

AREA ESTIMATES FOR HIGH GENUS LAWSON SURFACES VIA DPW

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ABSTRACT. Starting at a saddle tower surface, we give a new existence proof of the Lawson surfaces $\xi_{m,k}$ of high genus by deforming the corresponding DPW potential. As a byproduct, we obtain for fixed m estimates on the area of $\xi_{m,k}$ in terms of their genus $g = mk \gg 1$.

INTRODUCTION

Minimal surfaces are important objects in differential geometry which have fascinated geometers for centuries. Depending on the curvature of the ambient space, different techniques were developed to prove existence, uniqueness (possibly under certain geometric constraints), and to study the space of embedded minimal surfaces. In Euclidean space, minimal surfaces can be explicitly parametrised via Weierstrass representation. Constructing minimal surfaces in a compact symmetric space – such as the round 3-sphere – is much more involved.

Examples of compact embedded minimal surfaces in the 3-sphere of all genera were first found by Lawson [13] using the solution of the Plateau problem with respect to a polygonal boundary curve. Though enormous achievements have been made in the theory of minimal surfaces in positively curved 3-manifolds by Min-Max theory in recent years (see for example [14] and references therein), we still lack knowledge about the simplest compact minimal surfaces of genus $g \geq 2$ in the round 3-sphere. For example, the area of these surfaces is still unknown and the index and stability for Lawson $\xi_{1,g}$ -surfaces were only recently computed [8].

It is well known that the Lawson surfaces $\xi_{m,k}$ converge for fixed m and $k \rightarrow \infty$ to the union of $m + 1$ great spheres intersecting in a great circle. In this paper, we go backwards and construct Lawson surfaces $\xi_{m,k}$ for $k \gg 1$ by desingularizing the union of $m + 1$ great spheres using a Karcher saddle tower, a minimal surface generalizing the classical Scherk surface (see Section 1.2). As a consequence, our analysis determines the asymptotic behaviour of the area of the Lawson surfaces $\xi_{m,k}$ for $k \gg 1$ up to second order. In particular, in the case $m = 1$, Theorem 4 gives

$$\text{Area}(\xi_{1,g}) = 8\pi \left(1 - \frac{\ln 2}{2g} + \frac{\ln 2}{2g^2} + O\left(\frac{1}{g^3}\right) \right)$$

for the area of the Lawson surface $\xi_{1,g}$ of genus g with $g \gg 1$. The Lawson surfaces $\xi_{1,g}$ are conjectured to minimize the Willmore energy for surfaces of genus g ([11], Conjecture 8.4). Since the area of a minimal surface in \mathbb{S}^3 is its Willmore energy, the above equation yields estimates for the conjectured minimum Willmore energy of compact surfaces of genus $g \gg 1$. In [12], the large genus limit of the minimal Willmore energy is shown to be 8π , giving some evidence to the Kusner conjecture.

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Desingularization is a well established and productive method to construct minimal surfaces in various spaces using PDE methods (see for example [7] for an example of such construction in the 3-sphere). However, these methods would not give such fine area estimates as the ones that we obtain in this paper. We shall carry out the construction using integrable system methods, which in essence allow for more explicit formulas.

In this paper we consider a conformally parametrised minimal immersion f from a Riemann surface Σ into the round 3-sphere. The harmonicity of f gives rise to a symmetry of the Gauss–Codazzi equations in the 3-sphere inducing an associated family of (isometric) minimal surfaces on the universal covering of Σ with rotated Hopf differential. This family of surfaces is the geometric counterpart of an associated \mathbb{C}_* -family of flat $\mathrm{SL}(2, \mathbb{C})$ -connections ∇^λ [6] on the trivial \mathbb{C}^2 -bundle over Σ satisfying

- (i) conformality: $\nabla^\lambda = \lambda^{-1}\Phi + \nabla + \lambda\Psi$ for a nilpotent $\Phi \in \Omega^{1,0}(\Sigma, \mathfrak{sl}(2, \mathbb{C}))$;
- (ii) intrinsic closing: ∇^λ is unitary for all $\lambda \in \mathbb{S}^1$, i.e., ∇ is unitary and $\Psi = \Phi^*$ with respect to the standard hermitian metric on \mathbb{C}^2 ;
- (iii) extrinsic closing: ∇^λ is trivial for $\lambda = \pm 1$.

The minimal surface can be reconstructed from the associated family of connections as the gauge between ∇^{-1} and ∇^1 . Constructing minimal surfaces is thus equivalent to writing down appropriate families of flat connections.

The DPW method [3], which can be viewed as a generalisation of the Weierstrass representation for minimal surfaces in Euclidean space, is a way to generate families of flat connections from so-called *DPW potentials* on Σ , denoted by $\eta = \eta^\lambda$, using loop group factorisation methods. We summarise the basic procedure in Section 1.4. On simply connected domains Σ , all DPW potentials give rise to minimal surfaces. Whenever the domain has non-trivial topology, finding DPW potentials satisfying conditions equivalent to (i),(ii) and (iii) is difficult. So far, only special surface classes, such as trinoids [17], tori, and more recently n -noids were constructed using DPW [18, 19]. In this paper we give the first existence proof of closed embedded minimal surfaces of high genus in the 3-sphere via DPW.

The outline of the paper is as follows. We start with recalling the classical construction of Lawson surfaces, the Weierstrass representation of Karcher saddle tower surfaces, and some general facts concerning loop groups and DPW method in Section 1. In Section 2, we propose a family of putative DPW potentials for Lawson surfaces. Because of symmetries, Lawson surface $\xi_{m,k}$ is a $(k+1)$ -sheeted branched cover of the Riemann sphere. We choose the DPW potential η to be well defined on the Riemann sphere, with simple poles at the branch points of the covering. Our potential $\eta = \eta_t$ actually depends on a small real parameter t and closed minimal surfaces are recovered when $t = \frac{1}{2k+2}$. The Monodromy Problem is solved using the Implicit Function Theorem at $t = 0$. The strategy here is analogous to [18], [19] and similar to [4]. In the DPW setup, the area of a minimal surface can be computed explicitly from the DPW potential, see Corollary 18. Thus we compute the time derivative of η_t at $t = 0$ up to order 2 in Section 3. The constructed family of surfaces are identified to be the Lawson surfaces $\xi_{m,k}$ in Section 4. Finally, using the derivatives of η_t computed in Section 3 we obtain an asymptotic expansion of the area of high genus Lawson surfaces.

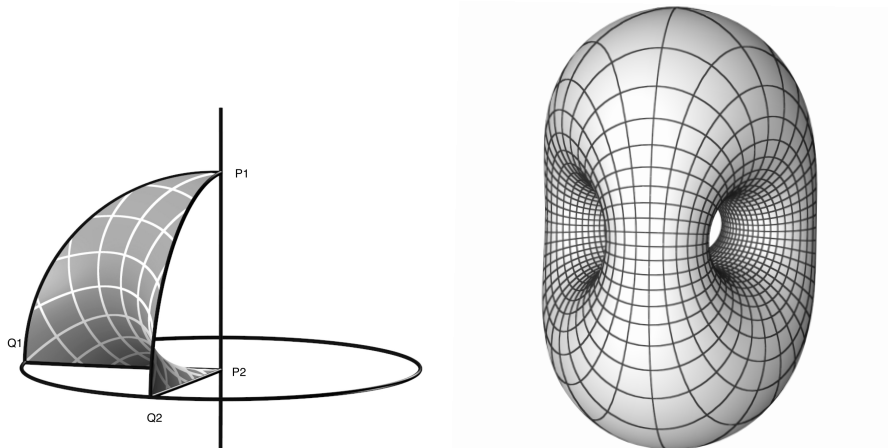


FIGURE 1. The Plateau solution of a geodesic 4-gon in the 3-sphere and the Lawson surface of genus 2, stereographically projected to the Euclidean space. Images by Nicholas Schmitt with xLab.

1. PRELIMINARY

In order to fix notations and to be self-contained we shortly recall the construction of Lawson surfaces, saddle towers as well as general facts about loop groups and DPW.

1.1. Lawson surfaces.

The original construction of the Lawson surfaces [13]

$$\xi_{m,k} : \Sigma \longrightarrow \mathbb{S}^3$$

uses the solution to the Plateau problem. Consider two orthogonal great circles C_1 and C_2 in the round 3-sphere. Let P_1, \dots, P_{2m+2} denote $(2m+2)$ equidistant points on C_1 , and Q_1, \dots, Q_{2k+2} denote $(2k+2)$ equidistant points on C_2 . For the convex geodesic polygon

$$\overline{P_1 Q_1 P_2 Q_2}$$

the corresponding Plateau solution, see Figure 1, is a minimal surface in S^3 . A closed minimal surface is obtained from this fundamental piece by repeatedly reflecting it across its geodesic boundaries. The resulting surfaces are called Lawson surfaces $\xi_{m,k}$, are embedded and of genus $g = m \cdot k$.

By construction the Lawson surfaces possess a large symmetry group. The subgroup of orientation preserving symmetries (both on the surface and in 3-space) contains

$$\mathbb{Z}_{m+1} \times \mathbb{Z}_{k+1},$$

where the action is the natural rotation in the planes spanned by the circles C_1 and C_2 , respectively.

The minimal surface $\xi_{m,k}$ induces a Riemann surface structure on Σ . The quotient of the Riemann surface by the symmetries \mathbb{Z}_{m+1} and \mathbb{Z}_{k+1} , respectively, is $\mathbb{C}P^1$ and the covering $\Sigma \rightarrow \mathbb{C}P^1$ is totally branched over $2k+2$ and respectively $2m+2$ points. Using the additional

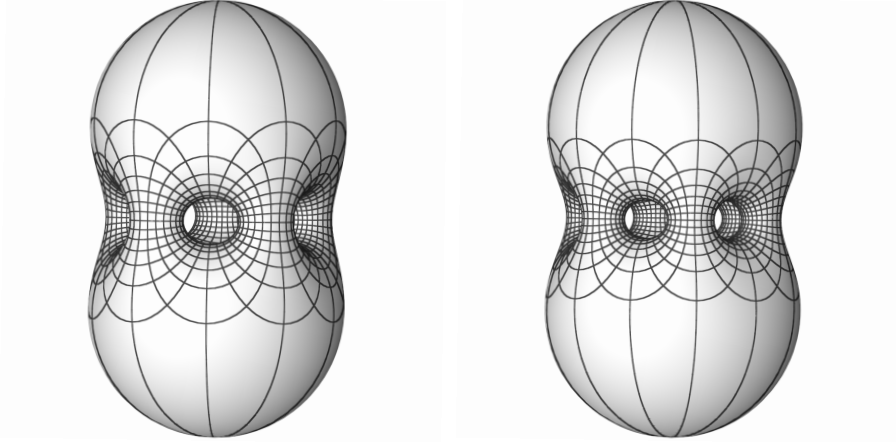


FIGURE 2. The Lawson surfaces of genus 4 and 5.

reflection symmetries, these $2k + 2$ and respectively $2m + 2$ points are in equidistance on the unit circle of the (round) 2-sphere.

Remark 1. *Since the surfaces $\xi_{m,k}$ and $\xi_{k,m}$ are isometric, the Lawson surfaces $\xi_{k,k}$ admit an additional orientation preserving symmetry.*

All Lawson surfaces admit additional symmetries which are not orientation preserving in space or not orientation preserving on the surfaces. They are given by reflections across geodesics contained in the surfaces (e.g., the polygonal boundary of the fundamental piece) or a by reflection across geodesic 2-spheres which intersect the surface orthogonally, e.g., symmetry planes of the geodesic polygon.

1.2. Saddle Tower Surfaces.

Karcher [9] generalised Scherk's singly periodic surface to surfaces with with $n = 2m + 2$ Scherk type ends and constant angle $\frac{2\pi}{2m+2}$ between consecutive ends, see the figure in [9]. These surfaces are called saddle tower surfaces and their Weierstrass data are given by

$$(1) \quad g = \frac{i}{z^m} \quad \text{and} \quad \omega = \frac{2nz^{2m} dz}{z^{2m+2} - 1}.$$

1.3. Loop groups.

In the following we give a comprehensive introduction to the theory of loop groups which contains only relevant theorems and facts with regard to the paper. For details we refer to [16]. Let G be a finite dimensional real Lie group with Lie algebra \mathfrak{g} . We define the loop spaces

- $\Lambda G := \{ \text{real analytic maps (loops)} \Phi: \mathbb{S}^1 \longrightarrow G, \quad \lambda \longmapsto \Phi^\lambda \};$
- $\Lambda \mathfrak{g} := \{ \text{real analytic maps (loops)} \eta: \mathbb{S}^1 \longrightarrow \mathfrak{g}, \quad \lambda \longmapsto \eta^\lambda \}.$

ΛG is an infinite dimensional Frechet Lie group via pointwise multiplication with $\Lambda \mathfrak{g}$ as its Lie algebra. For a complex Lie group $G^{\mathbb{C}}$ we denote

$$\Lambda_+ G^{\mathbb{C}} = \{ \Phi \in \Lambda G^{\mathbb{C}} \mid \Phi \text{ extends holomorphically to } \lambda = 0 \}$$

and

$$\Lambda_+ \mathfrak{g}^{\mathbb{C}} = \{\eta \in \Lambda \mathfrak{g}^{\mathbb{C}} \mid \eta \text{ extends holomorphically to } \lambda = 0\}.$$

In the particular case of $G^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$ we denote

$$\mathcal{B} = \{B \in \mathrm{SL}(2, \mathbb{C}) \mid B \text{ is upper triangular with positive diagonal entries}\}$$

and

$$\Lambda_+^{\mathbb{R}} \mathrm{SL}(2, \mathbb{C}) = \{B \in \Lambda_+ \mathrm{SL}(2, \mathbb{C}) \mid B(0) \in \mathcal{B}\}.$$

We will make use of the following theorem, often referred to as Iwasawa decomposition:

Theorem 1 ([16]). *Let $\Phi \in \Lambda \mathrm{SL}(2, \mathbb{C})$. Then there exist a splitting*

$$\Phi = F \cdot B$$

with $F \in \Lambda \mathrm{SU}(2)$ and $B \in \Lambda_+^{\mathbb{R}} \mathrm{SL}(2, \mathbb{C})$. This splitting is unique and depends real analytically on Φ . The pair (F, B) is called the Iwasawa decomposition of Φ .

