, it suffices to check that there exists a subsequence of $\{M_n\}_n$ verifying our assertions. Given $n \in \mathbb{N}$, let $p_n(1), \dots, p_n(2k) \in M_n$ be the $2k$ distinct points applied by the Gauss map N_n of M_n on a prescribed regular value $\theta_n \in \mathbb{S}^2 \cap \{x_2 = 0\}$ such that a spherical disk $D_{\mathbb{S}^2}(\theta_n, \varepsilon)$ of sufficiently small uniform radius ε centered at θ_n does not contain neither branch values of N_n nor the North and South Poles. Let $\widetilde{M}_n \subset \mathbb{R}^3$ be the doubly periodic surface obtained by lifting M_n and $\widetilde{p}_n(i)$ the point that corresponds to $p_n(i)$ in a fixed fundamental domain of \widetilde{M}_n , $i = 1 \dots, 2k$. By Proposition 5 and Lemma 5, the sequence $\{\widetilde{M}_n - \widetilde{p}_n(1)\}_n$ has uniform curvature and local area bounds. After extracting a subsequence, $\widetilde{M}_n - \widetilde{p}_n(1)$ converges uniformly on compact subsets of \mathbb{R}^3 to a properly embedded minimal surface $\widetilde{M}_\infty(1) \subset \mathbb{R}^3$. The same argument as in the

proof of Proposition 6 proves that $\widetilde{M}_\infty(1)$ is not flat, so it lies in one of the six cases in Proposition 4. Using that $a_n \rightarrow \infty$ as $n \rightarrow \infty$, we discard the cases (iv) , (v) , (vi) of that Proposition. Since the vertical part of the flux of all the M_n along a compact horizontal section is 2π and the vertical part of the conormal vector to M_n along such a horizontal section cannot vanish, we conclude that $\widetilde{M}_\infty(1)$ is not a vertical helicoid. Now suppose that $\widetilde{M}_\infty(1)$ is a Riemann minimal example, and let $F \in \mathbb{C}^*$ be the horizontal part of its flux along a compact horizontal section. Then for n large there exists a closed curve $\Gamma_n \subset \widetilde{M}_n - \widetilde{p}_n(1)$ such that $F(\Gamma_n)$ tends to F as $n \rightarrow \infty$. But $F(\Gamma_n) - i\overline{b}_n = t(n)\pi a_n$ for certain even integer $t(n)$, which gives a contradiction after taking limits. Hence $\widetilde{M}_\infty(1)$ is a vertical catenoid with flux $(0, 0, 2\pi)$.

Reasoning as in the proof of Proposition 6 we conclude that for n large, the points $\widetilde{p}_n(i) - \widetilde{p}_n(1)$, $2 \leq i \leq 2k$, are outside a large compact domain of $\widetilde{M}_n - \widetilde{p}_n(1)$ arbitrarily close to a vertical catenoid, and that after passing to a subsequence, $\{\widetilde{M}_n - \widetilde{p}_n(2)\}_n, \dots, \{\widetilde{M}_n - \widetilde{p}_n(2k)\}_n$ converge to different translations of $\widetilde{M}_\infty(1)$. Since an arbitrarily large compact region $\Omega(i)$ of $\widetilde{M}_\infty(1)$ can be uniformly approximated by compact regions $\widetilde{\Omega}_n(i) \subset \widetilde{M}_n - \widetilde{p}_n(i)$, it follows that $(T_n)_3 \rightarrow \infty$ as $n \rightarrow \infty$. Also note that the regions $[\widetilde{\Omega}_n(i) + \widetilde{p}_n(i)]/H_n$ can be chosen as the intersection of \widetilde{M}_n/H_n with disjoint horizontal slabs $S_n(i) \subset \mathbb{R}^3/H_n$ whose widths go to ∞ as $n \rightarrow \infty$.

For $j = 1, \dots, 2k$, let $C_n(j)$ be the component of $M_n - (\cup_{i=1}^{2k} (S_n(i)/T_n))$ that glues to $[\widetilde{\Omega}_n(j) + \widetilde{p}_n(j)]/\mathcal{P}_n$, $[\widetilde{\Omega}_n(j+1) + \widetilde{p}_n(j+1)]/\mathcal{P}_n$ (we can assume that $S_n(j+1)$ is directly above $S_n(j)$). A straightforward modification of the argument in Proposition 6 using the injectivity of the Gauss map of a catenoid shows that each $C_n(j)$ can be compactified by adding two ends $e_1(n), e_2(n)$ of M_n to obtain a compact annulus with two single branch points of the Gauss map N_n of M_n , and that N_n applies bijectively each boundary curve of $C_n(j)$ on the boundary of a small spherical disk centered at $(0, 0, \pm 1)$. As the fluxes around $e_1(n), e_2(n)$ cancel (by the Divergence Theorem and because the flux of M_n along each boundary curve of $C_n(j)$ tends to finite vertical but the flux at a Scherk-type end of M_n is horizontal and arbitrarily large), we deduce that $e_1(n)$ is a right end of M_n and $e_2(n)$ a left end (or vice versa), both with the same limit normal vector. Finally, the horizontal projection of $C_n(j)$ onto its image in $\{x_3 = 0\}/H_n$ extends smoothly across $e_1(n), e_2(n)$ giving rise to a proper local diffeomorphism hence a finite covering map, whose degree is one because $C_n(j)$ has one left end and one right end. This gives the graphical property of $C_n(j)$ and finishes the proof of the Proposition. \square

