Weierstrass representation of some simply-periodic minimal surfaces

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In [12] we proved the existence of simply-periodic minimal surfaces in $\mathbb{R}^3$ which could be described as desingularization of a set of vertical planes. These surfaces were constructed by gluing Scherk surfaces using the technique of Kapouleas [9], which amounts to solve a nonlinear partial differential equation on a manifold.

In this paper we obtain the Weierstrass representation of these surfaces. This gives an independent (and mostly algebraic) proof of the existence of these surfaces. It also proves that they form a smooth family, whereas with the former method, it was not known that the family is continuous.

Let $D = (D_1, \cdots D_n)$ be a finite set of distinct lines in the plane, $n \geq 2$. We assume that at least two lines are non parallel, and the intersection of any three lines is always empty (no triple intersection).

In this paper we construct a family of minimal surfaces $M_{D,\tau}$ where $\tau \in]0, \varepsilon[$, ε small enough, such that:

i) $M_{D,\tau}$ *converges to* $D \times \mathbb{R}$ *on compact subsets of* $\mathbb{R}^3$ *when* $\tau \to 0$.

We say that $M_{D,\tau}$ desingularize the set $D \times \mathbb{R}$.

ii) $M_{D,\tau}$ *is a complete minimal surface, periodic with period* $(0, 0, \tau^2)$, *with finite total curvature in the quotient.* $M_{D,\tau}$ *has* $2n$ *Scherk-type ends (i.e. asymptotic to vertical half-planes).*

The asymptotic half-planes of the ends of $M_{D,\tau}$ are not parallel to the lines of D as picture 1 below suggests. They are asymptotically parallel to the lines of D when $\tau \to 0$. For this reason, the ends may intersect. However:

iii) If the lines of D *are pairwise non parallel,* $M_{D,\tau}$ *is embedded.*

iv) Let D_i and D_j be two non parallel lines of D and $p = D_i \cap D_j$. There exists horizontal vectors $p(\tau)$ such that $p(\tau) \to p$ and $\tau^{-2}(M_{D,\tau} - p(\tau))$ converges on compact subsets of $\mathbb{R}^3$ to a simply-periodic Scherk surface, with period $(0,0,1)$, whose ends are parallel to the lines D_i and D_j.

This means that in a neighborhood of the intersection of two vertical planes of $D \times \mathbb{R}$, $M_{D,\tau}$ looks like a Scherk surface scaled by τ^2.

v) The map $(D,\tau) \mapsto M_{D,\tau}$ is smooth in the sense that the Weierstrass representation of $M_{D,\tau}$ depends smoothly on (D,τ).

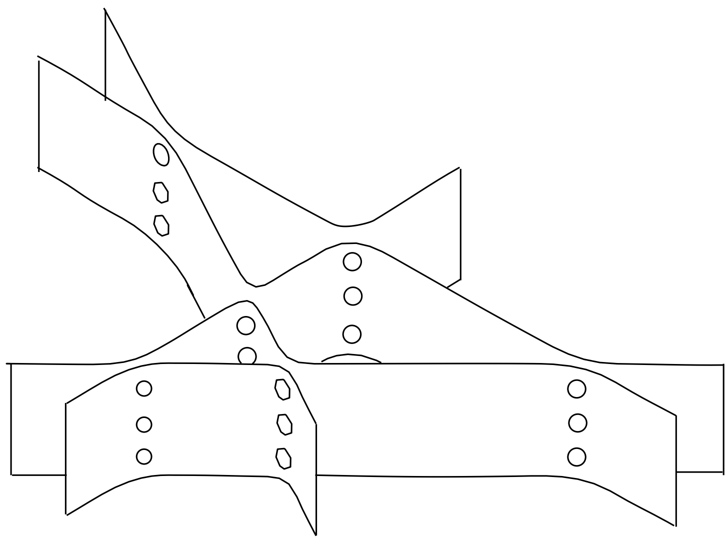

Fig. 1: A rough sketch of the surface we get in the case of four lines

We recall the principle of the Weierstrass representation of simply-periodic minimal surfaces. The reference is [11]. Let M be a simply-periodic complete minimal surface with period $(0,0,1)$. We see M as a minimal surface in the flat 3-manifold $\mathbb{R}^3/(0,0,1)$. It is well known that if M has finite total curvature, then M is conformally a compact Riemann surface Σ minus a finite number of point called the punctures, or ends. Moreover, there exists a meromorphic map g and a meromorphic 1-form η on Σ, such that if we define

$$\phi = (\phi_1, \phi_2, \phi_3) = \left(\frac{1}{2}(g^{-1} - g)\eta, \frac{i}{2}(g^{-1} + g)\eta, \eta\right) \tag{1}$$

$$X(p) = \left(\mathrm{Re} \int_{p_0}^{p} \phi_1, \mathrm{Re} \int_{p_0}^{p} \phi_2, \mathrm{Re} \int_{p_0}^{p} \phi_3 \right) \tag{2}$$

then $X = (X_1, X_2, X_3)$ is a minimal immersion from Σ minus the ends to $\mathbb{R}^3/(0,0,1)$ whose image is M. The triple (Σ, g, η) is called the Weierstrass representation of M. The function g is the Gauss map.