1.4. The DPW method.

Let Σ be a Riemann surface. A DPW potential on Σ is a closed 1-form

$$\eta \in \Omega^{1,0}(\Sigma, \Lambda \mathfrak{sl}(2, \mathbb{C}))$$

with

$$\lambda \eta \in \Omega^{1,0}(\Sigma, \Lambda_+ \mathfrak{sl}(2, \mathbb{C}))$$

such that its residuum at $\lambda = 0$

$$\eta_{-1} = \mathrm{Res}_{\lambda=0}(\eta)$$

is a nowhere vanishing and nilpotent 1-form.

A DPW potential η gives rise to a loop of flat $\mathrm{SL}(2, \mathbb{C})$ -connections. Let $\tilde{\Sigma}$ denote the universal covering of Σ and let

$$\Phi: \tilde{\Sigma} \longrightarrow \Lambda \mathrm{SL}(2, \mathbb{C})$$

be the solution of the ODE

$$(2) \quad d_{\Sigma} \Phi = \Phi \cdot \eta$$

with initial value $\Phi(p) \in \Lambda \mathrm{SL}(2, \mathbb{C})$. Then the Iwasawa decomposition (F, B) of Φ gives smooth maps

$$F: \tilde{\Sigma} \longrightarrow \Lambda \mathrm{SU}(2) \quad \text{and} \quad B: \tilde{\Sigma} \longrightarrow \Lambda_+^{\mathbb{R}} \mathrm{SL}(2, \mathbb{C})$$

and the associated family of flat connections of a minimal surface [1, 6]

$$f: \tilde{\Sigma} \longrightarrow \mathbb{S}^3.$$

is given by $\nabla^\lambda = d_\Sigma + (F^\lambda)^{-1}d_\Sigma F^\lambda$ satisfying

$$(3) \quad d_\Sigma + F^{-1}d_\Sigma F = (d_\Sigma + \eta).B^{-1} = d_\Sigma + B\eta B^{-1} - d_\Sigma B B^{-1}.$$

Identifying $\mathbb{S}^3 \cong \text{SU}(2)$, the surface can therefore be reconstructed by the Sym-Bobenko formula

$$(4) \quad f = F^{\lambda=1}(F^{\lambda=-1})^{-1}.$$

In this paper we are interested in constructing compact minimal surfaces with non trivial topology. Thus we start with a DPW potential defined on such a Riemann surface Σ . The so-constructed minimal surface is well-defined on Σ if its associated family of flat connections ∇^λ satisfies the closing conditions (i)-(iii). For the corresponding DPW potential it is sufficient to have

- (i) B has trivial monodromy, i.e., B is well-defined on Σ ;
- (ii) the connections $d_\Sigma + \eta^{\lambda=\pm 1}$ have trivial monodromy.

Let $\gamma \in \pi_1(\Sigma, z_0)$ and let $\mathcal{M}(\Phi, \gamma)$ denotes the monodromy of Φ with respect to γ . In terms of Φ , the condition on the DPW potential is equivalent to:

$$(5) \quad \begin{cases} \mathcal{M}(\Phi, \gamma) \in \Lambda \text{SU}(2) \\ \mathcal{M}(\Phi, \gamma)|_{\lambda=1} = \mathcal{M}(\Phi, \gamma)|_{\lambda=-1} = \pm \text{Id}_2 \end{cases}$$

We refer to these conditions in (5) as the *Monodromy Problem*.

1.4.1. Gauge freedom and apparent singularities.

A DPW potential η is not uniquely determined by its minimal immersion f . We rather have a gauge freedom. Consider a DPW potential η on Σ , and a holomorphic map

$$\tilde{B}: \Sigma \longrightarrow \Lambda_+ \text{SL}(2, \mathbb{C}).$$

The gauged potential is defined to be

$$(6) \quad \tilde{\eta} = \eta.\tilde{B} := \tilde{B}^{-1}\eta\tilde{B} + \tilde{B}^{-1}d_\Sigma\tilde{B}.$$

Due to the positivity of \tilde{B} , $\tilde{\eta}$ is again a DPW potential. Moreover,

$$\tilde{\Phi} = \Phi\tilde{B}.$$

is the unique solution of

$$d_\Sigma\tilde{\Phi} = \tilde{\Phi}\tilde{\eta} \quad \text{with initial condition} \quad \tilde{\Phi}(p) = \Phi(p)\tilde{B}(p).$$

Let $F_0 B_0 = B^{\lambda=0}\tilde{B}^{\lambda=0}$ be the finite dimensional Iwasawa decomposition into a unitary and an upper triangular matrix with positive diagonal entries. Then

$$\tilde{\Phi} = (F F_0) \left(B_0 (F_0 B_0)^{-1} B \tilde{B} \right)$$

and

$$\tilde{F} = FF_0 \quad \tilde{B} = B_0(F_0B_0)^{-1}B\tilde{B}$$

is the Iwasawa decomposition of $\tilde{\Phi}$. Therefore, the two DPW potentials η and $\tilde{\eta}$ yield the same minimal immersion f via Sym-Bobenko formula (4).

In particular, certain singularities of η can be removed using the gauge freedom. Let η be a meromorphic potential with a singularity at $q \in \Sigma$. If there exist a positive gauge $\tilde{B}: \Sigma \setminus \{q\} \rightarrow \Lambda_+ \text{SL}(2, \mathbb{C})$ such that $\tilde{B}.\eta$ extends holomorphically to q , then the surface obtained from η extends real analytically to q . Singularities of this type are called *apparent singularities*.

Remark 2. *In order to obtain a compact minimal surface $f: \Sigma \rightarrow \mathbb{S}^3$, its DPW potential necessarily has apparent singularities. This follows for instance from the area formula in Corollary 18.*

Remark 3. *The DPW method can be generalised to potentials η^λ that are only defined for $\lambda \in D_r = \{\lambda \in \mathbb{C}^* \mid |\lambda| \leq r\}$ for $r \in (0, 1]$. Details and proofs can be found in [17] and [10].*

2. A DPW POTENTIAL FOR LAWSON SURFACES OF HIGH GENUS

To choose our potential we take advantage of the symmetries of Lawson surface $\xi_{m,k}$. We assume the potential to be invariant under the \mathbb{Z}_{k+1} action as in [5]. In this section we show the existence of DPW potentials on a $(2m+3)$ -punctured sphere

$$\mathbb{C} \setminus \{p_0, \dots, p_{2m+1}\}$$

with an apparent singularity at $z = \infty$ such that the Monodromy Problem (5) is solved on a finite cover Σ of the punctured sphere, branched at $p_j, j \in \{0, \dots, 2m+1\}$. This gives rise to countably infinite many compact and embedded minimal surfaces in \mathbb{S}^3 . In section 4 we show that these minimal surfaces coincide with the Lawson surfaces $\xi_{m,k}$ for $k \gg 1$.

2.1. Notations for functional spaces.

We follow the notations set in [18]: For $f \in L^2(\mathbb{S}^1, \mathbb{C})$ consider its Fourier series

$$f = \sum_k f_k \lambda^k.$$

For $\rho > 1$ define

$$\|f\|_\rho = \sum |f_k| \rho^{|k|} \leq \infty$$

and let

$$\mathcal{W}_\rho := \{f \in L^2 \mid \|f\|_\rho < \infty\}$$

be the set of Fourier series absolutely convergent on the annulus

$$\mathbb{A}_\rho = \{\lambda \in \mathbb{C} \mid \frac{1}{\rho} < |\lambda| < \rho\}.$$

Remark 4. *The notation is also used for arbitrary loop spaces \mathcal{H} : \mathcal{H}_ρ denotes the subspace of \mathcal{H} of loops whose entries are in \mathcal{W}_ρ . Then $\Lambda SL(2, \mathbb{C})_\rho$, $\Lambda SU(2)_\rho$ and $\Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho$ are Banach Lie groups and Iwasawa decomposition is a smooth diffeomorphism from $\Lambda SL(2, \mathbb{C})_\rho$ to $\Lambda SU(2)_\rho \times \Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho$ (see Theorem 5 in [19]).*

Moreover, let

$$\mathcal{W}_\rho^{\geq 0} := \{f = \sum_k f_k \lambda^k \in \mathcal{W}_\rho \mid f_k = 0 \forall k < 0\}$$

denote the space of those loops $f \in L^2(\mathbb{S}^1, \mathbb{C})$ that can be extended to a holomorphic function on the unit disc. Similarly, let

$$\mathcal{W}_\rho^{> 0} := \{f = \sum_k f_k \lambda^k \in \mathcal{W}_\rho \mid f_k = 0 \forall k \leq 0\}$$

$$\mathcal{W}_\rho^{< 0} := \{f = \sum_k f_k \lambda^k \in \mathcal{W}_\rho \mid f_k = 0 \forall k \geq 0\}$$

denote the positive and negative space, respectively. Therefore we can decompose every $f \in \mathcal{W}_\rho$

$$f = f^+ + f^0 + f^-$$

into its positive and negative component $f^\pm \in \mathcal{W}_\rho^{\geq 0}$, and a constant component $f^0 = f_0$.

On \mathcal{W}_ρ there exists two important involutions. The first is

$$*: \mathcal{W}_\rho \longrightarrow \mathcal{W}_\rho; f \longmapsto f^*,$$

where f^* is determined by

$$f^*(\lambda) = \overline{f\left(\frac{1}{\bar{\lambda}}\right)} \quad \text{for } \lambda \in \mathbb{A}_\rho.$$

The second involution is the conjugation of $f \in \mathcal{W}_\rho$ defined by:

$$(7) \quad \bar{f}(\lambda) = \overline{f(\bar{\lambda})}.$$

Let $\mathcal{W}_{\mathbb{R}}$, $\mathcal{W}_{\mathbb{R}}^{\geq 0}$ etc. denote real subspaces of \mathcal{W}_ρ , $\mathcal{W}_\rho^{\geq 0}$ satisfying $\bar{f} = f$. Functions in $\mathcal{W}_{\mathbb{R}}$ can be decomposed as $f(\lambda) = \sum f_k \lambda^k$ with real coefficients $f_k \in \mathbb{R}$. Observe that conjugation and star commute:

$$\overline{u^*}(\lambda) = \overline{u^*}(\lambda) = u\left(\frac{1}{\lambda}\right).$$

Remark 5. *The notations for the decomposition of \mathcal{W}_ρ into $\mathcal{W}_\rho^{\geq 0}$ etc, the involutions and the real subspaces carry over to loop spaces \mathcal{H} .*

2.2. Convergence to a Saddle Tower.

The Lawson surfaces $\xi_{m,k}$ converge for m fixed and $k \rightarrow \infty$ to the union of $m + 1$ great spheres intersecting in a great circle. Moreover, the blow-up of $\xi_{m,k}$ converges for $k \rightarrow \infty$ to a saddle tower with $2m + 2$ ends. The following blow-up result is adapted from Theorem 4 in [19]. Though written for CMC surfaces in \mathbb{R}^3 , an analogue statement also holds for the ambient space \mathbb{S}^3 . We omit its proof, as we will only use it as a heuristic to construct our potential for Lawson surfaces.

Theorem 2. *Let Σ be a Riemann surface, $\epsilon > 0$ and $I = (-\epsilon, \epsilon) \subset \mathbb{R}$. Moreover, let $(\eta_t)_{t \in I}$ a family of DPW potentials on Σ and $(\Phi_t)_{t \in I}$ the corresponding family of solutions. Fix a base point $z_0 \in \tilde{\Sigma}$ and assume*

- (1) $(t, z) \mapsto \eta_t(z, \cdot)$ and $t \mapsto \Phi_t(z_0, \cdot)$ are C^1 maps into $(\Lambda\mathfrak{sl}(2, \mathbb{C}))_\rho$ and $(\Lambda SL(2, \mathbb{C}))_\rho$, respectively.
- (2) Φ_t solves the Monodromy Problem (5) for all $t \in I$.
- (3) $\Phi_0(z, \lambda)$ is independent of λ :

$$\Phi_0(z, \lambda) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix}.$$

Let $f_t : \Sigma \rightarrow \mathbb{S}^3 \cong SU(2)$ be the corresponding family of minimal immersions via DPW. (since $F_0(z)$ is independent of λ , $f_0 \equiv \text{Id}$.) Then

$$\psi : \Sigma \longrightarrow T_{\text{Id}}SU(2) \cong \mathbb{R}^3, \quad \psi(z) := \lim_{t \rightarrow 0} \frac{1}{t} (f_t(z) - \text{Id})$$

is a well-defined and (possibly branched) minimal immersion with the following Weierstrass data (with "vertical" axis x_2 and "horizontal" axes x_3, x_4 in the tangent plane $x_1 = 1$ of \mathbb{S}^3 at Id):

$$g(z) = \frac{i\alpha(z)}{\gamma(z)} \quad \text{and} \quad \omega = -4\gamma(z)^2 \text{Res}_\lambda \left(\frac{\partial \eta_{t,12}}{\partial t} \Big|_{t=0} \right),$$

where $\eta_{t,12}$ is the upper right entry of the 2×2 potential η_t and the residue taken with respect to its expansion in λ . The convergence is hereby uniform C^1 on compact subsets of Σ .