7.1 Weierstrass data.

Following the line of arguments in Section 6, we next show a local chart for \mathcal{W} around the boundary point that appears in Proposition 8. Given $i = 1, \dots, k$, choose points a_{2i-1}, b_{2i-1} (resp. a_{2i}, b_{2i}) in a small punctured neighborhood of 0 (resp. of ∞) in $\overline{\mathbb{C}}$, such that $a_j \neq b_j$ for any j . These unordered couples can be considered as the branch values of a meromorphic map g of degree $2k$, and a cut-and-paste construction analogous to the one in Section 6 gives rise to a marked meromorphic map $g \in \mathcal{W}$. Since the roles of a_j and b_j are symmetric, their elementary symmetric functions are right parameters in this setting. We introduce the parameters

$$\begin{aligned} x_j &= \frac{1}{2}(a_j + b_j), & y_j &= a_j b_j & \text{if } j \text{ is odd} \\ x_j &= \frac{1}{2} \left(\frac{1}{a_j} + \frac{1}{b_j} \right), & y_j &= \frac{1}{a_j b_j} & \text{if } j \text{ is even} \end{aligned}$$

so all parameters x_j, y_j , $1 \leq j \leq 2k$, are close to 0. Also the conditions on a_j, b_j translate into $y_j \neq x_j^2$ and $y_j \neq 0$. In what follows, we abbreviate $\mathbf{x} = (x_1, \dots, x_{2k})$, $\mathbf{y} = (y_1, \dots, y_{2k})$. Given $\varepsilon > 0$, we let

$$\begin{aligned} D(0, \varepsilon)^{4k} &= \{(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^{4k} \mid |x_j|, |y_j| < \varepsilon \text{ for all } j = 1, \dots, 2k\}, \\ \mathcal{B} &= \{(\mathbf{x}, \mathbf{y}) \in D(0, \varepsilon)^{4k} \mid x_j^2 = y_j \text{ for some } j\}, \\ \widehat{\mathcal{B}} &= \{(\mathbf{x}, \mathbf{y}) \in D(0, \varepsilon)^{4k} \mid y_j = 0 \text{ for some } j\}. \end{aligned}$$

$\mathcal{B} \cup \widehat{\mathcal{B}}$ is an analytic subvariety of the polydisk $D(0, \varepsilon)^{4k}$ and the map $(\mathbf{x}, \mathbf{y}) \in D(0, \varepsilon)^{4k} - (\mathcal{B} \cup \widehat{\mathcal{B}}) \mapsto \chi(\mathbf{x}, \mathbf{y}) = g \in \mathcal{W}$ is a local chart for \mathcal{W} . We also let $(\mathbf{0}, \mathbf{0})$ be the tuple $(\mathbf{x}, \mathbf{y}) = (0, \dots, 0) \in D(0, \varepsilon)^{4k}$.

Remark 8 *An argument as in Remark 7-(ii) shows that given a sequence $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ with $C(M_n) \rightarrow (\infty, 0)$, there exists another sequence of marked surfaces $\{M'_n\}_n$ inside the image of the chart χ such that for each n , M_n, M'_n only differ in the homology class in the last component of the marked surface.*

Next we find the equations to solve in order to produce an immersed minimal surface. Let $\overline{\mathbb{C}}_j$ be the j^{th} copy of $\overline{\mathbb{C}}$ and let Γ_j be the circle defined by $|z| = 1$ in $\overline{\mathbb{C}}_j$, with the positive orientation if j is odd and the negative orientation if j is even. All these curves are homologous in $g^{-1}(\overline{\mathbb{C}})$, and $[\Gamma_1]$ is the last component of the marked meromorphic map g . We also write $0_j, \infty_j$ for the points $z = 0$ and $z = \infty$ in $\overline{\mathbb{C}}_j$, see Figure 8 for the