Conversely, given a triple (Σ, g, η), there are some well known conditions so that the formula 2 defines a minimal immersion. The standard way to construct minimal surfaces with the Weierstrass representation is to define (Σ, g, η) depending on some parameters, and then prove that one can adjust the parameters so that the conditions are satisfied (which is of course the hard work).

In this paper, we prove that the equations have solutions using the implicit function theorem. We consider Riemann surfaces Σ in a neighborhood of *degenerated* (i.e. singular) Riemann surfaces with *ordinary double points* (also called *nodes*). We apply the implicit function theorem at a point (in the space of the parameters) where the Riemann surface Σ degenerates into a set of Riemann spheres connected by ordinary double points, and on each Riemann sphere, (g, η) is the Weierstrass representation of a Scherk surface.

The overall idea of the paper, and in particular the idea to parametrise the couples (Σ, g) by the branching values of g are taken from [10]. In section 6 of this paper, they prove that the Riemann Minimal Examples are unique in a neighborhood of the boundary of the moduli space using the implicit function theorem. Our paper started as an attempt to construct minimal surfaces using their idea. However, the arguments in [10] do not carry over to our case because they are rather specific to the genus one case.

The method we use in section 2 to show that the Weierstrass data converges when the Riemann surface degenerates is inspired from [2]. In this paper the author studies the "period matrices" of Riemann surfaces in a neighborhood of degenerated Riemann surfaces.

I would like to thank Florence Gaja for helping me with the algebraic geometric aspects of the paper in section 4, and Marc Soret for reading the first draft of this paper and for many useful suggestions.

1 The Weierstrass data

1.1 The Riemann surface and the Gauss map

Consider a finite set of $n \geq 2$ lines in the plane as in the introduction. We see the union of these lines as a planar graph. Note that each vertex

has valence 4. We use this graph mostly as a combinatorial object to give names to various quantities. Let n_v be the number of vertices, n_e be the number of bounded edges, n_f the number of bounded faces, n_∞ the number of unbounded edges (i.e. half-lines). We label:
- the vertices V_i, $i = 1, \cdots n_v$,
- the bounded edges E_i, $i = 1, \cdots n_e$,
- the unbounded edges E_i, $i = n_e + 1, \cdots n_e + n_\infty$,
- the bounded faces F_i, $i = 1, \cdots n_f$.

We orient our graph as follows. Label each face (bounded or not) with a $+$ or $-$ sign, in such a way that any two faces sharing an edge have opposite signs. (One way to do this is to note that each line divides the plane in two half-planes. Put a $+$ sign on one and a $-$ sign on the other. Label each face with the product of the signs of the half-planes it lies on.) We orient the boundary of each face with the positive orientation if the face is labelled $+$, and the negative orientation in the other case. This gives each edge an orientation. We identify $\mathbb{R}^2$ with $\mathbb{C}$. For each $i = 1, \cdots n_e + n_\infty$, let $e^{i\theta_i}$ be the normal to the oriented edge E_i (i.e. pointing to the $+$ face). We write $\theta = (\theta_1, \cdots \theta_{n_e+n_\infty})$.

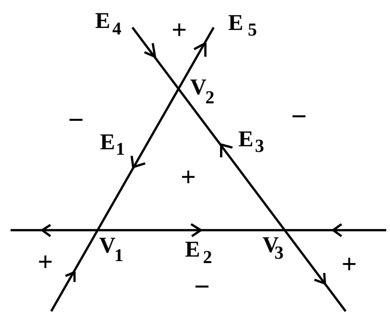

Fig. 2: Orientation of the edges.

We consider a small number $\varepsilon \in (0, 1)$. Given $\mathbf{x} = (x_1, \cdots x_{n_e+n_\infty}) \in \mathbb{C}^{n_e+n_\infty}$ and $\mathbf{y} = (y_1, \cdots y_{n_e}) \in (\mathbb{C} \setminus \{0\})^{n_e}$, such that $||\mathbf{x} - \theta|| < \varepsilon$ and $||\mathbf{y}|| < \varepsilon$, we construct a Riemann surface $\Sigma_{\mathbf{x},\mathbf{y}}$ as follows.

Consider n_v copies of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, labelled $\overline{\mathbb{C}}_1, \cdots \overline{\mathbb{C}}_{n_v}$. For each bounded edge E_i, with endpoints V_{i_1} and V_{i_2}: Cut $\overline{\mathbb{C}}_{i_1}$ along the arc joining $e^{ix_i - \sqrt{y_i}}$ and $e^{ix_i + \sqrt{y_i}}$. Also cut $\overline{\mathbb{C}}_{i_2}$ along the same arc. Glue $\overline{\mathbb{C}}_{i_1}$ and $\overline{\mathbb{C}}_{i_2}$ along this arc in the usual way (i.e. in the same way one constructs the Riemann surface $w^2 = (z - a_1)(z - a_2)$ by gluing two copies of $\overline{\mathbb{C}}$ along the cut $[a_1, a_2]$.) We assume that ε is small enough so that the cuts are contained in disjoint disks. This defines a compact Riemann surface that we call $\Sigma_{\mathbf{x},\mathbf{y}}$.