We aim at finding a family of DPW potentials η_t , $t \sim 0$, with a saddle tower (see 1.2) as its blow-up limit ψ at $t = 0$. The Gauss map g of the saddle tower (1) suggest to choose

$$\eta_0 = \begin{pmatrix} 0 & 0 \\ mz^{m-1}dz & 0 \end{pmatrix}.$$

The corresponding solution with initial value $\Phi_0(z = 0) = \text{Id}$ is then given by

$$\Phi_0(z) = \begin{pmatrix} 1 & 0 \\ z^m & 1 \end{pmatrix}$$

which is independent of λ and yields the correct g according to Theorem 2. The meromorphic 1-form ω of the saddle tower suggests that η_t should have simple poles with residue of order t at the $2m + 2$ roots of unity.

2.3. The potential.

Let $m \in \mathbb{N}^*$ be fixed and define $n = 2m + 2$. We consider the ansatz

$$\eta_t = \begin{pmatrix} 0 & 0 \\ mrz^{m-1}dz & 0 \end{pmatrix} + t \sum_{j=0}^{n-1} A_j(\lambda) \frac{dz}{z - p_j},$$

where $A_i \in (\Lambda \mathfrak{sl}(2, \mathbb{C}))_\rho$ and the initial condition

$$\Phi_t(z=0) = \text{Id}.$$

Here r and t are real parameters with $r \in (1 - \epsilon, 1 + \epsilon)$ and $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$, and

$$p_j = e^{2\pi i j/n} \quad \text{for } 0 \leq j \leq 2m + 1.$$

The parameter r will be later determined by solving the Monodromy Problem. Its initial value at $t = 0$ is $r = 1$.

2.4. Symmetries.

Due to the symmetries of the Lawson surfaces $\xi_{m,k}$, we also assume the potentials η_t to be symmetric. Let

$$A_0(\lambda) = \begin{pmatrix} a(\lambda) & \lambda^{-1}b(\lambda) \\ \lambda c(\lambda) & -a(\lambda) \end{pmatrix}$$

with functions a, b, c in $\mathcal{W}_{\mathbb{R}}^{\geq 0}$. We assume

$$A_{j+1}(\lambda) = D^{-1}A_j(-\lambda)D \quad \text{for } 0 \leq j \leq n-2 \text{ with}$$

$$D = \begin{pmatrix} e^{i\pi m/n} & 0 \\ 0 & e^{-i\pi m/n} \end{pmatrix}.$$

Observe that

$$e^{2i\pi m/n} = e^{i\pi(n-2)/n} = -e^{-2i\pi/n}.$$

Hence writing $A_j = \begin{pmatrix} a_j & \lambda^{-1}b_j \\ \lambda c_j & -a_j \end{pmatrix}$ with functions a_j, b_j, c_j in $\mathcal{W}_{\mathbb{R}}^{\geq 0}$ we obtain

$$\begin{cases} a_{j+1}(\lambda) = a_j(-\lambda) \\ b_{j+1}(\lambda) = e^{2i\pi/n}b_j(-\lambda) \\ c_{j+1}(\lambda) = e^{-2i\pi/n}c_j(-\lambda) \end{cases}$$

and

$$(8) \quad \begin{aligned} \sum_{j=0}^{n-1} \frac{a_j}{z - p_j} &= \frac{nz^m}{2} \left(\frac{a(\lambda)}{z^{m+1} - 1} + \frac{a(-\lambda)}{z^{m+1} + 1} \right) \\ \sum_{j=0}^{n-1} \frac{b_j}{z - p_j} &= \frac{n}{2} \left(\frac{b(\lambda)}{z^{m+1} - 1} - \frac{b(-\lambda)}{z^{m+1} + 1} \right) \\ \sum_{j=0}^{n-1} \frac{c_j}{z - p_j} &= \frac{nz^{m-1}}{2} \left(\frac{c(\lambda)}{z^{m+1} - 1} + \frac{c(-\lambda)}{z^{m+1} + 1} \right). \end{aligned}$$

The symmetries of η_t are induced by $\delta(z) = e^{2\pi i/n}z$ and $\sigma(z) = \bar{z}$. We have

$$\delta^* \eta_t(z, \lambda) = D^{-1} \eta_t(z, -\lambda) D, \quad \delta^* \Phi_t(z, \lambda) = D^{-1} \Phi_t(z, -\lambda) D$$

and

$$\sigma^* \overline{\eta_t} = \eta_t, \quad \sigma^* \overline{\Phi_t} = \Phi_t$$

which, remembering notation (7), means

$$\sigma^* \overline{\eta_t(\cdot, \bar{\lambda})} = \eta_t(\cdot, \lambda)$$

and likewise for Φ_t .

2.5. The Monodromy Problem.

Let $\gamma_0, \dots, \gamma_{n-1}$ be generators of the fundamental group $\pi_1(\mathbb{C} \setminus \{p_0, \dots, p_{n-1}\}, 0)$, with γ_j enclosing the singularity p_j . Let $M_j(t) = \mathcal{M}(\Phi_t, \gamma_j)$ be the monodromy of Φ_t along γ_j . We want to solve the following problem for all j :

$$(9) \quad \begin{cases} M_j(t) \in \Lambda SU(2) \\ M_j(t)|_{\lambda=\pm 1} \text{ diagonal} \\ M_j(t) \text{ has eigenvalues } e^{\pm 2\pi i t}. \end{cases}$$

We will see in Section 4.2 that provided Problem (9) is solved, taking $t = \frac{1}{2(k+1)}$, the potential η_t pulls back on a $(k+1)$ -branched cover to a potential with apparent singularities solving the Monodromy Problem (5). This yields the desired closed minimal surface.

Regarding symmetries, we have since $\delta(\gamma_j) = \gamma_{j+1}$:

$$(10) \quad M_{j+1}(t)(\lambda) = D^{-1}M_j(t)(-\lambda)D.$$

Because D is unitary and diagonal, it suffices to solve Problem (9) for $j = 0$. From now on, we write $M = M_0$. Since $\sigma(\gamma_0) = \gamma_0^{-1}$, we also have, using the notation (7)

$$(11) \quad M(t) = \left(\overline{M(t)} \right)^{-1}.$$

Remark 6. *It will turn out that provided Problem (9) is solved, the singularity at $z = \infty$ is apparent, see Section 2.7.*

At $t = 0$ the solution of

$$(12) \quad d_\Sigma \Phi_t = \Phi_t \eta_t \quad \text{and} \quad \Phi_t(0) = \text{Id}$$

is given by

$$\Phi_0(z) = \begin{pmatrix} 1 & 0 \\ rz^m & 1 \end{pmatrix}.$$

with trivial monodromy, i.e.,

$$M(0) = \text{Id}.$$

Hence

$$\widetilde{M}(t) := \frac{1}{t} \log M(t)$$

extends smoothly at $t = 0$, with $\widetilde{M}(0) = M'(0)$. When $t \neq 0$, Problem (9) is equivalent to

$$(13) \quad \begin{cases} \widetilde{M}(t) \in \Lambda \mathfrak{su}(2) \\ \widetilde{M}(t)|_{\lambda=\pm 1} \text{ diagonal} \\ \widetilde{M}(t) \text{ has eigenvalues } \pm 2\pi i. \end{cases}$$

From the symmetry (11) of M we deduce

$$\widetilde{M}(t) = -\overline{\widetilde{M}(t)}.$$

Following [18] we compute

$$\begin{aligned} \widetilde{M}(0) &= \int_{\gamma_0} \Phi_0 \frac{\partial \eta_t}{\partial t} \Big|_{t=0} \Phi_0^{-1} = 2\pi i \text{Res}_{p_0} \left[\begin{pmatrix} 1 & 0 \\ rz^m & 1 \end{pmatrix} A_0 \begin{pmatrix} 1 & 0 \\ -rz^m & 1 \end{pmatrix} \frac{dz}{z - p_0} \right] \\ &= 2\pi i \begin{pmatrix} a - \lambda^{-1}rb & \lambda^{-1}b \\ 2ra - \lambda^{-1}r^2b + \lambda c & -a + \lambda^{-1}rb \end{pmatrix}. \end{aligned}$$

Let $\mathbf{x} = (r, a, b, c)$ denote the vector of parameters. To highlight the parameters, we denote the potential determined by \mathbf{x} as

$$\eta_t = \eta_t^{\mathbf{x}}.$$

The initial value of \mathbf{x} , denoted by \mathbf{x}_0 , is taken to be

$$(14) \quad r = 1, \quad a = \lambda, \quad b = \frac{\lambda^2 - 1}{2} \quad \text{and} \quad c = -2.$$

For these value of the parameters, we obtain at $t = 0$

$$(15) \quad \widetilde{M}(0) = \pi i \begin{pmatrix} \lambda + \lambda^{-1} & \lambda - \lambda^{-1} \\ \lambda^{-1} - \lambda & -\lambda - \lambda^{-1} \end{pmatrix}$$

so Problem (13) is solved at $t = 0$.

Remark 7. *Assuming $r = 1$, one can prove as in [18] that (14) is the only solution to Problem (13) at $t = 0$, up to $(a, b, c) \mapsto (-a, -b, -c)$.*

2.6. Solving the Monodromy Problem for $t \neq 0$.

For a parameter (t, \mathbf{x}) and the corresponding solution Φ_t of (12) and its monodromy $M(t) \in \text{ASL}(2, \mathbb{C})$ we define

$$\begin{aligned} \mathcal{F}(t, \mathbf{x}) &= \frac{1}{2\pi i} (\widetilde{M}_{11}(t) + \widetilde{M}_{11}(t)^*) \\ \mathcal{G}(t, \mathbf{x}) &= \frac{1}{2\pi i} (\widetilde{M}_{21}(t) + \widetilde{M}_{12}(t)^*) \\ \mathcal{H}_1(t, \mathbf{x}) &= \frac{1}{2\pi i} \widetilde{M}_{12}(t)(\lambda = 1) \\ \mathcal{H}_2(t, \mathbf{x}) &= \frac{1}{2\pi i} \widetilde{M}_{12}(t)(\lambda = -1) \\ \mathcal{K}(\mathbf{x}) &= -\det(A_0)|(\lambda = 0) = (a^0)^2 + b^0 c^0. \end{aligned}$$

Proposition 8. *Problem (13) is equivalent to*

$$(16) \quad \begin{cases} \mathcal{F}(t, \mathbf{x}) = 0 \\ \mathcal{G}(t, \mathbf{x}) = 0 \\ \mathcal{H}_1(t, \mathbf{x}) = 0 \\ \mathcal{H}_2(t, \mathbf{x}) = 0 \\ \mathcal{K}(\mathbf{x}) = 1 \end{cases}$$

Proof. the first four equations of (16) are clearly equivalent to the first two equations of (13). Regarding the last one, by standard fuchsian system theorem, $\widetilde{M}(t)$ has the same eigenvalues as $2\pi i A_0$, so the last equation of (13) is equivalent to $\det(A_0) = -1$. Provided $\widetilde{M}(t) \in \Lambda \mathfrak{su}(2)$, its eigenvalues are imaginary for λ on the unit circle, so the eigenvalues of A_0 are real. Now $\det(A_0)$ is a homorphic function of λ in the unit disk which is real on the unit circle, so it must be constant. \square

From the symmetries we have

$$\overline{\mathcal{F}} = -\frac{1}{2\pi i} (\overline{\widetilde{M}_{11}} + \overline{\widetilde{M}_{11}^*}) = -\frac{1}{2\pi i} (-\widetilde{M}_{11} - \widetilde{M}_{11}^*) = \mathcal{F}.$$

Hence $\mathcal{F}(t, \mathbf{x}) \in \mathcal{W}_{\mathbb{R}}$. In the same way, $\mathcal{G}(t, \mathbf{x}) \in \mathcal{W}_{\mathbb{R}}$. Further, since $\mathcal{F}^* = -\mathcal{F}$ by definition, we obtain $\mathcal{F}^+(\lambda) = -\mathcal{F}^-(\frac{1}{\lambda})$, and therefore we do not have to solve $\mathcal{F}^- = 0$ separately. Moreover,

$$\mathcal{F} \in \mathcal{W}_{\mathbb{R}} \Rightarrow \mathcal{F}^0 \in \mathbb{R}$$

$$\overline{\mathcal{F}^0} = (\mathcal{F}^*)^0 = -\mathcal{F}^0 \Rightarrow \mathcal{F}^0 \in i\mathbb{R}.$$

Hence $\mathcal{F}^0(t, \mathbf{x}) = 0$ automatically holds by symmetry.