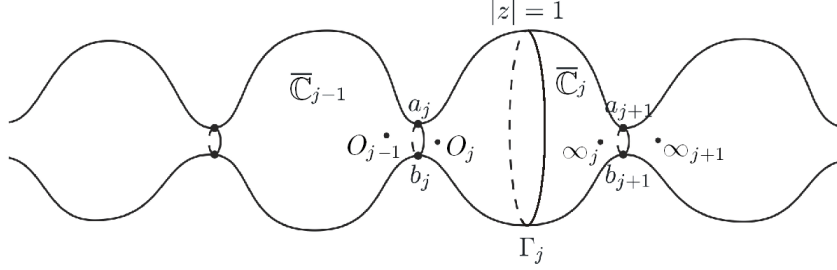


Figure 8: In this case, j is odd.

case j odd. Recall that ϕ is defined as the unique holomorphic 1-form with $\int_{\gamma} \phi = 2\pi i$. Define for $1 \leq j \leq 2k$

$$A_j = \begin{cases} \int_{\Gamma_j} g^{-1}\phi & (j \text{ odd}) \\ \int_{\Gamma_{j+1}} g\phi & (j \text{ even}) \end{cases} \quad B_j = \begin{cases} \text{Res}_{0_{j-1}}(g^{-1}\phi) \cdot \text{Res}_{0_j}(g^{-1}\phi) & (j \text{ odd}) \\ \text{Res}_{\infty_{j-1}}(g\phi) \cdot \text{Res}_{\infty_j}(g\phi) & (j \text{ even}) \end{cases}$$

In this definition and in the sequel we will adopt a cyclic convention on the subindexes, so when $j = 1$, $j - 1$ must be understood as $2k$. By the residue theorem, we have for odd j

$$\begin{aligned} A_j - A_{j-2} &= 2\pi i (\text{Res}_{0_j} g^{-1}\phi + \text{Res}_{0_{j-1}} g^{-1}\phi), \\ A_{j+1} - A_{j-1} &= 2\pi i (\text{Res}_{\infty_{j+1}} g\phi + \text{Res}_{\infty_j} g\phi). \end{aligned}$$

It follows easily that g closes periods if and only if there exist $a \in \mathbb{R}$, $b \in \mathbb{C}$ such that

$$\left. \begin{aligned} A_{2i-1} &= b, A_{2i} = \bar{b} && \text{for all } i = 1, \dots, k, \\ B_j &= -a^2 && \text{for all } j = 1, \dots, 2k, \end{aligned} \right\} \quad (18)$$

which are the equations we have to solve.

Remark 9 *The definition of A_j and B_j are motivated by the following fact: Fix j , say odd. There is no natural way to distinguish between the two zeros of g which are close to the branch points a_j, b_j . We call them 0_{j-1} and 0_j for convenience, but this is a little bit artificial. Indeed the fact that one zero belongs to \bar{C}_j or \bar{C}_{j-1} depends on the choice of the common cut β_j connecting the points a_j and b_j , but there is no way to choose β_j depending continuously on the parameters (this is a homotopy issue). In other words, the residue of $g^{-1}\phi$ at 0_j is not a well defined function of the parameters. On the other hand, the unordered pair $\{0_{j-1}, 0_j\}$ depends continuously on the parameters. For this reason, we should consider the elementary symmetric functions of the residues at 0_{j-1} and 0_j . This is essentially what we do in the definition of A_j and B_j .*

7.2 Holomorphic extension.

When $a_j = b_j$ (which corresponds to $y_j = x_j^2$) for a given j , the definition of g gives a Riemann surface with a node between $\overline{\mathbb{C}}_{j-1}$ and $\overline{\mathbb{C}}_j$. More precisely, each $(\mathbf{x}, \mathbf{y}) \in \mathcal{B}$ gives rise to the following objects:

- A Riemann surface with nodes, consisting of l spheres S_j joined by node points P_j, Q_j so that $P_j = Q_{j+1}$.
- l nonconstant meromorphic maps $g(j) : S_j \rightarrow \overline{\mathbb{C}}$ with $\sum_j \deg(g(j)) = 2k$ and $g(j)(\{P_j, Q_j\}) \subset \{0, \infty\}$.
- l meromorphic differentials $\phi(j)$ on S_j with just two simple poles at P_j, Q_j and residues 1 at P_j , -1 at Q_j .

If $a_j = 0$ and $b_j \neq 0$ (or vice versa) for j odd, then the conformal structure between the copies $\overline{\mathbb{C}}_{j-1}$ and $\overline{\mathbb{C}}_j$ does not degenerate, but the corresponding z -map has a double zero. For j even we have a similar behavior exchanging zero by pole. Thus each $(\mathbf{x}, \mathbf{y}) \in \widehat{\mathcal{B}} - \mathcal{B}$ produces a conformal torus \mathbb{M} and a single meromorphic degree $2k$ map $g : \mathbb{M} \rightarrow \overline{\mathbb{C}}$ with at least a double zero or pole. ϕ extends to the (unique) holomorphic differential on \mathbb{M} with $\int_{\Gamma_1} \phi = 2\pi i$.