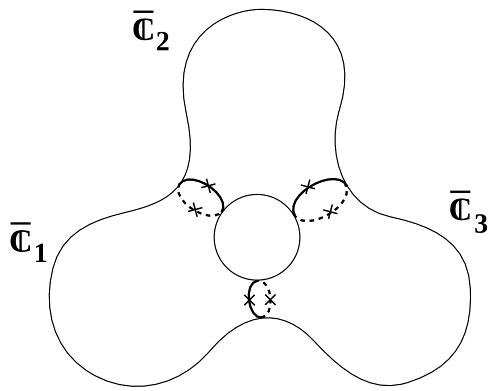

Fig. 3: The compact Riemann surface $\Sigma_{\mathbf{x},\mathbf{y}}$ in the case of three lines. The three curves are the lifts of the cuts to $\Sigma_{\mathbf{x},\mathbf{y}}$. The two branch points are marked on each curve.

We define a meromorphic function $z : \Sigma_{\mathbf{x},\mathbf{y}} \to \overline{\mathbb{C}}$ by $z(\zeta) = \zeta$ for $\zeta \in \overline{\mathbb{C}}_i$. It is clearly well defined on $\Sigma_{\mathbf{x},\mathbf{y}}$. z will be the Gauss map.

We use the notation 0_i and ∞_i for the points 0 and ∞ on $\overline{\mathbb{C}}_i$. z has one zero at each 0_i and one pole at each ∞_i. Therefore the degree of z is n_v.

The function z has two branch points per edge, with branching values $e^{ix_i \pm \sqrt{y_i}}$. The total branching order of z is $2n_e$. By the Riemann Hurwitz formula, the genus of $\Sigma_{\mathbf{x},\mathbf{y}}$ is $1 - n_v + n_e$. Since the Euler characteristic of the plane is 1, the genus of $\Sigma_{\mathbf{x},\mathbf{y}}$ is n_f. (This is also clear from the topological point of view.)

Remark 1 $\Sigma_{\mathbf{x},\mathbf{y}}$ does not depend on the determination of the square root $\sqrt{y_i}$, since replacing $\sqrt{y_i}$ by $-\sqrt{y_i}$ does not change the cut. Actually the square root is there precisely so that different values of $\mathbf{y}$ give different branching values of z.

Remark 2 For the surface we want to construct, all the parameters x_i and y_i are real. The reason we define $\Sigma_{\mathbf{x},\mathbf{y}}$ for complex parameters x_i and y_i is that in section 2 we need to have a family of Riemann surfaces depending on *complex* parameters in order to use results from algebraic geometry.

1.2 The meromorphic 1-form

We have not used the parameters $x_{n_e+1}, \cdots x_{n_e+n_\infty}$ yet. For each unbounded edge E_i, $i = n_e + 1, \cdots n_e + n_\infty$, with endpoint V_j, let q_i be the point e^{ix_i} in $\overline{\mathbb{C}}_j$. The q_i will be the ends of our surface. $\eta_{\mathbf{x},\mathbf{y}}$ will be a meromorphic 1-form with a simple pole at each q_i.

There are a lot of such meromorphic 1-forms. By standard Riemann surface theory, we can prescribe:
i) The residue of $\eta_{\mathbf{x},\mathbf{y}}$ at each end q_i, with the only condition that the sum of the residues is zero, which is the condition for the existence of a meromorphic 1-form with prescribed principal parts ([3], Theorem 18.11).
ii) The integrals of $\eta_{\mathbf{x},\mathbf{y}}$ on the g curves $A_1, \cdots A_g$ of a "canonical basis" of $H_1(\Sigma_{\mathbf{x},\mathbf{y}}, \mathbb{Z})$, where g is the genus of $\Sigma_{\mathbf{x},\mathbf{y}}$.

Recall that a canonical basis of the homology is a set of $2g$ closed curves $A_1, \cdots A_g, B_1, \cdots B_g$ such that the intersection numbers satisfy

$$A_i.A_j = 0 \ , \ B_i.B_j = 0 \ , \ A_i.B_j = \delta_{ij}$$

By standard Riemann surface theory, the map $\omega \mapsto (\int_{A_1} \omega, \cdots \int_{A_g} \omega)$ is then an isomorphism from the space of holomorphic 1-forms to $\mathbb{C}^g$ (see [5] page 231). Prescribing the residues clearly defines $\eta_{\mathbf{x},\mathbf{y}}$ up to a holomorphic 1-form. Hence prescribing the A-periods defines $\eta_{\mathbf{x},\mathbf{y}}$ uniquely.