Differentiating \mathcal{F} and \mathcal{G} with respect to \mathbf{x} at $(0, \mathbf{x}_0)$ given by (14) gives

$$(17) \quad \begin{aligned} d\mathcal{F} &= da - \lambda^{-1}db - da^* + \lambda db^* + (\lambda^{-1} - \lambda)dr \\ d\mathcal{G} &= 2da - \lambda^{-1}db + \lambda dc - \lambda db^* + (\lambda + \lambda^{-1})dr. \end{aligned}$$

Write

$$b(\lambda) = b^0 + \lambda \tilde{b}(\lambda) \quad \text{with } \tilde{b} \in \mathcal{W}_{\mathbb{R}}^{\geq 0}.$$

Then (recalling that $b^0 \in \mathbb{R}$)

$$(18) \quad \begin{aligned} d\mathcal{F}^+ &= da^+ - d\tilde{b}^+ + \lambda db^0 - \lambda dr \\ d\mathcal{G}^+ &= 2da^+ - d\tilde{b}^+ + \lambda dc - \lambda db^0 + \lambda dr \\ d\mathcal{G}^- &= -\lambda^{-1}db^0 - (d\tilde{b}^+)^* + \lambda^{-1}dr \\ (d\mathcal{G}^-)^* &= -\lambda db^0 - d\tilde{b}^+ + \lambda dr. \end{aligned}$$

The Jacobian of $(\mathcal{F}^+, \mathcal{G}^+, (\mathcal{G}^-)^*)$ with respect to $(a^+, \tilde{b}^+, \lambda c)$ is

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

so this operator is an automorphism of $(\mathcal{W}_{\mathbb{R}}^{\geq 0})^3$. (Both variables and functions are in $\mathcal{W}_{\mathbb{R}}^{\geq 0}$ by definition and the previous symmetry arguments.) Therefore, applying the Implicit Function Theorem, the equations $\mathcal{F}^+ = 0$, $\mathcal{G}^+ = 0$ and $\mathcal{G}^- = 0$ uniquely determine the parameters a^+ , \tilde{b}^+ and c as functions of t and the remaining parameters r, a^0, b^0, \tilde{b}^0 .

It remains to solve four real equations $\mathcal{G}^0 = 0$, $\mathcal{H}_1 = 0$, $\mathcal{H}_2 = 0$ and $\mathcal{K} = 0$ with the remaining four parameters $(r, a^0, b^0, \tilde{b}^0) \in \mathbb{R}^4$. The derivatives of the functions a^+ , \tilde{b}^+ and c with respect to these parameters satisfy

$$\begin{aligned} d\tilde{b}^+ &= -\lambda db^0 + \lambda dr, \\ da^+ &= -2\lambda db^0 + 2\lambda dr, \\ dc &= 4db^0 - 4dr, \end{aligned}$$

which is obtained by inserting $d\mathcal{F}^+ = 0$, $d\mathcal{G}^+ = 0$ and $d\mathcal{G}^- = 0$ into (18). With these we obtain

$$\begin{aligned} d\mathcal{G}^0 &= 2da^0 - 2d\tilde{b}^0 \\ d\mathcal{H}_1 &= db(1) = db^0 + d\tilde{b}^0 + d\tilde{b}^+(1) = db^0 + d\tilde{b}^0 - db^0 + dr = d\tilde{b}^0 + dr \\ d\mathcal{H}_2 &= -db(-1) = -db^0 + d\tilde{b}^0 + d\tilde{b}^+(-1) = -db^0 + d\tilde{b}^0 + db^0 - dr = d\tilde{b}^0 - dr \\ d\mathcal{K} &= -\frac{1}{2}dc^0 - 2db^0 = 4db^0 - 4dr. \end{aligned}$$

The Jacobian of $(\mathcal{G}^0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{K})$ with respect to $(a^0, b^0, \tilde{b}^0, r)$ is an automorphism of \mathbb{R}^4 . Therefore, these equations uniquely determine the remaining parameters $(a^0, b^0, \tilde{b}^0, r)$ as smooth

functions for $t \sim 0$ by Implicit Function Theorem. So we have proven the following proposition:

Proposition 9. *For $t > 0$ small, there exists a unique $\mathbf{x}(t)$ in a neighbourhood of \mathbf{x}_0 such that (16) holds. In other words, the DPW potential $\eta_t^{\mathbf{x}(t)}$ solves the Monodromy Problem (9).*

We shall need the value of the monodromies at $\lambda = \pm 1$:

Proposition 10. *The monodromy of the solution of (12) for $\eta_t^{\mathbf{x}}$ determined by Proposition 9 satisfies*

$$M_j(t)(\lambda = 1) = \begin{pmatrix} e^{2\pi it} & 0 \\ 0 & e^{-2\pi it} \end{pmatrix}^{(-1)^j} \quad \text{and} \quad M_j(t)(\lambda = -1) = \begin{pmatrix} e^{-2\pi it} & 0 \\ 0 & e^{2\pi it} \end{pmatrix}^{(-1)^j}$$

for $0 \leq j \leq n-1$. Moreover, $\det(A_j(t))(\lambda) = -1$ for all $\lambda \in \mathbb{S}^1$.

Proof. By equation (13), $\widetilde{M}(t)(\pm 1)$ is diagonal with eigenvalues $\pm 2\pi i$. From Equation (15) at $t = 0$, we obtain by continuity

$$\widetilde{M}(t)(\lambda = 1) = 2\pi i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \widetilde{M}(t)(\lambda = -1) = 2\pi i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The proposition follows from $M(t) = \exp(t\widetilde{M}(t))$ and Equation (10). \square

2.7. Regularity at $z = \infty$. The following proposition guarantees that the surfaces constructed by $\eta_t^{\mathbf{x}}$ in Proposition 9 extends smoothly to $z = \infty$, see also Subsection 1.4.1.

Proposition 11. *For $t \sim 0$ let $\eta_t = \eta_t^{\mathbf{x}(t)}$ be the unique solution of the Monodromy Problem with parameters $\mathbf{x}(t)$. Then $z = \infty$ is an apparent singularity of η_t .*

Proof. let $\gamma_\infty = \prod_{i=0}^{n-1} \gamma_i$ and $M_\infty(t)$ be the monodromy of Φ_t corresponding to γ_∞ . By Proposition 10, Φ_t solves the following Monodromy Problem:

$$\begin{cases} M_\infty(t) \in \Lambda SU(2) \\ M_\infty(t)(\pm 1) = \text{Id} \end{cases}$$

Consider the gauge

$$G_0(z) = \begin{pmatrix} z^{-m} & \frac{-1}{r} \\ 0 & z^m \end{pmatrix}.$$

Then

$$\eta_0 \cdot G_0 = \begin{pmatrix} 0 & 0 \\ mz^{-m-1}dz & 0 \end{pmatrix}$$

which is holomorphic at ∞ since $m \geq 1$. We introduce a parameter $s \in \mathcal{W}_\rho^{\geq 0}$ and define

$$G_s(z, \lambda) = \begin{pmatrix} z^{-m} & \frac{1}{r}(s(\lambda) - 1) \\ 0 & z^m \end{pmatrix}.$$

Let $\widehat{\eta}_t = \eta_t \cdot G_s$. A computation reveals that

$$\widehat{\eta}_{t;21} = \frac{mr dz}{z^{m+1}} + t \sum_{i=0}^{n-1} \frac{\lambda c_i dz}{z^{2m}(z-p_i)}$$

is holomorphic at $z = \infty$ and

$$\widehat{\eta}_{t;11} = \frac{-ms dz}{z} + t \sum_{i=0}^{n-1} \left[\frac{a_i dz}{z-p_i} + \frac{\lambda c_i(1-s)dz}{rz^m(z-p_i)} \right],$$

which is holomorphic at $z = \infty$ by choosing

$$s = \frac{t}{m} \sum_{i=0}^{n-1} a_i$$

Finally,

$$\widehat{\eta}_{t;12} = \frac{s}{r}(1-s)mz^{m-1}dz + t \sum_{i=0}^{n-1} \left[\frac{2a_i(s-1)z^m dz}{r(z-p_i)} + \frac{b_i z^{2m} dz}{\lambda(z-p_i)} - \frac{\lambda c_i(s-1)^2 dz}{r^2(z-p_i)} \right].$$

We use $w = \frac{1}{z}$ as a local coordinate. From Equation (8), we obtain

$$\widehat{\eta}_{t;12} = \lambda^{-1} B(\lambda) \frac{dw}{w^{m+1}} + O(w^0 dw)$$

with

$$B(\lambda) = -\lambda r^{-1} s(1-s)m - \lambda r^{-1} t(s-1)n(a(\lambda) + a(-\lambda)) - t(b(\lambda) - b(-\lambda)).$$

In particular, $B(0) = 0$. By Theorem 6 in the Appendix 4.2, $\widehat{\eta}_t$ is holomorphic at $z = \infty$. (Note that $\widehat{\eta}_{0;21} = -mw^{m-1}dw$ so to apply Theorem 6, we make the change of coordinate $v = kw$ with $k^m = -1$.) \square

Remark 12. The coefficient $\widehat{\beta}$ of λ^{-1} in $\widehat{\eta}_{t;12}$ is obtained from the coefficient of λ^{-1} in $\eta_{t;12}$ multiplied by $(G_{s;22})^2$. Equation (8) then gives

$$\widehat{\beta} = \frac{nb^0 z^{n-2} dz}{z^n - 1}$$

which does not vanish at ∞ . Hence the immersion obtained from the DPW method will be unbranched at ∞ .

3. DERIVATIVES OF THE PARAMETERS

In this section, we consider the unique family $\eta_t^{\mathbf{x}(t)}$ from Proposition 9 solving the Monodromy Problem. Let $\mathbf{x}(t) = (r(t), a(t), b(t), c(t))$. We want to compute the time derivatives of the parameters.

3.1. Time parity of the potential.

The following proposition facilitates the computations of the derivatives of the parameters.

Proposition 13. *Assume that $\eta_t = \eta_t^{\mathbf{x}(t)}$ is the unique family from Proposition 9. Then*

$$\eta_{-t}(z, -\lambda) = \eta_t(z, \lambda).$$

This is equivalent to

$$\begin{cases} a(-t)(-\lambda) = -a(t)(\lambda) \\ b(-t)(-\lambda) = b(t)(\lambda) \\ c(-t)(-\lambda) = c(t)(\lambda) \\ r(-t) = r(t) \end{cases}$$

In particular, $\frac{d^k}{dt^k}a(t=0)$ is an odd function of λ and vanishes at $\lambda = 0$, for all even k .

Proof. Let

$$\widehat{\eta}_t(z, \lambda) = \eta_{-t}(z, -\lambda).$$

Then

$$\widehat{\eta}_t(z, \lambda) = \begin{pmatrix} 0 & 0 \\ m\widehat{r}(t)z^{m-1}dz & 0 \end{pmatrix} + t \sum_{j=0}^{n-1} \widehat{A}_j(t)(\lambda) \frac{dz}{z - p_j}$$

with

$$\widehat{r}(t) = r(-t) \quad \text{and} \quad \widehat{A}_j(t)(\lambda) = -A_j(-t)(-\lambda).$$

Hence

$$\widehat{A}_0(t) = \begin{pmatrix} \widehat{a}(t) & \lambda^{-1}\widehat{b}(t) \\ \lambda\widehat{c}(t) & -\widehat{a}(t) \end{pmatrix} \quad \text{with} \quad \begin{cases} \widehat{a}(t)(\lambda) = -a(-t)(-\lambda) \\ \widehat{b}(t)(\lambda) = b(-t)(-\lambda) \\ \widehat{c}(t)(\lambda) = c(-t)(-\lambda) \end{cases}$$

Let $\widehat{\mathbf{x}}(t) = (\widehat{a}(t), \widehat{b}(t), \widehat{c}(t), \widehat{r}(t))$. Observe that at $t = 0$, $\widehat{\mathbf{x}}(0) = \mathbf{x}(0)$. Let $\widehat{\Phi}_t$ be the solution of $d\widehat{\Phi}_t = \widehat{\Phi}_t \widehat{\eta}_t$ with initial condition $\widehat{\Phi}_t(0) = \text{Id}$. Then $\widehat{\Phi}_t(z, \lambda) = \Phi_{-t}(z, -\lambda)$. Hence $\widehat{\Phi}_t$ solves Equation (16). By uniqueness in the Implicit Function Theorem, $\widehat{\mathbf{x}}(t) = \mathbf{x}(t)$ for all t in a neighbourhood of 0. \square

3.2. First order derivatives.

Proposition 14. *The t -derivatives of the parameters $\mathbf{x}(t)$ solving (16) at $t = 0$ are given by*

$$(19) \quad a'(0) = (1 - \lambda^2)\kappa_m, \quad b'(0) = (\lambda - \lambda^3)\kappa_m, \quad c'(0) = 0, \quad r'(0) = 0.$$

where

$$(20) \quad \kappa_m = \frac{n}{2} \int_0^1 \frac{(1 - x^m)^2}{1 - x^n} dx, \quad n = 2m + 2.$$

The values of κ_m for small values of m are tabulated below:

m	κ_m
1	$\ln 2$
2	$\frac{3}{2} \ln 3$
3	$2 \ln 2 + \sqrt{2} \ln(1 + \sqrt{2})$
4	$\frac{5}{4} \ln 5 + \frac{\sqrt{5}}{2} \ln(2 + \sqrt{5})$
5	$\ln 2 + \frac{3}{2} \ln 3 + \sqrt{3} \ln(2 + \sqrt{3})$