Remark 10 *Although we will not use it, the points of the form $(\mathbf{x}, \mathbf{0}) \in \widehat{\mathcal{B}} - \mathcal{B}$ with $\mathbf{x} = (\lambda, -\lambda, \dots, \lambda, -\lambda)$ for $\lambda > 0$ small, represent boundary points of \mathcal{W} corresponding to Riemann minimal examples close to a stack of vertical catenoids.*

Proposition 9 *Both g and ϕ depend holomorphically on all parameters (\mathbf{x}, \mathbf{y}) in a neighborhood of $(\mathbf{0}, \mathbf{0})$ (including at points of $\mathcal{B} \cup \widehat{\mathcal{B}}$).*

Proof. Same as Proposition 7. □

Proposition 10 *For $1 \leq j \leq 2k$, the functions A_j and $\widetilde{B}_j = y_j B_j$ extend holomorphically through $\mathcal{B} \cup \widehat{\mathcal{B}}$.*

Proof. The extendability of A_j through a point $(\mathbf{x}_0, \mathbf{y}_0) \in \mathcal{B} \cup \widehat{\mathcal{B}}$ is a consequence of Proposition 9 since the curves Γ_h stay in the limit Riemann surface minus its nodes (if

any). For \tilde{B}_j we cannot apply Proposition 9 directly because some of the points O_{j-1}, O_j or ∞_{j-1}, ∞_j where we compute the residues may collapse either into node points (when $(\mathbf{x}_0, \mathbf{y}_0) \in \mathcal{B}$) or into branch points of g (when $(\mathbf{x}_0, \mathbf{y}_0) \in \widehat{\mathcal{B}} - \mathcal{B}$). Rather we estimate the rate at which these residues blow-up.

We prove the extendability of \tilde{B}_j when j is odd, the proof for j even is similar. Consider a tuple $(\mathbf{x}, \mathbf{y}) \in D(0, \varepsilon)^{4k} - (\mathcal{B} \cup \widehat{\mathcal{B}})$ close to $(\mathbf{0}, \mathbf{0})$, and let $\Omega \in g^{-1}(\overline{\mathbb{C}})$ be the annulus bounded by Γ_{j-1} and Γ_j , where $g = \aleph(\mathbf{x}, \mathbf{y})$. We shall first make a conformal representation of this domain into a standard annulus. Ω contains the branch points $a = a_j, b = b_j$. Recall that a, b are close to 0. We introduce the functions

$$u = \sqrt{\frac{z-a}{z-b}}, \quad v = \frac{1+u}{1-u}$$

and fix the sign of the square root by asking that $u \sim 1$ on Γ_j and consequently $u \sim -1$ on Γ_{j-1} . Then both u and v are conformal representations of Ω . On Γ_j we may write $z = e^{i\theta}$

$$u = \sqrt{1 + \frac{(b-a)e^{-i\theta}}{1-be^{-i\theta}}} \sim 1 + \frac{1}{2}(b-a)e^{-i\theta}$$

when a, b are close to 0. Consequently, $u(\Omega)$ is close to $\overline{\mathbb{C}}$ minus the two disks $D(\pm 1, r/2)$ where $r = |b-a|$, and $v(\Omega)$ is close to the annulus $D(0, 4/r) - D(0, r/4)$. So we may write the Laurent series of ϕ with respect to the variable v in the annulus $D(0, 1/r) - D(0, r)$:

$$\phi = \sum_{n \in \mathbb{Z}} c_n v^n dv$$

where $c_n = \frac{1}{2\pi i} \int_{|v|=1} \frac{\phi}{v^{n+1}}$ depends holomorphically on all parameters. Since ϕ extends holomorphically, the integral of $|\phi|$ on each unit circle $|z|=1$ extends analytically. Hence it is bounded by some constant $C > 0$ (for the remainder of the proof we shall denote by C different positive constants). On Γ_j we have $|v| \sim 4/r$, hence $|v| \geq 1/r$. This gives if $n+1 \geq 0$

$$|c_n| = \frac{1}{2\pi} \left| \int_{\Gamma_j} \frac{\phi}{v^{n+1}} \right| \leq \frac{1}{2\pi} \int_{\Gamma_j} \frac{|\phi|}{|v|^{n+1}} \leq Cr^{n+1}.$$

On Γ_{j-1} we have $|v| \sim r/4$ hence $|v| \leq r$. This gives if $n+1 \leq 0$

$$|c_n| \leq \frac{1}{2\pi} \int_{\Gamma_{j-1}} |\phi| |v|^{|n+1|} \leq Cr^{|n+1|}.$$