For any bounded edge E_i, $i = 1, \cdots n_e$, with endpoints V_{i_1} and V_{i_2}, oriented from V_{i_1} to V_{i_2}, let γ_i be a small circle in $\overline{\mathbb{C}}_{i_1}$, enclosing the cut corresponding to the edge, oriented positively. Note that this circle is homotopic in $\Sigma_{\mathbf{x},\mathbf{y}}$ to a small circle in $\overline{\mathbb{C}}_{i_2}$, enclosing the cut, oriented negatively.

For each unbounded edge E_i, $i = n_e + 1, \cdots n_e + n_\infty$, with endpoint V_j, let γ_i be a small circle around the pole q_i, oriented positively if the edge is oriented from V_j to infinity, and negatively otherwise.

In both cases we call γ_i the γ-curve associated to the edge E_i. We would like to say that $\eta_{\mathbf{x},\mathbf{y}}$ is the unique meromorphic 1-form with simples poles at q_i and whose integral on any γ-curve is 1. It is not clear a priori that such a 1-form exists. So we choose a canonical basis, we define $\eta_{\mathbf{x},\mathbf{y}}$ by prescribing its residues and A-periods, and then we prove that the integral of $\eta_{\mathbf{x},\mathbf{y}}$ on any γ-curve is 1.

For each bounded edge E_i, from V_{i_1} to V_{i_2}, let Γ_i be a curve from the point $z = 2$ in $\overline{\mathbb{C}}_{i_1}$ to the point $z = 2$ in $\overline{\mathbb{C}}_{i_2}$, such that Γ_i does not intersect any γ-curve other than γ_i. We call Γ_i the Γ-curve associated to the edge E_i. The reason to choose $z = 2$ here is that we need a point far from the cuts, and neither a pole nor a zero of z.

For each face F_i, let B_i be the product of the Γ-curves associated to the edges on the boundary of the face F_i. We think of B_i as a curve which goes around the face F_i, even if this does not really make sense.

Remark 3 There is no canonical way to choose Γ_i, hence no canonical way to define B_i. However all choices of Γ_i are homotopic modulo γ_i. We will return to this problem in section 2.5.

We now define the curves A_i. Without loss of generality, we may suppose that all vertices have distinct abscissa. We order the bounded faces by the abscissa of their leftmost point. For each face F_i, let V_j be the leftmost point of F_i. Let E_k be one of the two edges on ∂F_i, with endpoint V_j. Clearly the face on the other side of E_k is either unbounded, or a face F_l with $l < i$. Let A_i be the curve γ_k.

By construction, the intersection numbers satisfy

$$A_i.A_j = 0 \ , \ B_i.B_j = 0 \ , \ A_i.B_i = 1 \text{ and } (A_i.B_j = 0 \text{ if } i < j)$$

Hence $\{A_1, \cdots A_g, B_1, \cdots B_g\}$ is not a "canonical basis" but if we define $m_{ij} = A_j.B_i$, the matrix m_{ij} is invertible in $SL(g, \mathbb{Z})$. Let $B'_i = \sum m^{ij} B_j$, where m^{ij} is the inverse matrix of m_{ij}. We have $A_i.B'_j = \delta_{ij}$ and $B'_i.B'_j = 0$ so $\{A_1, \cdots A_g, B'_1, \cdots B'_g\}$ is a canonical basis.

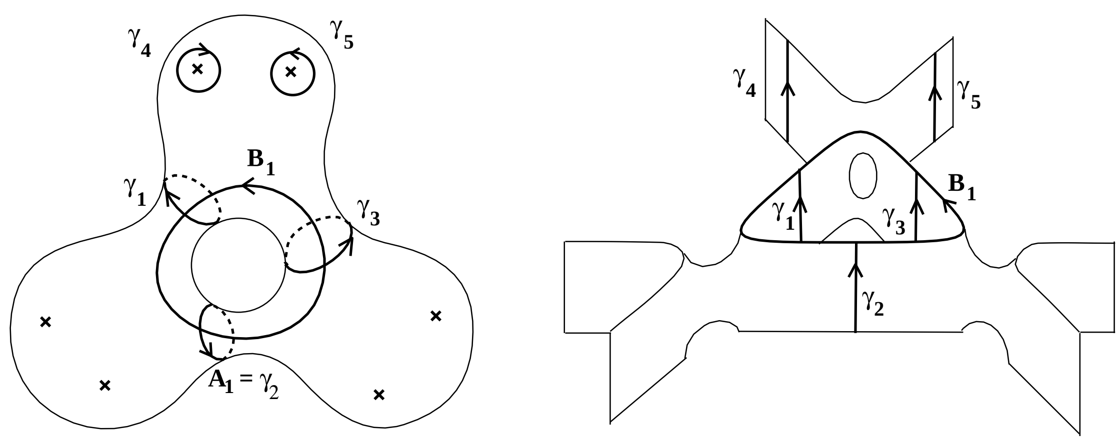

Fig. 4: Left: the curves γ_i, A_1 and B_1 on $\Sigma_{\mathbf{x},\mathbf{y}}$. The marked points are the punctures. Right: this is what we expect the minimal surface to look like. The surface is oriented by its normal which points toward the + faces.