Proof of Proposition 14: First of all, by Proposition 13, $r'(0) = 0$. Let

$$N(t) := M'(t)M(t)^{-1}.$$

Since the Monodromy Problem is solved for $\eta_t^{\mathbf{x}(t)}$, we have by Proposition 10:

$$\begin{cases} N(t) \in \Lambda \mathfrak{su}(2) \\ N(t)(\lambda = \pm 1) = \pm 2\pi i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{cases}$$

from which we deduce

$$(21) \quad \begin{cases} N'(0) \in \Lambda \mathfrak{su}(2) \\ N'(0)(\lambda = \pm 1) = 0 \end{cases}$$

as $\Lambda \mathfrak{su}(2)$ is a \mathbb{R} -vector space. Our first goal is to compute $N'(0)$ in terms of the parameters $\mathbf{x}(0)$ and its derivatives. Then the derivatives of the parameters is obtained by solving (21). Recall that Φ_t is the solution of $d_\Sigma \Phi_t = \Phi_t \eta_t$ in the universal cover $\tilde{\Sigma}$ of $\mathbb{C} \setminus \{p_0, \dots, p_{n-1}\}$ with initial condition $\Phi_t(0) = \text{Id}$. Also recall that $\gamma = \gamma_1$ is a closed curve enclosing the point $p_0 = 1$ and such that $\gamma(0) = \gamma(1) = 0$. By Proposition 8 in [18], we have for all t

$$N(t) = \int_\gamma \Phi_t \eta'_t \Phi_t^{-1}$$

where we denote the lift of γ to $\tilde{\Sigma}$ still by γ . Hence

$$N'(0) = \int_\gamma \Phi'_0 \eta'_0 \Phi_0^{-1} + \Phi_0 \eta''_0 \Phi_0^{-1} - \Phi_0 \eta'_0 \Phi_0^{-1} \Phi'_0 \Phi_0^{-1}.$$

Let

$$U = \Phi'_0 \Phi_0^{-1}.$$

It is easy to check that (for details compare with the proof of Proposition 8 in [18])

$$dU = \Phi_0 \eta'_0 \Phi_0^{-1}.$$

Thus

$$(22) \quad N'(0) = \int_\gamma U dU + \Phi_0 \eta''_0 \Phi_0^{-1} - dU U.$$

Our next goal is to compute the commutator $[U, dU]$. Using Equations (8), (14) and $r'(0) = 0$, we compute

$$\eta'_0 = \frac{n dz}{2(z^n - 1)} \begin{pmatrix} 2\lambda z^m & \lambda - \lambda^{-1} \\ -4\lambda z^{2m} & -2\lambda z^m \end{pmatrix}.$$

This gives

$$dU = \frac{n dz}{2(z^n - 1)} \begin{pmatrix} (\lambda + \lambda^{-1})z^m & \lambda - \lambda^{-1} \\ (\lambda^{-1} - \lambda)z^{2m} & -(\lambda + \lambda^{-1})z^m \end{pmatrix}.$$

Let

$$I_k(z) = \int_0^z \frac{w^k dw}{w^n - 1}.$$

Since $\Phi_t(0) = \text{Id}$, we have $U(0) = 0$, so integration yields

$$U = \frac{n}{2} \begin{pmatrix} (\lambda + \lambda^{-1})I_m & (\lambda - \lambda^{-1})I_0 \\ (\lambda^{-1} - \lambda)I_{2m} & -(\lambda + \lambda^{-1})I_m \end{pmatrix}$$

which gives

$$(23) \quad [U, dU] = \frac{n^2 dz}{4(z^n - 1)} \begin{pmatrix} (\lambda - \lambda^{-1})^2(I_{2m} - z^{2m}I_0) & 2(\lambda^2 - \lambda^{-2})(I_m - z^m I_0) \\ 2(\lambda^2 - \lambda^{-2})(z^{2m}I_m - z^m I_{2m}) & (\lambda - \lambda^{-1})^2(z^{2m}I_0 - I_{2m}) \end{pmatrix}.$$

To proceed, we compute the integrals involved in $\int_\gamma [U, dU]$.

Lemma 15. *With $J_{k,\ell} = \int_0^1 \frac{x^k - x^\ell}{x^n - 1} dx$ we have*

$$\int_\gamma \frac{I_k z^\ell - I_\ell z^k}{z^n - 1} dz = \frac{4\pi i}{n} J_{k,\ell}.$$

Proof. Let D be the disk bounded by γ . Then

$$f_k(z) = \int_0^z \frac{w^k - w^{n-1}}{w^n - 1} dw$$

is holomorphic in D because the integrand extends holomorphically to 1, and

$$I_k(z) = f_k(z) + \int_0^z \frac{w^{n-1}}{w^n - 1} dw = f_k(z) + \frac{1}{n} \log(1 - z^n).$$

Therefore, we have

$$\int_\gamma \frac{I_k z^\ell - I_\ell z^k}{z^n - 1} = \int_\gamma \frac{f_k z^\ell - f_\ell z^k}{z^n - 1} + \frac{1}{n} \int_\gamma \frac{z^\ell - z^k}{z^n - 1} \log(1 - z^n).$$

The first term on the right hand side can be computed via the Residue Theorem

$$\int_\gamma \frac{f_k z^\ell - f_\ell z^k}{z^n - 1} = \frac{2\pi i}{n} (f_k(1) - f_\ell(1)) = \frac{2\pi i}{n} J_{k,\ell}.$$

The second term can be computed via integration by parts and then applying the Residue Theorem:

$$\begin{aligned}
 \int_{\gamma} \frac{z^{\ell} - z^k}{z^n - 1} \log(1 - z^n) &= \int_{\gamma} (f'_{\ell} - f'_k) \log(1 - z^n) \\
 &= [(f_{\ell} - f_k) \log(1 - z^n)]_{\gamma(0)}^{\gamma(1)} - \int_{\gamma} (f_{\ell} - f_k) \frac{nz^{n-1}}{z^n - 1} \\
 &= 0 - 2\pi i(f_{\ell}(1) - f_k(1)).
 \end{aligned}$$

□

Proof of Proposition 14 continued. Using Equation (23) and Lemma 15, we obtain

$$(24) \quad \int_{\gamma} [U, dU] = \pi i n \begin{pmatrix} (\lambda - \lambda^{-1})^2 J_{2m,0} & 2(\lambda^2 - \lambda^{-2}) J_{m,0} \\ 2(\lambda^2 - \lambda^{-2}) J_{m,2m} & -(\lambda - \lambda^{-1})^2 J_{m,0} \end{pmatrix},$$

and by Leibniz rule we have

$$\eta''_0 = \begin{pmatrix} 0 & 0 \\ r'' m z^{m-1} dz & 0 \end{pmatrix} + 2 \sum_{i=0}^{n-1} \begin{pmatrix} a'_i & \lambda^{-1} b'_i \\ \lambda c'_i & -a'_i \end{pmatrix} \frac{dz}{z - p_i}$$

where a', b', c' are evaluated at $t = 0$. By the Residue Theorem

$$\begin{aligned}
 \int_{\gamma} \Phi_0 \eta''_0 \Phi_0^{-1} &= 4\pi i \operatorname{Res}_1 \Phi_0 \begin{pmatrix} a' & \lambda^{-1} b' \\ \lambda c' & -a' \end{pmatrix} \Phi_0^{-1} \frac{dz}{z - 1} \\
 (25) \quad &= 4\pi i \begin{pmatrix} a' - \lambda^{-1} b' & \lambda^{-1} b' \\ 2a' - \lambda^{-1} b' + \lambda c' & -a' + \lambda^{-1} b' \end{pmatrix}.
 \end{aligned}$$

Recall from Equation (22) that $N'(0)$ is the sum of (24) and (25). We now solve Problem (21) by the method of Section 2.6. By Proposition 13, $b'_{\lambda=0} = 0$, so we may write $b' = \lambda \tilde{b}'$. We then have

$$(26) \quad 0 = N'_{11} + N'_{11}{}^* = 4\pi i(a' - \tilde{b}' - a'^* + \tilde{b}'^*)$$

$$(27) \quad 0 = N'_{21} + N'_{12}{}^* = 4\pi i(\kappa_m(\lambda^2 - \lambda^{-2}) + 2a' - \tilde{b}' + \lambda c' - \tilde{b}'^*).$$

Projecting Equation (27) on $\mathcal{W}^{<0}$, $\mathcal{W}^{>0}$ and \mathcal{W}^0 and Equation (26) on $\mathcal{W}^{>0}$ we obtain

$$\begin{cases} \tilde{b}'^+ = a'^+ = -\lambda^2 \kappa_m \\ c' = 0 \\ a'^0 = \tilde{b}'^0 \end{cases}$$

Then

$$0 = N'_{12}|_{\lambda=1} = 4\pi i \tilde{b}'(1) = 4\pi i(\tilde{b}'^0 - \kappa_m)$$

gives

$$\tilde{b}'^0 = \kappa_m$$

concluding the proof. □

4. AREA ESTIMATES FOR LAWSON SURFACES

In this section we first compute the area in terms of the DPW potential and then show that the surfaces we construct by Proposition 9 yields Lawson surfaces for certain rational values of t .

4.1. The area of a minimal surface via DPW.

Proposition 16. *Let η be a holomorphic DPW potential on a compact domain Ω such that a solution Φ of $d_\Omega \Phi = \Phi \eta$ solves the Monodromy Problem (5). Let (F, B) the Iwasawa decomposition of Φ and f the resulting minimal immersion in \mathbb{S}^3 . Then*

$$(28) \quad \text{Area}(f(\Omega)) = -2i \int_{\partial\Omega} \text{trace}(\eta_{-1} B_0^{-1} B_1),$$

where $B = \sum_{k=0}^{\infty} \lambda^k B_k$ and $\eta = \sum_{k=-1}^{\infty} \lambda^k \eta_k$.

Proof. First, observe that B is globally defined on Ω , because Φ solves the Monodromy Problem. The minimal surface f comes with an associated family of flat connections given by

$$d_\Omega + F^{-1} d_\Omega F.$$

In a local coordinate z , we can split the connection 1-form into its complex linear and anti-linear parts

$$F^{-1} d_\Omega F = U dz + V d\bar{z},$$

and compute (compare with [1, 5])

$$U = \begin{pmatrix} \rho^{-1} \rho_z & \lambda^{-1} \rho^2 a_{-1} \\ b_0 \rho^{-2} & -\rho^{-1} \rho_z \end{pmatrix} \quad V = \begin{pmatrix} -\rho^{-1} \rho_{\bar{z}} & -\bar{b}_0 \rho^{-2} \\ -\lambda \rho^2 \bar{a}_{-1} & \rho^{-1} \rho_{\bar{z}} \end{pmatrix},$$

for some real valued and positive function ρ with

$$\eta_k = \begin{pmatrix} c_k & a_k \\ b_k & -c_k \end{pmatrix} dz.$$

Then the induced volume form dA of the minimal immersion f is computed to be

$$(29) \quad \begin{aligned} dA &= 4\rho^4 |a_{-1}|^2 dx \wedge dy \\ &= -2i \text{trace} \begin{pmatrix} 0 & \rho^2 a_{-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\rho^2 \bar{a}_{-1} & 0 \end{pmatrix} dz \wedge d\bar{z} \\ &= -2i \text{trace}(U_{-1} dz \wedge V_1 d\bar{z}) \end{aligned}$$

Let ∂B and $\bar{\partial} B$ denote the complex linear and complex anti-linear part of dB . Then we have by (3)

$$U_{-1} dz = B_0 \eta_{-1} B_0^{-1} \quad \text{and} \quad V_1 d\bar{z} = -\bar{\partial} B_1 B_0^{-1} + \bar{\partial} B_0 B_0^{-1} B_1 B_0^{-1}.$$

Using properties of the trace we obtain

$$\text{trace}(U_{-1}dz \wedge V_1d\bar{z}) = \text{trace}(-\eta_{-1}B_0^{-1} \wedge \bar{\partial}B_1 + \eta_{-1} \wedge B_0^{-1}\bar{\partial}B_0 B_0^{-1}B_1).$$

Moreover, because η is holomorphic

$$d(\eta_{-1}B_0^{-1}B_1) = \eta_{-1} \wedge (B_0^{-1}\bar{\partial}B_0 B_0^{-1}B_1 - B_0^{-1}\bar{\partial}B_1).$$

Therefore,

$$\text{Area}(f(\Omega)) = -2i \int_{\Omega} d \text{trace}(\eta_{-1}B_0^{-1}B_1) = -2i \int_{\partial\Omega} \text{trace}(\eta_{-1}B_0^{-1}B_1)$$

by Stokes' theorem. \square

In our case (the pull-back of) η will have apparent singularities at p_j and the corresponding boundary terms in Equation (28) can be computed as residues.

Proposition 17. *Let η be a DPW potential with a singularity at $z = p$ and G a gauge such that $\eta.G$ extends holomorphically to the disc $D(p, r)$ of radius $r > 0$ around p . Then*

$$\lim_{r \rightarrow 0} \int_{\partial D(p, r)} \text{trace}(\eta_{-1}B_0^{-1}B_1) = -2\pi i \text{Res}_p \text{trace}(\eta_{-1}G_1G_0^{-1}).$$

Proof. Let $\hat{\eta} = \eta.G$, $\hat{\Phi} = \Phi G$ and (\hat{F}, \hat{B}) be the Iwasawa decomposition of $\hat{\Phi}$. Then

$$B = D\hat{B}G^{-1},$$

where D is the unitary part of $G(0)$, i.e., it is a constant and diagonal matrix. We have

$$\begin{aligned} \eta_{-1} &= G_0\hat{\eta}_{-1}G_0^{-1}, \\ B_0 &= D\hat{B}_0G_0^{-1}, \\ B_1 &= D(\hat{B}_1G_0^{-1} - \hat{B}_0G_0^{-1}G_1G_0^{-1}), \\ \eta_{-1}B_0^{-1}B_1 &= G_0\hat{\eta}_{-1}\hat{B}_0^{-1}\hat{B}_1G_0^{-1} - \eta_{-1}G_1G_0^{-1}. \end{aligned} \tag{30}$$

Therefore,

$$\int_{\partial D(p, r)} \text{trace}(\eta_{-1}B_0^{-1}B_1) = \int_{\partial D(p, r)} \text{trace}(\hat{\eta}_{-1}\hat{B}_0^{-1}\hat{B}_1) - \int_{\partial D(p, r)} \text{trace}(\eta_{-1}G_1G_0^{-1}).$$

The first integral on the right hand side goes to 0 as $r \rightarrow 0$, because $\hat{\eta}$ and \hat{B} are smooth in $D(p, r)$. The proposition then follows from the Residue Theorem. \square

Corollary 18. *Let Σ be a compact Riemann surface and η a DPW potential with n apparent singularities at p_0, \dots, p_{n-1} solving the Monodromy problem (5). Then*

$$\text{Area}(f(\Sigma)) = 4\pi \sum_{j=0}^{n-1} \text{Res}_{p_j} \text{trace}(\eta_{-1}G_1^j(G_0^j)^{-1})$$

where G^j is a local gauge such that $\eta.G^j$ extends holomorphically to p_j .