In summary,

$$|c_n| \leq Cr^{|n+1|} \quad \text{for all } n \in \mathbb{Z}. \quad (19)$$

We now compute the residues appearing in B_j . First note that $z = 0$ is equivalent to $v = \alpha$ or $v = 1/\alpha$, where

$$\alpha = \frac{1 + \sqrt{a/b}}{1 - \sqrt{a/b}}.$$

Furthermore, $g = z = \frac{a-b}{4v}(v - \alpha)(v - 1/\alpha)$ hence

$$\text{Res}_{v=\alpha} (g^{-1}\phi) = \frac{4}{(a-b)(\alpha - 1/\alpha)} \sum_{n \in \mathbb{Z}} c_n \alpha^{n+1} = -\frac{1}{\sqrt{ab}} \sum_{n \in \mathbb{Z}} c_n \alpha^{n+1}.$$

(Observe here that this residue is indeed a multivalued function as was explained in Remark 9). From the equality $(b-a)\alpha = (\sqrt{a} + \sqrt{b})^2$ we see that $|\alpha| \ll 1/r$ and $|\alpha|^{-1} \ll 1/r$. Using this information, the estimate (19) and the convergence of the series $\sum_{n \geq 1} t^n$ for any $t \in (0, 1)$, it is straightforward to check that $\sqrt{y_j} \text{Res}(g^{-1}\phi)$ is bounded, where the residue is computed at 0_j or 0_{j-1} . Hence $y_j B_j$ is bounded on the set $y_j \neq x_j^2, y_j \neq 0$. Since $y_j B_j$ is a well defined holomorphic function, it extends holomorphically by the Riemann extension theorem. \square

7.3 Partial derivatives and Inverse Function Theorem.

Proposition 11 *For each $j = 1, \dots, 2k$ it holds*

$$A_j(\mathbf{0}, \mathbf{0}) = 0, \quad \tilde{B}_j(\mathbf{0}, \mathbf{0}) = -1, \quad \frac{\partial A_{j-1}}{\partial x_j}(\mathbf{0}, \mathbf{0}) = 2\pi i$$

(with A_0 understood as A_{2k}), and all remaining partial derivatives of the A_h are zero. We will not need the partial derivatives of the \tilde{B}_h .

Proof. Let $(\mathbf{x}, \mathbf{y})_j = (0, \dots, 0, x_j, 0, \dots, 0, y_j, 0, \dots, 0)$ with $y_j \neq x_j^2$ and $y_j \neq 0$. The Riemann surface associated to $(\mathbf{x}, \mathbf{y})_j$ has $2k - 1$ nodes which disconnect it into $2k - 1$ genus zero components. On $2k - 2$ of these components (which we will call *simple spheres*), the corresponding meromorphic map g_m is the usual degree one z -map if we see the simple sphere as a copy of $\overline{\mathbb{C}}$, and the height differential ϕ becomes $\frac{dz}{z}$. The remaining component S is obtained from the copies $\overline{\mathbb{C}}_{j-1}, \overline{\mathbb{C}}_j$ (so we will call S a *doubly sphere*).

First consider the case j odd. Then S can be parametrized by $\{(z, w) \in \overline{\mathbb{C}}^2 \mid w^2 = (z - a)(z - b)\}$ where $a + b = 2x_j$, $ab = y_j$, and the corresponding meromorphic map is $g(z, w) = z$. Hence, $w = \sqrt{(z - a)(z - b)}$ is well defined on S , where we convine that the sign of the square root is determined by $w \sim z$ in $\overline{\mathbb{C}}_j$ and $w \sim -z$ in $\overline{\mathbb{C}}_{j-1}$. Since ϕ has simple poles at the nodes ∞_{j-1} and ∞_j with respective residues 1 and -1 , and

$$\text{Res}_{\infty_j} \frac{dz}{w} = \text{Res}_{z=\infty} \frac{dz}{z\sqrt{1 - 2x_j/z + y_j/z^2}} = -1,$$

we conclude that $\phi = dz/w$. Thus,

$$A_j(\mathbf{x}, \mathbf{y})_j = \int_{\Gamma_j} \frac{dz}{zw} = -2\pi i \text{Res}_{\infty_j} \frac{dz}{zw} = 0 \quad (20)$$

and

$$A_{j-1}(\mathbf{x}, \mathbf{y})_j = \int_{\Gamma_{j-1}} \frac{z dz}{w} = 2\pi i \text{Res}_{\infty_{j-1}} \frac{z dz}{-z\sqrt{1 - 2x_j/z + y_j/z^2}} = 2\pi i x_j. \quad (21)$$