Definition 1 *$\eta_{\mathbf{x},\mathbf{y}}$ is the unique meromorphic 1-form on $\Sigma_{\mathbf{x},\mathbf{y}}$ such that:*
- For each unbounded edge E_i, $i = n_e + 1, \cdots n_e + n_\infty$, with endpoint V_j: $\eta_{\mathbf{x},\mathbf{y}}$ has one simple pole at the point $q_i = e^{ix_i}$ of $\overline{\mathbb{C}}_j$, with residue $\dfrac{1}{2\pi i}$ if the edge E_i is oriented from V_j to infinity, and $\dfrac{-1}{2\pi i}$ otherwise. By definition of γ_i, this is equivalent to $\displaystyle\int_{\gamma_i} \eta_{\mathbf{x},\mathbf{y}} = 1$.
- For each closed curve A_i of the canonical basis, $\displaystyle\int_{A_i} \eta_{\mathbf{x},\mathbf{y}} = 1$.

Proposition 1 *For any $i = 1, \cdots n_e + n_\infty$, we have $\int_{\gamma_i} \eta_{\mathbf{x},\mathbf{y}} = 1$.*

Proof: we already know the period is 1 on the following curves: the A-curves and the curves γ_i around the ends, i.e. for $i = n_e + 1, \cdots n_e + n_\infty$.

For any vertex V_i, consider the domain obtained from $\overline{\mathbb{C}}_i$ by removing four small disks containing the four cuts (or ends in the case of unbounded edges). The boundary of this domain is homologous to $\displaystyle\sum_{V_i \in \partial E_j} \pm\gamma_j$, where $V_i \in \partial E_j$ means that the sum is taken on the four indices j such that E_j is an edge with endpoint V_i. By our choice of the orientations, there are two + signs and two − signs. By Cauchy theorem, $\displaystyle\sum_{V_i \in \partial E_j} \pm \int_{\gamma_j} \eta_{\mathbf{x},\mathbf{y}} = 0$. Hence if we already know that three of the periods are 1, the last one is also 1.

Using this, it is easy to see on a given particular example of graph that the proposition is true. Here is an argument in the general case (may be skipped at first reading). We mark the edges for which the γ-periods are known to be 1. At the beginning, the marked edges are the unbounded edges and the edges corresponding to the A-curves.

Claim 1 *If there is an unmarked edge, then there is a vertex with three marked edges and one unmarked edge.*

Thus we can mark the fourth edge, and by repeated use of the claim, we can mark all edges.

Proof of the claim: Assume there is no vertex with three marked edges. Then there is a vertex where at least two edges are unmarked. Follow one of these edges. At the end there is another vertex which also has at least two unmarked edges. Going on like this we eventually hit a vertex that we have already met. Thus there is a cycle of unmarked edges. This cycle bounds a compact region. Consider the leftmost vertex of this region. Then the two edges going to the right from this vertex are unmarked. By construction of the canonical basis, one of these edges is associated to a A-curve. This is a contradiction, which proves the claim.

1.3 The equations

So far, what we have done is parametrise all potential Weierstrass data $(\Sigma_{\mathbf{x},\mathbf{y}}, z, \eta_{\mathbf{x},\mathbf{y}})$ by the branching values of the Gauss map: $e^{ix_i \pm \sqrt{y_i}}$, $i = 1, \cdots n_e$, and the value of the Gauss map at the ends: e^{ix_i}, $i = n_e + 1, \cdots n_e + n_\infty$.

We recall the conditions so that (Σ, g, η) is the Weierstrass data for a simply-periodic minimal surface with period $(0,0,1)$ and Scherk-type ends:

- The zeroes of η are the zeroes and poles of g, with the same multiplicity.
- η has only simple poles with residue $\dfrac{\pm 1}{2\pi i}$ and $|g| = 1$ at the poles of η. This guarantees that the ends are asymptotic to vertical half-planes.
- $\text{Re}\int_c \phi_1 = 0$, $\text{Re}\int_c \phi_2 = 0$ and $\text{Re}\int_c \phi_3 = 0 \bmod 1$ for any closed curve c on Σ. Here ϕ_1, ϕ_2 and ϕ_3 are defined by equation 1. This guarantees that $X(p) = \text{Re}\int_{p_0}^p (\phi_1, \phi_2, \phi_3)$ is well defined, the so-called period problem.