Example: Consider the potential η for a great sphere and the gauge G given by

$$\eta = \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} dz \quad \text{and} \quad G = \begin{pmatrix} z & 0 \\ -\lambda & z^{-1} \end{pmatrix}.$$

We have $\eta.G$ is holomorphic at $z = \infty$. Then

$$\text{Res}_\infty \text{trace}(\eta_{-1} G_1 G_0^{-1}) = \text{Res}_\infty \text{trace} \begin{pmatrix} 0 & dz \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} = \text{Res}_\infty \frac{-dz}{z} = 1,$$

from which we obtain that the area of a great sphere is 4π .

4.2. Construction of compact minimal surfaces.

In Proposition 9 we have constructed a family of DPW potentials $\eta_t^{\mathbf{x}(t)}$ over $\mathbb{C}P^1$ with $n+1$ singularities at $z = p_j$, $j = 0, \dots, n-1$, and at $z = \infty$. By solving (16), the singularity at $z = \infty$ becomes apparent (Proposition 11), i.e., the corresponding minimal surface f extends smoothly to $z = \infty$. The monodromy at the other n singularities $M_j(\lambda = \pm 1)$ were computed in Proposition 10. For $t = \frac{1}{2k+2}$, we obtain $M_j^{k+1}(\lambda = 1) = M_j^{k+1}(\lambda = -1) = -\text{Id}$, for all $j = 0, \dots, 2m+1$. In other words, the Monodromy Problem (5) is solved on a $(k+1)$ -fold cover of $\mathbb{C}P^1$ branched at p_j

Thus let $t = \frac{1}{2k+2}$ for $k \in \mathbb{Z}$, $k \gg 1$ in following and consider the compact Riemann surface $\Sigma = \Sigma_{m,k}$ of genus $g = mk$ given by the algebraic equation

$$y^{k+1} = \frac{z^{m+1} - 1}{z^{m+1} + 1}.$$

The $(k+1)$ -fold covering given by

$$\pi: \Sigma \longrightarrow \mathbb{C}P^1, (y, z) \mapsto z$$

is totally branched over p_j , $j = 0, \dots, 2m+1$. Note that the monodromy μ (see Chapter II of [2]) of the covering $\Sigma \rightarrow \mathbb{C}P^1$ is given by an element of the permutation group

$$\sigma \in \mathcal{S}_{k+1}$$

of order $k+1$ such that

$$(31) \quad \mu(\gamma_{2j}) = \sigma \quad \mu(\gamma_{2j+1}) = \sigma^{-1},$$

for $j = 0, \dots, m$ and simple closed curves γ_j around the branch points p_j .

Consider the pull-back DPW potential $\pi^* \eta^\lambda$ on Σ . It can be locally desingularized around the preimages of the branch points $\hat{p}_j = \pi^{-1}(p_j)$ as follows: Let w be a local holomorphic coordinate on Σ centered at \hat{p}_j such that

$$w^{k+1} = z - p_j.$$

Since $t(k+1) = \frac{1}{2}$, the residue of the connection $d + \pi^*\eta$ at $w = 0$ is

$$(32) \quad t(k+1)A_j(\lambda) = \frac{1}{2} \begin{pmatrix} a_j(\lambda) & \lambda^{-1}b_j(\lambda) \\ \lambda c_j(\lambda) & -a_j(\lambda) \end{pmatrix}$$

for $a_j, b_j, c_j \in \mathcal{W}^{\geq 0}$ as in Section 2.4 satisfying, by Proposition 10

$$(33) \quad -a_j(\lambda)^2 - b_j(\lambda)c_j(\lambda) = -1$$

Consider the local gauge transformation

$$(34) \quad g_j = g_j(w, \lambda) = \begin{pmatrix} \frac{b_j(\lambda)}{1-a_j(\lambda)} & 0 \\ \lambda & \frac{1-a_j(\lambda)}{b_j(\lambda)} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{w}} & 0 \\ 0 & \sqrt{w} \end{pmatrix},$$

which is well-defined on a double covering of the w -disc (centered at $w = 0$) and some λ -disc centered at $\lambda = 0$. A computation gives

$$\hat{\eta} := \pi^*\eta \cdot g_j = \begin{pmatrix} 0 & \frac{(a_j-1)^2}{2\lambda b_j} \\ \frac{\lambda b_j(a_j^2 + b_j c_j - 1)}{(a_j-1)^2 w^2} & 0 \end{pmatrix} dw + O(w^{k-1})dw$$

which extends holomorphically to $w = 0$ thanks to Equation (33). Moreover, the λ^{-1} -term of $\hat{\eta}$ is non-zero at $w = 0$.

Remark 19. *The gauge (34) is not necessarily well-defined on the whole λ -unit-disc. Therefore, we need to apply the r -Iwasawa decomposition instead of the ordinary Iwasawa decomposition for the reconstruction. This does not alter the corresponding minimal surface.*

On the domain of the coordinate w the minimal surfaces \hat{f} and f obtained from the DPW potentials $\hat{\eta}$ and $\pi^*\eta$, respectively, coincide. Thus f extends smoothly to $w = 0$ ¹. We have shown the following

Proposition 20. *For $t = \frac{1}{2k+2}$ the pull-back potential $\tilde{\eta} = \pi^*\eta_t^{\mathbf{x}(t)}$ on $\Sigma_{m,k}$ has apparent singularities at $\pi^{-1}(p_j), j = 0, \dots, n-1$. In other words, the minimal immersion corresponding to the DPW potential $\tilde{\eta}$ extends smoothly to $p_j, j = 0, \dots, n-1$.*

Theorem 3. *For every $m \in \mathbb{N}^{\geq 1}$ fixed, there is a $K \in \mathbb{N}$ such that for every $k \geq K$ there exists an immersed compact minimal surface $f_{m,k}$ of genus $g = mk$ in \mathbb{S}^3 . Moreover, the symmetry group of $f_{m,k}$ contains $\mathbb{Z}_{m+1} \times \mathbb{Z}_{k+1}$.*

Proof. Proposition 9 shows the existence of DPW potentials $\eta_t = \eta_t^{\mathbf{x}(t)}$ for $t \sim 0$. Thus let $K \in \mathbb{N}$ such that η_t exist for all $t < \frac{1}{2K}$. Fix an integer $k \geq K$ and consider $\tilde{\eta} = \pi^*(\eta_t)$, $t = \frac{1}{2k+2}$ the pull-back DPW potential to $\Sigma_{m,k}$.

Let Φ be the solution of $d_{\Sigma_{m,k}}\Phi = \Phi\tilde{\eta}$, with initial condition $\Phi(\tilde{0}) = \text{Id}$, where $\tilde{0} \in \Sigma_{m,k}$ is a preimage of $z = 0$ under π . We claim that the Sym-Bobenko formula yields a well-defined

¹In order to see that one does actually obtain the same surface, one can first work on a double covering of the w -plane, and then prove that the unitary factor of the Iwasawa decomposition is already defined on the w -plane, while the gauge and the positive part of the Iwasawa decomposition have monodromy $-\text{Id}$ around $w = 0$.

minimal immersion $f: \Sigma_{m,k} \rightarrow \mathbb{S}^3$.

By Proposition 9 and Equation (31) the pull-back potential $\tilde{\eta}$ satisfies the closing conditions on $\Sigma_{m,k} \setminus S$, where $S = \pi^{-1}\{z \mid z = \infty \text{ or } z = p_j, j = 0, \dots, n-1\}$. Indeed, the extrinsic closing condition follows from the construction of the covering $\Sigma_{m,k} \rightarrow \mathbb{C}P^1$: A closed curve γ on the $(2m+2)$ -punctured sphere lifts to a closed curve $\hat{\gamma}$ in $\Sigma_{m,k}$ if and only if the monodromy μ in (31) of $\Sigma_{m,k} \rightarrow \mathbb{C}P^1$ along γ is trivial. Comparing μ with the monodromy representation of the potential η_t at $\lambda = \pm 1$ (see Proposition 10) we directly see that the monodromy of the potential η_t at $\lambda = \pm 1$ along a closed curve $\hat{\gamma}$ in $\Sigma_{m,k} \setminus S$ is $\pm \text{Id}$. We therefore obtain a well-defined minimal immersion

$$f: \Sigma_{m,k} \setminus S \longrightarrow \mathbb{S}^3.$$

By Proposition 20 the minimal immersion f extends as an immersion through the branch points p_j of π . Proposition 11 shows that the surface also extends smoothly through the preimages $\pi^{-1}(\infty)$ and we obtain a well-defined map $f_{m,k}: \Sigma_{m,k} \rightarrow \mathbb{S}^3$.

It remains to show that $f_{m,k}$ is immersed at $\pi^{-1}(\infty)$. This follows either by Remark 12 or from the following counting argument: On a branched minimal surface of genus $g = mk$ the Hopf differential Q has $4g - 4 - b$ zeros (counted with multiplicity), where b is the number of branch points (counted with multiplicity). On the other hand, for $f_{m,k}$ the form of the DPW potential and (8) gives that Q is a constant multiple of

$$\pi^* \frac{z^{m-1} (dz)^2}{z^{2m+2} - 1}.$$

This gives

$$(k-1)(2m+2) + (2k+2)(m-1) = 4km - 4 = 4g - 4$$

zeros of Q . Thus $b = 0$ and $f_{m,k}$ must be an immersion.

That the surface $f_{m,k}$ has a \mathbb{Z}_{m+1} and a \mathbb{Z}_{k+1} symmetry follows from the symmetries of the potential and by uniqueness of the Iwasawa decomposition. The \mathbb{Z}_{m+1} -action rotates the tangent plane of $f(\tilde{0}) \in \mathbb{S}^3 \subset \mathbb{R}^4$ and fixes its orthogonal complement, while the \mathbb{Z}_{k+1} -action fixes the tangent plane of $f(\tilde{0}) \in \mathbb{S}^3 \subset \mathbb{R}^4$ and rotates its orthogonal complement. Hence, the \mathbb{Z}_{m+1} and \mathbb{Z}_{k+1} -actions commute. \square

Theorem 4. *For $k \rightarrow \infty$, the asymptotic expansion of the area of the minimal surfaces*

$$f_{m,k}: \Sigma_{m,k} \longrightarrow \mathbb{S}^3$$

is given by

$$\text{Area}(f_{m,k}) = 4\pi(m+1) \left(1 - \frac{\kappa_m}{2(k+1)} + O\left(\frac{1}{(k+1)^3}\right) \right),$$

with κ_m as defined in Proposition 14

Proof. Recall that the area of $f_{m,k}$ is given by a sum of residues, see Corollary 18. The local gauges (34) have local monodromies $-\text{Id}$ around $\pi^{-1}(p_j)$ on $\Sigma_{m,k}$. Thus, we consider the double covering $\hat{\Sigma}_{m,k} \rightarrow \Sigma_{m,k}$ defined by the $(2k+2)$ -fold covering $\hat{\pi}$ of $\mathbb{C}P^1: t$

$$\hat{y}^{2k+2} = \frac{z^{m+1} - 1}{z^{m+1} + 1}.$$

Applying Corollary 18 to the potential $\hat{\pi}^* \eta := \hat{\pi}^* \eta_t^{\mathbf{x}^{(t)}}$ for $t = \frac{1}{2k+2}$, gives rise to a minimal surface

$$\tilde{f}_{m,k} : \hat{\Sigma}_{m,k} \longrightarrow \Sigma_{m,k} \rightarrow \mathbb{S}^3,$$

where $\tilde{f}_{m,k}$ is a double cover of $f_{m,k}$. A direct computation (using the gauge G_0 in Section 2.7) yields that there is no contribution of residues at the points over $z = \infty$. We claim that at each preimage \hat{p}_j of the branch points p_j , $j = 0, \dots, n-1$, the residue is

$$(35) \quad \text{Res}_{\hat{p}_j} \text{trace}(\hat{\pi}^* \eta_{-1} g_{j,1} g_{j,0}^{-1}) = 1 - a(t)|_{\lambda=0}$$

where $a(t)$ is provided by Proposition 9, and the gauge is given by (34). Indeed, using (32) and (34) and the coordinate $x = \sqrt{w}$ centered at \hat{p}_j we have

$$\hat{\pi}^* \eta_{-1} = \begin{pmatrix} 0 & b_j(0) \\ 0 & 0 \end{pmatrix} \frac{dx}{x} + \text{higher order terms in } x,$$

$$g_{j,1} = \begin{pmatrix} \frac{a'_j(0)b_j(0)+b'_j(0)(1-a_j(0))}{(1-a_j(0))^2} \frac{1}{x} & 0 \\ \frac{1}{x} & -\frac{a'_j(0)b_j(0)+b'_j(0)(1-a_j(0))}{b_j(0)^2} x \end{pmatrix}$$

and

$$g_{j,0} = \begin{pmatrix} \frac{b_j(0)}{1-a_j(0)} \frac{1}{x} & 0 \\ 0 & \frac{1-a_j(0)}{b_j(0)} x \end{pmatrix}$$

which yields (35). (In the above computations, t is fixed and therefore omitted, and a_j , b_j are seen as functions of λ .) By Propositions 14 and 13,

$$1 - a(t)|_{\lambda=0} = 1 - \kappa_m t + O(t^3).$$

By Corollary 18

$$2 \text{Area}(f_{m,k}) = \text{Area}(\tilde{f}_{m,k}) = 4\pi(2m+2)(1 - \kappa_m t + O(t^3)).$$