These two equations imply that $A_j(\mathbf{0}, \mathbf{0}) = 0$ for every $j = 1, \dots, 2k$ (not necessarily odd). Furthermore, given $h = 1, \dots, 2k$ it holds $\frac{\partial A_j}{\partial x_h}(\mathbf{0}, \mathbf{0}) = \lim_{y_h \rightarrow 0} \frac{d}{dx_h} \Big|_{x_h=0} A_j(\mathbf{x}, \mathbf{y})_h$. If $h \neq j - 1, j$ then the integral in $A_j(\mathbf{x}, \mathbf{y})_h$ can be computed in a simple sphere which does not depend on x_h , hence $\frac{\partial A_j}{\partial x_h}(\mathbf{0}, \mathbf{0}) = 0$. If $h = j$ then (20) implies $\frac{\partial A_j}{\partial x_j}(\mathbf{0}, \mathbf{0}) = 0$. If $h = j - 1$, then (21) gives $\frac{\partial A_{j-1}}{\partial x_j}(\mathbf{0}, \mathbf{0}) = 2\pi i$.

Concerning \tilde{B}_j , we write

$$\text{Res}_{0_j} (g^{-1}\phi) = \frac{1}{\sqrt{y_j}}, \quad \text{Res}_{0_{j-1}} (g^{-1}\phi) = \frac{-1}{\sqrt{y_j}},$$

hence $\tilde{B}_j(\mathbf{x}, \mathbf{y})_j = -1$ for all j odd. The computations in the case j even are similar, with the following modifications:

$$w = \sqrt{\left(\frac{1}{z} - \frac{1}{a}\right)\left(\frac{1}{z} - \frac{1}{b}\right)} = \sqrt{\frac{1}{z^2} - \frac{2x_j}{z} + y_j}, \quad \phi = \frac{-dz}{z^2 w},$$

$$A_j(\mathbf{x}, \mathbf{y})_j = \int_{\Gamma_j} \frac{-dz}{zw} = 2\pi i \text{Res}_{0_j} \frac{dz}{zw} = 0,$$

$$A_{j-1}(\mathbf{x}, \mathbf{y})_j = \int_{\Gamma_{j-1}} \frac{-dz}{z^3 w} = 2\pi i \text{Res}_{0_{j-1}} \frac{-dz}{-z^2 \sqrt{1 - 2x_j z + y_j z^2}} = 2\pi i x_j,$$

$$\operatorname{Res}_{\infty_j}(g\phi) = \frac{1}{\sqrt{y_j}}, \quad \operatorname{Res}_{\infty_{j-1}}(g\phi) = \frac{-1}{\sqrt{y_j}},$$

from where one obtains similar conclusions to the case j odd. The derivatives of the A_h with respect to the variables y_j clearly all vanish, and the proof is complete. \square

Recall that given $j = 1, \dots, 2j$, we have $1/B_j = y_j/\tilde{B}_j$. Since \tilde{B}_j extends holomorphically to the polydisk $D(0, \varepsilon)^{4k}$ (Proposition 10) and $\tilde{B}_j(\mathbf{0}, \mathbf{0}) = 0$ (Proposition 11), we deduce that $1/B_j$ extends holomorphically to $D(0, \varepsilon)^{4k}$ for $\varepsilon > 0$ small enough. Hence, the map $\Theta : D(0, \varepsilon)^{4k} \rightarrow \mathbb{C}^{4k}$ given by

$$\Theta(\mathbf{x}, \mathbf{y}) (A_1, \dots, A_{2k}, 1/B_1, \dots, 1/B_{2k}) \tag{22}$$

is holomorphic and $\Theta(\mathbf{0}, \mathbf{0}) = (\mathbf{0}, \mathbf{0})$.

Theorem 3 *There exists $\varepsilon > 0$ small such that Θ restricts to $D(0, \varepsilon)^{4k}$ as a biholomorphism onto its image.*

Proof. It follows from Proposition 10 that

$$\frac{\partial B_j^{-1}}{\partial x_i}(\mathbf{0}, \mathbf{0}) = 0, \quad \frac{\partial B_j^{-1}}{\partial y_i}(\mathbf{0}, \mathbf{0}) = -\delta_{ij},$$

Using these equations and the values of the partial derivatives of the functions A_h at $(\mathbf{0}, \mathbf{0})$ given by Proposition 10, it is straightforward to check that the Jacobian matrix of Θ at $(\mathbf{0}, \mathbf{0})$ is invertible. Now the Theorem follows from the Inverse function theorem. \square