With $(\Sigma, g, \eta) = (\Sigma_{\mathbf{x},\mathbf{y}}, z, \eta_{\mathbf{x},\mathbf{y}})$, these conditions become:

$$\eta_{\mathbf{x},\mathbf{y}}(0_i) = \eta_{\mathbf{x},\mathbf{y}}(\infty_i) = 0 \quad (i = 1, \cdots n_v) \tag{3}$$

$$x_i \in \mathbb{R} \quad (i = n_e + 1, \cdots n_e + n_\infty) \tag{4}$$

$$\text{Re}\int_{A_i} \phi_1 = 0, \quad \text{Re}\int_{A_i} \phi_2 = 0 \tag{5}$$

$$\text{Re}\int_{B_i} \phi_1 = 0, \quad \text{Re}\int_{B_i} \phi_2 = 0, \quad \text{Re}\int_{B_i} \phi_3 = 0 \bmod 1 \tag{6}$$

Indeed, the number of zeroes of $\eta_{\mathbf{x},\mathbf{y}}$ is the number of poles plus $2g - 2$, hence it is $n_\infty + 2n_f - 2$. Now each vertex has valence 4 so $4n_v = n_\infty + 2n_e$. Hence the number of zeroes is $4n_v - 2n_e + 2n_f - 2 = 2n_v$. So equation 3 says that the zeroes of $\eta_{\mathbf{x},\mathbf{y}}$ are the zeroes and poles of the Gauss map z. The other equations are immediate consequences of the definition of $\eta_{\mathbf{x},\mathbf{y}}$.

1.4 Outline of the paper

The goal of the paper is to prove that the above equations have solutions. These equations may be written as a system $F(\mathbf{x},\mathbf{y})=0$, where F is defined on $||\mathbf{x}-\theta||<\varepsilon$, $||\mathbf{y}||<\varepsilon$, $y_i\neq 0$. We prove that F extends holomorphically to the whole polydisk $||\mathbf{x}-\theta||<\varepsilon$, $||\mathbf{y}||<\varepsilon$, and then prove that $F(\theta,0)=0$ and F is a submersion at $(\theta,0)$. The implicit function theorem then says that $F^{-1}(0)$ is a nonempty submanifold of explicit dimension.

2 Convergence of $\eta_{\mathbf{x},\mathbf{y}}$ when $y_i\to 0$

When one or more of the components y_i of $\mathbf{y}$ is zero, the cut associated to the edge E_i degenerates into the point e^{ix_i}. There are two ways to define $\Sigma_{\mathbf{x},\mathbf{y}}$.

1) We identify the points e^{ix_i} in $\overline{\mathbb{C}}_{i_1}$ and $\overline{\mathbb{C}}_{i_2}$ and obtain an ordinary double point. $\Sigma_{\mathbf{x},\mathbf{y}}$ is a degenerate Riemann surface with ordinary double points, also called a Riemann surface with nodes (see [8] page 245 for the definition of a Riemann surface with nodes).

2) We do not identify these two points. $\Sigma_{\mathbf{x},\mathbf{y}}$ is a (possibly disconnected) compact Riemann surface.

Both points of view are useful. The first one is natural when one considers the family of all $\Sigma_{\mathbf{x},\mathbf{y}}$. The second one is convenient when talking about meromorphic 1-forms on $\Sigma_{\mathbf{x},\mathbf{y}}$. We use the same notation for both definitions. We will say whether we consider $\Sigma_{\mathbf{x},\mathbf{y}}$ as a Riemann surface with or without double points.

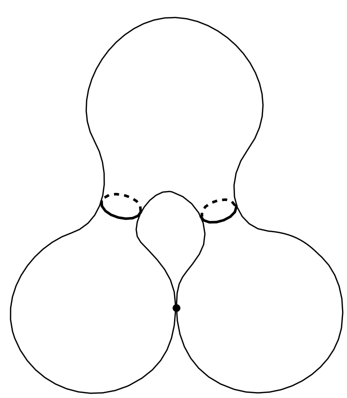

Fig. 5: $\Sigma_{\mathbf{x},\mathbf{y}}$ seen as a Riemann surface with a double point when $y_2 = 0$.

Proposition 2 *When* $(\mathbf{x}', \mathbf{y}') \to (\mathbf{x}, \mathbf{y})$ *and one or more of the components* y_i *of* $\mathbf{y}$ *is zero,* $\eta_{\mathbf{x}',\mathbf{y}'}$ *converges (away from the branch points and double points) to the unique meromorphic 1-form* $\eta_{\mathbf{x},\mathbf{y}}$ *on* $\Sigma_{\mathbf{x},\mathbf{y}}$*, seen as a Riemann surface without double points, such that:*

- $\eta_{\mathbf{x},\mathbf{y}}$ *has simple poles at the points* $q_j = e^{ix_j}$*,* $j = n_e + 1, \cdots n_e + n_\infty$*, with residue* $\dfrac{\pm 1}{2\pi i}$ *as in definition 1.*

- For any edge E_i *such that* $y_i = 0$*, oriented from* V_{i_1} *to* V_{i_2}*,* $\eta_{\mathbf{x},\mathbf{y}}$ *has a simple pole at the point* e^{ix_i} *of* $\overline{\mathbb{C}}_{i_1}$*, with residue* $\dfrac{1}{2\pi i}$*, and a simple pole at the point* e^{ix_i} *of* $\overline{\mathbb{C}}_{i_2}$*, with residue* $\dfrac{-1}{2\pi i}$*.*

- For any curve γ_i *on* $\Sigma_{\mathbf{x},\mathbf{y}}$*,* $\int_{\gamma_i} \eta_{\mathbf{x},\mathbf{y}} = 1$*.*