□

Remark 21. *Due to time parity (Proposition 13) the same minimal surface is obtained when choosing $t = \frac{-1}{2(k+1)}$, though the area computation differs in detail. Indeed, the residue in (32) will have the opposite sign, so the gauge g_j in (34) needs to be altered turning the right hand side of (35) into $1+a(t)|_{\lambda=0}$. This gives of course the same area for the surface, since $a(t)|_{\lambda=0}$ is odd in t .*

Theorem 5. *The minimal surfaces $f_{m,k} : \Sigma_{m,k} \rightarrow \mathbb{S}^3$ coincide with the Lawson surfaces $\xi_{m,k}$. In particular, the asymptotic expansion of the area of the Lawson surfaces is given by Theorem 4.*

Proof. Using the symmetries we first show that the geodesic polygon of the construction of the Lawson surface is contained in the surface $f_{m,k}$: By construction of the potential, η_t admits the symmetry $\sigma^* \eta_t = \bar{\eta}_t$ for $\sigma(z) = \bar{z}$, see Section 2.4. Analogously to [5] it can be shown (for the initial value $\Phi(0) = \text{Id}$) that the line

$$\{z \in \mathbb{R} \subset \mathbb{C} \mid |z| < 1\}$$

is mapped via $f = f_{m,k}$ to a geodesic in the 3-sphere. The symmetry $\delta \circ \delta$ of the surface is induced by rotational symmetry

$$x \in \mathbb{S}^3 = \text{SU}(2) \longmapsto D^{-2}x D^2$$

of the 3-sphere. Its fix point set is the circle

$$C_1 = \left\{ \left(\begin{array}{cc} w & 0 \\ 0 & \bar{w} \end{array} \right) \mid w \in \mathbb{S}^1 \subset \mathbb{C} \right\}.$$

The induced symmetry $\delta \circ \delta \circ \sigma$ on the z -plane fixes the line

$$\{z \in \mathbb{C} \mid \arg(z) = \frac{\pi}{m+1}, |z| < 1\}$$

and therefore f also maps this line to a geodesic in the 3-sphere using the same arguments as above. The analytic continuation of these two geodesics on the abstract Riemannian surface extend as geodesics in 3-space (contained in the surface) through the points $f(1)$ respectively $f(\exp \frac{\pi i}{m+1})$. By construction of the surface via (4) and Proposition 10, there is a rotational symmetry (induced from the monodromy around $z = 1$) of the surface which is given by

$$x \in \mathbb{S}^3 = \text{SU}(2) \longmapsto \begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & e^{-2\pi i t} \end{pmatrix} x \begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & e^{-2\pi i t} \end{pmatrix}$$

with $t = \frac{1}{2k+2}$. The fix point set of this rotation is the circle C_2 orthogonal to the circle C_1 . Clearly, $f(1)$ is a fixed point of the rotational symmetry, and hence lies on C_2 . Analogously, we find that $f(\exp \frac{\pi i}{m+1}) \in C_2$ by applying the symmetry induced from the monodromy around $z = \exp \frac{\pi i}{m+1}$. Applying the rotational symmetry at the point $f(1)$ together with the reflection symmetry across the geodesic in 3-space which contains $\{f(x) \mid x \in \mathbb{R}, |x| < 1\}$, we easily deduce that also $\{f(x) \mid x \in \mathbb{R}, x > 1\}$ is a geodesic in the 3-sphere. Analogously, f maps $\{z \in \mathbb{C} \mid \arg(z) = \frac{\pi}{m+1}, |z| > 1\}$ to a geodesic. Finally, being a fix point of the rotational symmetry $\delta \circ \delta$, f maps (points over) $z = \infty$ to a point on C_1 . The angles between the four different geodesics joining $f(0)$ and $f(1)$, $f(1)$ and $f(\infty)$, $f(\infty)$ and $f(\exp \frac{\pi i}{m+1})$, $f(\exp \frac{\pi i}{m+1})$ and $f(0)$, respectively, must be the same as the angles in the geodesic polygon of the Lawson surface by the very form of the symmetries. From these observations we obtain that $f_{m,k}$ maps the boundary of the sector

$$Se = \{z \in \mathbb{C} \mid 0 \leq \arg(z) \leq \frac{\pi}{m+1}\}$$

to the geodesic polygon Γ in the construction of a Lawson surface.

We want to prove that $f_{m,k}(Se)$ is contained in a hemisphere for k large enough. The contour Γ is contained in a ball $B(p, r) \subset \mathbb{S}^3$ of radius $r < \frac{\pi}{2}$. Assume by contradiction that there exists a point $q \in S$ such that $d(q, p) \geq \frac{\pi}{2}$. Then $d(q, \Gamma) \geq \frac{\pi}{2} - r$. By the Monotonicity Formula for minimal surfaces, the area of S is greater than $c(\frac{\pi}{2} - r)^2$ for some universal constant c . But we know that the area of S is equal to $\frac{1}{n(k+1)}$ of the area of $f_{m,k}(\Sigma)$, so is less than $\frac{2\pi}{k+1}$. Hence for k large enough, S is included in the hemisphere $B(p, \frac{\pi}{2})$. Then the solution of the Plateau Problem is unique by a standard application of the maximum principle (see [8, Theorem 4.1]). Hence $f_{m,k}(\Sigma)$ is the Lawson surface $\xi_{m,k}$. \square

APPENDIX A. ON REMOVABLE SINGULARITIES

The DPW method can also be applied to obtain CMC surfaces from Σ into 3-dimensional space forms. In this section, we want to give sufficient conditions for a singularity of a DPW potential to be apparent in this more general setup. Be aware of the slightly differing notations in this section.

The Monodromy Problem associated to the general Sym-Bobenko formula at Sym-points λ_1 and $\lambda_2 \in \mathbb{C}_*$ is: If Φ is a solution of $d_\Sigma \Phi = \Phi \eta$ and $M(\lambda)$ is the monodromy of Φ , then

$$(36) \quad \text{if } \lambda_2 \neq \lambda_1: \begin{cases} M \in \Lambda SU(2) \\ M(\lambda_1) = \text{Id}_2 \\ M(\lambda_2) = \text{Id}_2 \end{cases} \quad \text{if } \lambda_2 = \lambda_1: \begin{cases} M \in \Lambda SU(2) \\ M(\lambda_1) = \text{Id}_2 \\ M'(\lambda_1) = 0 \end{cases}$$

Example 22.

- $(\lambda_1, \lambda_2) = (1, 1)$ produces surfaces in \mathbb{R}^3 with $H \equiv 1$,
- $(\lambda_1, \lambda_2) = (e^{i\alpha}, -e^{-i\alpha})$ produces surfaces in \mathbb{S}^3 with $H \equiv \tan \alpha$,
- $(\lambda_1, \lambda_2) = (e^q, e^{-q})$ produces surfaces in \mathbb{H}^3 with $H \equiv \coth q$.

We assume that the Sym-points are chosen so that

$$\lambda_1 + \lambda_2 \in e^{i\theta} \mathbb{R} \quad \text{and} \quad \lambda_1 \lambda_2 = e^{2i\theta}$$

for some $e^{i\theta} \in \mathbb{S}^1$. For the above examples, $e^{i\theta}$ is respectively 1, i and -1 . This ensures that

$$(37) \quad \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{\lambda e^{i\theta}} \in \mathbb{R}, \quad \forall \lambda \in \mathbb{S}^1.$$

Moreover, we fix $\rho > 1$ such that both Sym-points satisfy $|\lambda_1| < \rho$, $|\lambda_2| < \rho$.

Theorem 6. Fix an integer $m \geq 1$. For $t \in (-\varepsilon, \varepsilon)$ let η_t be a family of DPW potentials on $D^*(0, r)$ and Φ_t a family of solutions of $d_{D^*(0, r)} \Phi_t = \Phi_t \eta_t$ on its universal cover, with C^1 -dependence on (t, z) as maps into $\Lambda \mathfrak{sl}(2, \mathbb{C})_\rho$ and $\Lambda SL(2, \mathbb{C})_\rho$, respectively. Assume

- (1) η_t has a pole of order at most $2m + 1$ at $z = 0$ with principal part

$$\eta_t(z, \lambda) = \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} \left(\frac{a_t(\lambda)}{z^{2m+1}} + \frac{b_t(\lambda)}{z^{m+1}} + \frac{c_t(\lambda)}{z} \right) dz + \Xi_t(z, \lambda)$$

where Ξ_t is holomorphic with respect to z in $D(0, r)$.

- (2)

$$\eta_0 = \begin{pmatrix} 0 & 0 \\ mz^{m-1} dz & 0 \end{pmatrix}.$$

- (3) Φ_t solves the Monodromy Problem (36).

- (4) $a_t^0 = \text{Re}(e^{-i\theta} b_t^0) = 0$.

Then $a_t \equiv b_t \equiv c_t \equiv 0$, for t small enough. In particular, η_t is holomorphic at $z = 0$.

Proof. We can write

$$\Phi_0(z, \lambda) = V(\lambda) \begin{pmatrix} 1 & 0 \\ z^m & 1 \end{pmatrix} \quad \text{with } V \in \Lambda SL(2, \mathbb{C}).$$

Let (F, B) be the Iwasawa decomposition of V . By Theorem 5 in [19] we have $F \in \Lambda SU(2)_\rho$ and $B \in \Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho$. Replacing Φ_t by $F^{-1}\Phi_t$ for all t , we can assume without loss of generality that $V \in \Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho$ (which does not change the monodromy properties of Φ_t).

For $\mathbf{x} = (a, b, c) \in (\mathcal{W}^{\geq 0})^3$, let $\eta_t^{\mathbf{x}}$ be the potential in $D^*(0, r)$ defined by

$$\eta_t^{\mathbf{x}}(z, \lambda) = \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} \left(\frac{a}{z^{2m+1}} + \frac{b}{z^{m+1}} + \frac{c}{z} \right) dz + \Xi_t(z, \lambda).$$

Let $\Phi_t^{\mathbf{x}}$ be the solution of $d\Phi_t^{\mathbf{x}} = \Phi_t^{\mathbf{x}}\eta_t^{\mathbf{x}}$ on the universal cover with initial condition $\Phi_t^{\mathbf{x}}(\tilde{z}_0, \lambda) = \Phi_t(\tilde{z}_0, \lambda)$. We consider the problem of finding $\eta_t^{\mathbf{x}}$ such that

$$(38) \quad \begin{cases} \Phi_t^{\mathbf{x}} \text{ solves the Monodromy Problem (36)} \\ a^0 = 0 \\ \operatorname{Re}(e^{-i\theta} b^0) = 0 \end{cases}$$

Writing $\mathbf{x}_t = (a_t, b_t, c_t)$, we have $\eta_t = \eta_t^{\mathbf{x}_t}$ and $\Phi_t = \Phi_t^{\mathbf{x}_t}$. We want to apply an Implicit Function argument to show that for (t, \mathbf{x}) in a neighbourhood of $(0, 0)$, solving Problem (38) is equivalent to $\mathbf{x}_t \equiv 0$, from which Theorem 6 follows.

Fix a base point $z_0 \in D^*(0, r)$. Let \tilde{z}_0 be a lift of z_0 to the universal cover $\widetilde{D}^*(0, r)$ and γ be a generator of $\pi_1(D^*(0, r), z_0)$. Let $M(t, \mathbf{x})$ be the monodromy of $\Phi_t^{\mathbf{x}}$ with respect to γ . Then the following Lemma holds.

Lemma 23.

- (1) $(t, \mathbf{x}) \mapsto M(t, \mathbf{x})$ is a C^1 map from $(-\epsilon, \epsilon) \times (\mathcal{W}^{\geq 0})^3$ to $\Lambda SL(2, \mathbb{C})_\rho$.
- (2) For all $t \in (-\epsilon, \epsilon)$, $M(t, 0) = \operatorname{Id}_2$.
- (3) The partial derivative of M with respect to \mathbf{x} at $(0, 0)$ is given by

$$d_{\mathbf{x}}M = \frac{2\pi i}{\lambda} V \begin{pmatrix} -db & dc \\ -da & db \end{pmatrix} V^{-1}.$$

Proof. (1) Point 1 follows from standard ODE theory.

(2) Point 2 follows from the fact that $\eta_t^{\mathbf{x}=0}$ is holomorphic in $D(0, r)$.