Remark 11 *As a consequence of Theorem 3, for $t < 0$ and $b \in \mathbb{C}$ close to $(\mathbf{0}, \mathbf{0})$ there exists a unique $(\mathbf{x}, \mathbf{y}) \in D(0, \varepsilon)^{4k}$ such that $\Theta(\mathbf{x}, \mathbf{y}) = (b, \bar{b}, \dots, b, \bar{b}, t, \dots, t)$. This uniqueness result and equation (18) imply that the space of immersed minimal surfaces sufficiently close to the boundary point of \mathcal{W} given by the $2k$ catenoids (see Proposition 8) has three real freedom parameters.*

8 Openness.

Recall that $\mathcal{K} \subset \mathcal{S}$ represents the space of standard examples with $4k$ ends. A direct consequence of its construction is that \mathcal{K} is closed in \mathcal{S} . We saw in Section 3 that \mathcal{K} is

open in \mathcal{S} , by the nondegeneracy of any standard example. Both closeness and openness remain valid for the space $\tilde{\mathcal{K}}$ of marked standard examples inside $\tilde{\mathcal{S}}$. By Lemma 4, $\tilde{\mathcal{K}}$ is closed and open in \mathcal{M} . Our Theorem 1 reduces to prove that $\tilde{\mathcal{K}} = \tilde{\mathcal{S}}$, which will be proved by contradiction in Section 9. For this reason, in what follows we will assume $\tilde{\mathcal{S}} - \tilde{\mathcal{K}} \neq \emptyset$.

Theorem 4 *The classifying map $C : \tilde{\mathcal{S}} - \tilde{\mathcal{K}} \rightarrow \mathbb{R}^* \times \mathbb{C}$ is open.*

Proof. Fix a marked surface $M \in \tilde{\mathcal{S}} - \tilde{\mathcal{K}}$. It suffices to see that C is open in a neighborhood of M in $\tilde{\mathcal{S}} - \tilde{\mathcal{K}}$. Let $(a, b) = C(M) \in \mathbb{R}^* \times \mathbb{C}$ and $\mathcal{M}(a, b) = L^{-1}(L_{(a,b)}) \subset \mathcal{M}$ (with the notation of Subsection 4.2). Since $\tilde{\mathcal{K}}$ is open and closed in $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}(a, b) = \tilde{\mathcal{S}} \cap \mathcal{M}(a, b)$ is an analytic subvariety of \mathcal{W} by Proposition 3, we conclude that $(\tilde{\mathcal{S}} - \tilde{\mathcal{K}})(a, b) = (\tilde{\mathcal{S}} - \tilde{\mathcal{K}}) \cap \tilde{\mathcal{S}}(a, b)$ is an analytic subvariety of \mathcal{W} .

Assertion 2 $(\tilde{\mathcal{S}} - \tilde{\mathcal{K}})(a, b)$ *is compact.*

To prove Assertion 2, take a sequence $\{M_n\}_n \subset (\tilde{\mathcal{S}} - \tilde{\mathcal{K}})(a, b)$. By Proposition 5, the sequence of geometric surfaces $\{M_n\}_n$ has uniformly bounded Gaussian curvature. Similarly as in the proof of Proposition 6, we can find $\delta > 0$ such that for every n , there exists a point $p_n \in M_n$ where the normal vector $N_n(p_n)$ lies in $\mathbb{S}^2 \cap \{x_2 = 0\}$ and stays at spherical distance at least δ from the branch locus of the Gauss map N_n of M_n . The usual limit process shows that, after passing to a subsequence, suitable liftings of $M_n - p_n$ converge smoothly as $n \rightarrow \infty$ to a properly embedded nonflat minimal surface $\tilde{M}_\infty \subset \mathbb{R}^3$ in one of the six cases listed in Proposition 4. As $\tilde{\mathcal{K}}$ is open in $\tilde{\mathcal{S}}$, if \tilde{M}_∞ were in case (vi) of Proposition 4 then its quotient would actually be in $\tilde{\mathcal{S}} - \tilde{\mathcal{K}}$, hence it only remains to discard the first five possibilities of Proposition 4 for \tilde{M}_∞ .

As the period vector H_n at the ends of M_n is $(0, \pi a, 0)$ for all n , the cases (i), (ii), (iii) of Proposition 4 are not possible for \tilde{M}_∞ . If \tilde{M}_∞ is a singly periodic Scherk surface S_ρ with two horizontal ends, then any choice of the nonhorizontal period vector T_n of M_n necessarily diverges to ∞ (case (iv) of Proposition 4). By Proposition 6 and Remark 7-(ii), for all n large we can find a new marked surface M'_n which can be viewed inside the open subset $\mathcal{U}(\varepsilon')$ of \mathbb{C}^{4k} appearing in Theorem 2, such that M_n, M'_n only differ in the homology class of the last component of the marked surfaces. Since $L|_{\mathcal{U}(\varepsilon')}$ is a biholomorphism (Theorem 2), the space of tuples in $\mathcal{U}(\varepsilon')$ producing immersed minimal surfaces has three real freedom parameters. Since $\tilde{\mathcal{K}}$ has real dimension three and $C|_{\tilde{\mathcal{K}}}$ takes values arbitrarily