Moreover, the map $(\mathbf{x}, \mathbf{y}) \mapsto \eta_{\mathbf{x},\mathbf{y}}$ *is meromorphic on the whole polydisk* $||\mathbf{x} - \theta|| < \varepsilon$*,* $||\mathbf{y}|| < \varepsilon$*.*

By convergence away from the branch points and double points, we mean the following. Let $\rho > 0$ be a small number. If $(\mathbf{x}', \mathbf{y}')$ is close enough to $(\mathbf{x}, \mathbf{y})$, the points $e^{ix'_i \pm \sqrt{y'_i}}$ are in the disks of radius ρ and center $e^{ix_i \pm \sqrt{y_i}}$. $\Sigma_{\mathbf{x}',\mathbf{y}'}$ minus all these disks is the same thing as $\Sigma_{\mathbf{x},\mathbf{y}}$ minus the same disks, so does not depend on $(\mathbf{x}', \mathbf{y}')$. What we mean in the proposition is that for all $\rho > 0$, $\eta_{\mathbf{x}',\mathbf{y}'}$ converges to $\eta_{\mathbf{x},\mathbf{y}}$ on the above domain. The last statement of the proposition should be understood in a similar way.

The fact that $\eta_{\mathbf{x},\mathbf{y}}$ depends holomorphically on $(\mathbf{x}, \mathbf{y})$ when all y_i are nonzero may be considered standard. The main point in the proof of this proposition is to prove that $\eta_{\mathbf{x}',\mathbf{y}'}$ converges to a meromorphic 1-form when $y'_i \to 0$. Note that from the fact that $\int_{\gamma_i} \eta_{\mathbf{x}',\mathbf{y}'} = 1$, it is clear that $\eta_{\mathbf{x},\mathbf{y}}$ must have poles at the two points e^{ix_i}, with the indicated residues.

To prove the proposition, we take a more algebraic point of view. We have a family of (possibly degenerated) Riemann surfaces $\Sigma_{\mathbf{x},\mathbf{y}}$ which we see as abstract complex curves. From an algebraic point of view, a family of curves is defined as a morphism $\pi : X \to Y$ of complex analytic varieties, where Y is the space of parameters and the fibers are the curves. Note that some of the fibers may be singular. A way to define a meromorphic form on a singular fiber is to say that it is the restriction to the fiber of a meromorphic form on X. If we have a meromorphic 1-form η on each curve, a convenient way to say that η depends holomorphically on the parameters is that η is the restriction to the curve of a meromorphic form on X.

We note that to prove the proposition, we may look at the variables x_i and y_i separately since a function which is holomorphic with respect to each variable separately is actually holomorphic as a function of several complexe variables. In the following sections, we look at the variable y_i. All the other variables x_j and y_k, $k \neq i$ have fixed value, with $y_k \neq 0$.

2.1 Preliminaries

To ease the notation we write $y = y_i$. Since all the other variables have fixed value we write $\Sigma_y = \Sigma_{\mathbf{x},\mathbf{y}}$ and $\eta_y = \eta_{\mathbf{x},\mathbf{y}}$. We introduce the following functions on Σ_y.

$$v = z - \frac{1}{2}\left(e^{ix_i+\sqrt{y_i}} + e^{ix_i-\sqrt{y_i}}\right) = z - e^{ix_i}\cosh\sqrt{y_i}$$

$$w = \sqrt{\left(z - e^{ix_i+\sqrt{y_i}}\right)\left(z - e^{ix_i-\sqrt{y_i}}\right)}$$

A straightforward computation shows that $w^2 = v^2 - t$ with

$$t = \left(\frac{e^{ix_i+\sqrt{y_i}} - e^{ix_i-\sqrt{y_i}}}{2}\right)^2 = e^{2ix_i}\left(\sinh\sqrt{y_i}\right)^2$$

Consider a small, fixed number $r > 0$. If ε is small enough, the cut $[e^{ix_i-\sqrt{y_i}}, e^{ix_i+\sqrt{y_i}}]$ is contained in the disk of radius r and center $e^{i\theta_i}$. Let Σ'_y be Σ_y minus this disk. More precisely, Σ'_y is Σ_y minus the connected component of the set of points such that $|z - e^{i\theta_i}| \leq r$, containing the two branch points under consideration. Σ'_y does not depend on y.

Let Σ''_y be the connected component of the set $|v| < 2r$ containing the branch points. Σ'_y and Σ''_y cover Σ_y. We see Σ'_y as the points that are far from the branch point, and Σ''_y as the points that are close to the branch points.

The function w is well defined on Σ''_y up to a global definition of its sign. We choose the sign as follows: If ε is small compared to r, then on the boundary of Σ''_y we have $w^2 \sim v^2$. We choose the sign of the square root such that $w \sim v$ on the component of the boundary included in $\overline{\mathbb{C}}_{i_1}$. Then w is well defined and holomorphic on Σ''_y, and $w \sim -v$ on the component of the boundary included in $\overline{\mathbb{C}}_{i_2}$.