(3) Let $\Psi_t^{\mathbf{x}}$ be the solution of $d\Psi_t^{\mathbf{x}} = \Psi_t^{\mathbf{x}}\eta_t^{\mathbf{x}}$ in the universal cover with initial condition $\Psi_t^{\mathbf{x}}(\tilde{z}_0) = \operatorname{Id}_2$. Then $\Phi_t^{\mathbf{x}} = \Phi_t(\tilde{z}_0)\Psi_t^{\mathbf{x}}$ and thus

$$M(t, \mathbf{x}) = \Phi_t(\tilde{z}_0)\mathcal{M}_\gamma(\Psi_t^{\mathbf{x}})\Phi_t(\tilde{z}_0)^{-1}.$$

By Proposition 8 in [18], the partial derivative of $\mathcal{M}_\gamma(\Psi_t^{\mathbf{x}})$ with respect to \mathbf{x} at $(t, \mathbf{x}) = (0, 0)$ is given by

$$d_{\mathbf{x}}\mathcal{M}_\gamma(\Psi_t^{\mathbf{x}}) = \int_\gamma \Psi_0^0 d_{\mathbf{x}}\eta_t^{\mathbf{x}}(\Psi_0^0)^{-1}.$$

Hence since $\mathcal{M}_\gamma(\Psi_0^0) = \text{Id}_2$:

$$\begin{aligned}
 d_{\mathbf{x}}M &= \Phi_0(\tilde{z}_0)d_{\mathbf{x}}\mathcal{M}_\gamma(\Psi_t^{\mathbf{x}})\Phi_0(\tilde{z}_0)^{-1} \\
 &= \int_\gamma \Phi_0 d_{\mathbf{x}}\eta_t^{\mathbf{x}}\Phi_0^{-1} \\
 &= \int_\gamma V \begin{pmatrix} 1 & 0 \\ z^m & 1 \end{pmatrix} \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} \left(\frac{da}{z^{2m+1}} + \frac{db}{z^{m+1}} + \frac{dc}{z} \right) \begin{pmatrix} 1 & 0 \\ -z^m & 1 \end{pmatrix} V^{-1} \\
 &= 2\pi i V \text{Res}_0 \begin{pmatrix} -z^m & 1 \\ -z^{2m} & z^m \end{pmatrix} \left(\frac{da}{z^{2m+1}} + \frac{db}{z^{m+1}} + \frac{dc}{z} \right) V^{-1} \\
 &= \frac{2\pi i}{\lambda} V \begin{pmatrix} -db & dc \\ -da & db \end{pmatrix} V^{-1}.
 \end{aligned}$$

□

We define for (t, \mathbf{x}) in a neighbourhood of $(0, 0)$:

$$\begin{aligned}
 (39) \quad \widetilde{M}(t, \mathbf{x})(\lambda) &= \frac{1}{\lambda - \lambda_1} (\log M(t, \mathbf{x})(\lambda) - \log M(t, \mathbf{x})(\lambda_1)) \quad \text{if } \lambda \neq \lambda_1 \\
 \widehat{M}(t, \mathbf{x})(\lambda) &= \frac{\lambda e^{i\theta}}{\lambda - \lambda_2} (\widetilde{M}(t, \mathbf{x})(\lambda) - \widetilde{M}(t, \mathbf{x})(\lambda_2)) \quad \text{if } \lambda \neq \lambda_1, \lambda_2.
 \end{aligned}$$

Then $\widetilde{M}(t, \mathbf{x})$ extends holomorphically to $\lambda = \lambda_1$ and $\widehat{M}(t, \mathbf{x})$ extends holomorphically to $\lambda = \lambda_1, \lambda_2$. Moreover, by Proposition 5 in [20] \widetilde{M} and \widehat{M} are smooth maps taking values in $\Lambda\mathfrak{sl}(2, \mathbb{C})_\rho$.

Lemma 24. *The Monodromy Problem (36) for $\Phi_t^{\mathbf{x}}$ is equivalent to:*

$$(40) \quad \begin{cases} \widehat{M}(t, \mathbf{x}) \in \Lambda\mathfrak{su}(2) \\ M(t, \mathbf{x})(\lambda_1) = \text{Id}_2 \\ \widetilde{M}(t, \mathbf{x})(\lambda_2) = 0. \end{cases}$$

Proof. The second equation in (36) and (40) are the same. The third one of (36) and (40) are equivalent: While for $\lambda_2 \neq \lambda_1$ the equation is the same, we use for $\lambda_1 = \lambda_2$ that

$$\widetilde{M}(t, \mathbf{x})(\lambda_1) = \frac{\partial}{\partial \lambda} M(t, \mathbf{x})(\lambda)|_{\lambda=\lambda_1}.$$

As for the first equation of (36) and (40),

$$\widehat{M}(t, \mathbf{x})(\lambda) = \frac{\lambda e^{i\theta}}{(\lambda - \lambda_1)(\lambda - \lambda_2)} \log M(t, \mathbf{x})(\lambda).$$

Thus Lemma 24 follows from Equation (37). □

We introduce the auxiliary variables (p, q, r) defined as functions of (a, b, c) by

$$\begin{pmatrix} -q & r \\ -p & q \end{pmatrix} = V \begin{pmatrix} -b & c \\ -a & b \end{pmatrix} V^{-1}.$$

This change of variables is an automorphism of $(\mathcal{W}^{\geq 0})^3$ because $V \in \Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_{\rho}$ and hence its entries are in $\mathcal{W}^{\geq 0}$. Then

$$(41) \quad d_{\mathbf{x}}M = \frac{2\pi i}{\lambda} \begin{pmatrix} -dq & dr \\ -dp & dq \end{pmatrix}.$$

Writing $V(0) = \begin{pmatrix} \mu & \nu \\ 0 & \mu^{-1} \end{pmatrix}$ with $\mu > 0$ and $\nu \in \mathbb{C}$, we have

$$\begin{pmatrix} -q^0 & r^0 \\ -p^0 & q^0 \end{pmatrix} = \begin{pmatrix} -b^0 - \frac{\nu}{\mu}a^0 & \nu^2 a^0 + 2\mu\nu b^0 + \mu^2 c^0 \\ -\frac{1}{\mu^2}a^0 & b^0 + \frac{\nu}{\mu}a^0 \end{pmatrix}.$$

Therefore,

$$(42) \quad \begin{cases} a^0 = 0 \\ \operatorname{Re}(e^{-i\theta}b^0) = 0 \end{cases} \Leftrightarrow \begin{cases} p^0 = 0 \\ \operatorname{Re}(e^{-i\theta}q^0) = 0 \end{cases}$$

We decompose the parameter $q \in \mathcal{W}^{\geq 0}$ into

$$q(\lambda) = q(\lambda_1) + (\lambda - \lambda_1)\tilde{q}(\lambda) \quad \text{with } \tilde{q} \in \mathcal{W}^{\geq 0}.$$

Then we further decompose \tilde{q} into

$$\tilde{q}(\lambda) = \tilde{q}(\lambda_2) + (\lambda - \lambda_2)\hat{q}(\lambda) \quad \text{with } \hat{q} \in \mathcal{W}^{\geq 0}.$$

This gives by Equation (41):

$$\begin{aligned} d_{\mathbf{x}}\widetilde{M}_{11} &= \frac{2\pi i}{\lambda - \lambda_1} (d_{\mathbf{x}}M_{11}(\lambda) - d_{\mathbf{x}}M_{11}(\lambda_1)) \\ &= \frac{-2\pi i}{\lambda - \lambda_1} \left(\frac{dq(\lambda_1) + (\lambda - \lambda_1)d\tilde{q}(\lambda)}{\lambda} - \frac{dq(\lambda_1)}{\lambda_1} \right) \\ &= -2\pi i \left(\frac{d\tilde{q}(\lambda)}{\lambda} - \frac{dq(\lambda_1)}{\lambda\lambda_1} \right). \\ d_{\mathbf{x}}\widehat{M}_{11} &= \frac{-2\pi i \lambda e^{i\theta}}{\lambda - \lambda_2} \left(\frac{d\tilde{q}(\lambda_2) + (\lambda - \lambda_2)d\hat{q}(\lambda)}{\lambda} - \frac{dq(\lambda_1)}{\lambda\lambda_1} - \frac{d\tilde{q}(\lambda_2)}{\lambda_2} + \frac{dq(\lambda_1)}{\lambda_1\lambda_2} \right) \\ &= -2\pi i e^{i\theta} \left(d\hat{q}(\lambda) - \frac{d\tilde{q}(\lambda_2)}{\lambda_2} + \frac{dq(\lambda_1)}{\lambda_1\lambda_2} \right). \end{aligned}$$

By decomposing the other parameters q and r in the same way and we obtain similar formulas for the other entries of $d_{\mathbf{x}}\widetilde{M}$ and $d_{\mathbf{x}}\widehat{M}$. Let

$$(43) \quad \begin{aligned} \mathcal{E}_1(t, \mathbf{x}) &= \widehat{M}_{11}(t, \mathbf{x}) + \widehat{M}_{11}(t, \mathbf{x})^* \in \mathcal{W} \\ \mathcal{E}_2(t, \mathbf{x}) &= \widehat{M}_{12}(t, \mathbf{x}) + \widehat{M}_{21}(t, \mathbf{x})^* \in \mathcal{W} \\ \mathcal{E}_3(t, \mathbf{x}) &= (M_{11}(t, \mathbf{x})(1) - 1, M_{12}(t, \mathbf{x})(1), M_{21}(t, \mathbf{x})(1)) \in \mathbb{C}^3 \\ \mathcal{E}_4(t, \mathbf{x}) &= \left(\widetilde{M}_{11}(t, \mathbf{x})(1), \widetilde{M}_{12}(t, \mathbf{x})(1), \widetilde{M}_{21}(t, \mathbf{x})(1) \right) \in \mathbb{C}^3. \end{aligned}$$

The Monodromy Problem (40) is then equivalent to $\mathcal{E}_k(t, \mathbf{x}) = 0$ for $1 \leq k \leq 4$. To put everything together we define

$$\mathcal{F}(t, \mathbf{x}) = \left[\mathcal{E}_1^+, \mathcal{E}_2^+, (\mathcal{E}_2^-)^*, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_2^0, p^0, \operatorname{Re}(e^{-i\theta} q^0) + i\operatorname{Re}(\mathcal{E}_1^0) \right] (t, \mathbf{x}) \in (\mathcal{W}^{>0})^3 \times \mathbb{C}^9.$$

By Equation (42), Problem (38) is equivalent to $\mathcal{F}(t, \mathbf{x}) = 0$. Indeed, since $\mathcal{E}_1 = \mathcal{E}_1^*$ we have $\mathcal{E}_1 = 0$ is equivalent to $\mathcal{E}_1^+ = 0$ and $\operatorname{Re}(\mathcal{E}_1^0) = 0$.

Lemma 25. *The derivative of \mathcal{F} at $(0, 0)$ with respect to the parameters*

$$(\widehat{p}^+, \widehat{q}^+, \widehat{r}^+, p(\lambda_1), q(\lambda_1), r(\lambda_1), \widetilde{p}(\lambda_2), \widetilde{q}(\lambda_2), \widetilde{r}(\lambda_2), \widehat{p}^0, \widehat{q}^0, \widehat{r}^0).$$

is an \mathbb{R} -linear automorphism of $(\mathcal{W}^{>0})^3 \times \mathbb{C}^9$.

Proof. We have

$$d(\mathcal{E}_1^+, \mathcal{E}_2^+, (\mathcal{E}_2^-)^*) = 2\pi i e^{i\theta} (-d\widehat{q}^+, d\widehat{r}^+, -d\widehat{p}^+),$$

so the derivative of $(\mathcal{E}_1^+, \mathcal{E}_2^+, (\mathcal{E}_2^-)^*)$ with respect to $(\widehat{q}^+, \widehat{r}^+, \widehat{p}^+)$ is an automorphism of $(\mathcal{W}^{>0})^3$. Let L be the partial derivative of the remaining components of \mathcal{F} with respect to the remaining variables. It suffices to prove that L is an automorphism of \mathbb{C}^9 . Let

$$X = (P(\lambda_1), Q(\lambda_1), R(\lambda_1), \widetilde{P}(\lambda_2), \widetilde{Q}(\lambda_2), \widetilde{R}(\lambda_2), \widehat{P}^0, \widehat{Q}^0, \widehat{R}^0) \in \operatorname{Ker}(L).$$

Then

$$d\mathcal{E}_3 X = 2\pi i (-Q(\lambda_1), R(\lambda_1), -P(\lambda_1)) \Rightarrow P(\lambda_1) = Q(\lambda_1) = R(\lambda_1) = 0.$$

$$d\mathcal{E}_4 X = \frac{2\pi i}{\lambda_2} (-\widetilde{Q}(\lambda_2), \widetilde{R}(\lambda_2), -\widetilde{P}(\lambda_2)) \Rightarrow \widetilde{P}(\lambda_2) = \widetilde{Q}(\lambda_2) = \widetilde{R}(\lambda_2) = 0.$$

From

$$p(\lambda) = p(\lambda_1) + (\lambda - \lambda_1)\widetilde{p}(\lambda_2) + (\lambda - \lambda_1)(\lambda - \lambda_2)\widehat{p}(\lambda)$$

we obtain

$$p^0 = p(\lambda_1) - \lambda_1\widetilde{p}(\lambda_2) + e^{2i\theta}\widehat{p}^0.$$

Hence

$$\widehat{P}^0 = 0 \quad \text{and} \quad \operatorname{Re}(e^{i\theta}\widehat{Q}^0) = 0.$$

Then

$$\begin{aligned} d\mathcal{E}_2^0 X &= 2\pi i e^{i\theta} \widehat{R}^0 \Rightarrow \widehat{R}^0 = 0 \\ \operatorname{Re}(d\mathcal{E}_1^0 X) &= 4\pi \operatorname{Im}(e^{i\theta}\widehat{Q}^0) \Rightarrow \widehat{Q}^0 = 0 \end{aligned}$$

Hence $X = 0$ so L is an automorphism of \mathbb{C}^9 .

□

By Implicit Function Theorem, Problem (38) uniquely determines \mathbf{x} as a function of t for (t, \mathbf{x}) in a neighbourhood of $(0, 0)$. By Point 2 of Lemma 23 the unique solution is given by $\mathbf{x} \equiv 0$, proving Theorem 6. \square

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