close to (a, b) , we deduce that if $M' \in \widetilde{\mathcal{S}}$ lies in $\mathcal{U}(\varepsilon')$, then $C(M')$ coincides with the value of C at a certain standard example $M_0 \in \widetilde{\mathcal{K}}$. In particular $L(M') = L(M_0)$ and as $L|_{\mathcal{U}(\varepsilon')}$ is injective, we have $M' = M_0$. As this last equality cannot hold for M' being $M'_n \in \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$, we conclude that \widetilde{M}_∞ is not a singly periodic Scherk minimal surface.

Now assume that \widetilde{M}_∞ lies in case (v) of Proposition 4. Let Γ be a component of the intersection of \widetilde{M}_∞ with a horizontal plane $\{x_3 = c\}$ whose height does not coincide with the heights of the horizontal ends of \widetilde{M}_∞ . Since \widetilde{M}_∞ has exactly two nonhorizontal ends, Γ is an embedded U -shaped curve with two almost parallel divergent ends, and if we denote by $\Pi \subset \mathbb{R}^3$ the plane passing through the origin parallel to the nonhorizontal ends of \widetilde{M}_∞ , then the conormal vector to \widetilde{M}_∞ along each of the divergent branch of Γ becomes arbitrarily close to the upward pointing unit vector $\eta \in \Pi$ such that η is orthogonal to $\Pi \cap \{x_3 = c\}$. Since translated liftings of the M_n converge smoothly to \widetilde{M}_∞ , we deduce that M_n contains arbitrarily large arcs at constant height along which the conormal vector η_n is arbitrarily close to η . In particular, the integral of the third component of η_n along such arcs becomes arbitrarily large. As the conormal vector of M_n along any compact horizontal section misses the horizontal values (the Gauss map of M_n is never vertical), it follows that the vertical component of the flux of M_n along a compact horizontal section diverges to ∞ as $n \rightarrow \infty$. This contradicts our normalization on the surfaces of \mathcal{S} , and proves Assertion 2.

We now finish the proof of Theorem 4. By Assertion 2 and Lemma 2, $(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})(a, b)$ is a finite subset, hence we can find an open set \mathcal{U} of \mathcal{W} containing M such that $(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})(a, b) \cap \mathcal{U} = \mathcal{M}(a, b) \cap \mathcal{U} = \{M\}$. In terms of the ligature map $L : \mathcal{W} \rightarrow \mathbb{C}^{4k}$, the last equality writes as $L^{-1}(L_{(a,b)}) \cap \mathcal{U} = \{M\}$. Since L is holomorphic, we can apply the Openness Theorem for finite holomorphic maps (see [4] page 667) to conclude that $L|_{\mathcal{U}}$ is an open map. Finally, the relationship between the ligature map L and the map C gives the existence of a neighborhood of M in $\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$ where the restriction of C is open. \square

The same argument in the proof of Assertion 2 remains valid under the weaker hypothesis on $C(M_n)$ to converge to some $(a, b) \in \mathbb{R}^* \times \mathbb{C}$ instead of being constant on a sequence $\{M_n\}_n \subset \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$. This proves the validity of the following statement.

Theorem 5 *The classifying map $C : \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}} \rightarrow \mathbb{R}^* \times \mathbb{C}$ is proper.*

9 The proof of Theorem 1.

Recall that we were assuming $\tilde{\mathcal{S}} - \tilde{\mathcal{K}} \neq \emptyset$. By Theorems 4 and 5, $C : \tilde{\mathcal{S}} - \tilde{\mathcal{K}} \rightarrow \mathbb{R}^* \times \mathbb{C}$ is an open and proper map. Thus, $C(\tilde{\mathcal{S}} - \tilde{\mathcal{K}})$ is an open and closed subset of $\mathbb{R}^* \times \mathbb{C}$. Since $C(\tilde{\mathcal{S}} - \tilde{\mathcal{K}})$ has points in both connected components of $\mathbb{R}^* \times \mathbb{C}$, we deduce that $C|_{\tilde{\mathcal{S}} - \tilde{\mathcal{K}}}$ is surjective. In particular, we can find a sequence $\{M_n\}_n \subset \tilde{\mathcal{S}} - \tilde{\mathcal{K}}$ such that $\{C(M_n)\}_n$ tends to $(\infty, 0)$ as n goes to infinity. Now the argument is similar to the one in the proof of Assertion 2 when we discarded the singly periodic Scherk limit, using Proposition 8, Remark 8 and Theorem 3 instead of Proposition 6, Remark 7-(ii) and Theorem 2.

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