We introduce the function $V = v + w$ and $W = v - w$ on Σ''_y. We have

$$VW = t$$

When $y = 0$, Σ''_0, seen as a Riemann surface with a double point, is the union of the two disks $D(e^{ix_i}, 2r)$ in $\overline{\mathbb{C}}_{i_1}$ and $\overline{\mathbb{C}}_{i_2}$ with the two points e^{ix_i} identified. In the first disk we have $w = v$, $V = 2v$ and $W = 0$. In the other one we have $w = -v$, $V = 0$ and $W = 2v$.

2.2 The complex 2-manifold X

We consider the disjoint union of all complex curves Σ_y

$$X = \bigcup_{|y|<\varepsilon} \Sigma_y$$

We make X into a complex analytic 2-manifold as follows. X is covered by the two sets $X' = \bigcup \Sigma'_y$ and $X'' = \bigcup \Sigma''_y$. Since Σ'_y does not depend on y, X' is in a natural way the product manifold $D(\varepsilon) \times \Sigma'_0$, where $D(\varepsilon)$ is the disk of radius ε in $\mathbb{C}$. Consider the map

$$\varphi : X'' \to \mathbb{C}^2 \qquad p \in \Sigma''_y \mapsto (V(p), W(p))$$

φ is one to one. Indeed, if $\varphi(p) = \varphi(p')$ then $VW = t$ implies $t(y) = t(y')$ hence $y = y'$ because the function $y \mapsto t$ is well defined and one to one in a neighborhood of 0. So p and p' are on the same curve. On the other hand $\varphi(p) = \varphi(p')$ implies $p = p'$ because (V, W) separate points on the curve $VW = t$.

$\varphi(X'')$ is an open subset of $\mathbb{C}^2$. Indeed this is the set of points (V, W) such that $|V + W| < 2r$ and $VW \in t(D(\varepsilon))$. Hence we may take φ as a chart on X''.

The manifold structures on X' and X'' are compatible. This comes from the fact that the map $(y, p) \mapsto (V(p), W(p))$ is holomorphic. (Recall that a holomorphic bijection is biholomorphic). This makes X into a complex analytic 2-manifold. We define the projection

$$\pi : X \to \mathbb{C} \qquad p \in \Sigma_y \mapsto y$$

π is holomorphic. This is clear on X'', and on X', π is the composition $p \mapsto (V, W) \mapsto VW = t \mapsto y$ which is holomorphic. The fiber $\pi^{-1}(y)$ is Σ_y.

The canonical injection $\Sigma_y \hookrightarrow X$ is holomorphic. This is clear on Σ'_y, and on Σ''_y, this comes from the fact that $p \mapsto (V(p), W(p))$ is holomorphic. This means that the complex structure on Σ_y, seen as a curve in X, is the same as its original complex structure.

2.3 Restriction of a meromorphic form

Given a meromorphic 2-form ω on X, we define a meromorphic 1-form on Σ_y, called the restriction of ω to Σ_y and written $\omega|_y$.

Consider a curve Σ_y and $p \in \Sigma_y$. I claim that unless $y = 0$ and p is the double point, there exists a function ζ in a neighborhood of p such that (ζ, t) are complex coordinates in a neighborhood of p in X. If $p \in X'$, this comes from the fact that the function $y \mapsto t$ is biholomorphic in a neighborhood of 0. If $p \in X''$, from $VW = t$, we see that (V, t) are coordinates in a neighborhood of p if $V(p) \neq 0$, and (W, t) are coordinates if $W(p) \neq 0$. So unless $V(p) = W(p) = 0$, we may take either $\zeta = V$ or $\zeta = W$.

Write $\omega = f(t, \zeta) dt \wedge d\zeta$ where f is a meromorphic function. Let $\omega|_y = f(t(y), \zeta) d\zeta$. It is straightforward to check that this definition does not depend on the chosen coordinate ζ, so this defines a meromorphic form on Σ_y if $y \neq 0$, and on Σ_0 minus the double point if $y = 0$.

Remark 4 In fact $\omega|_y$ is the Poincaré residue of the meromorphic 2-form $\dfrac{-\omega}{t - t(y)}$ which has a pole along Σ_y (see [5] page 147).

Now assume that ω is holomorphic in a neighborhood of the double point of Σ_0. Write $\omega = f(V, W) dV \wedge dW$ where f is holomorphic. From $VW = t$ we get

$$\omega = -f\left(V, \frac{t}{V}\right) dt \wedge \frac{dV}{V} = f\left(\frac{t}{W}, W\right) dt \wedge \frac{dW}{W}$$

Recall that Σ_0 minus the double point has two components, one where $V = 0$ and one where $W = 0$. On the component $W = 0$, we get $\omega|_0 = -f(V, 0) dV/V$ hence $\omega|_0$ (restricted to this component) has a simple pole at the double point, with residue $-f(0, 0)$. On the component $V = 0$, we get $\omega|_0 = f(0, W) dW/W$ hence $\omega|_0$ has a simple pole at the double point, with residue $f(0, 0)$. In other words, $\omega|_0$ has a simple pole on each side of the double point, with opposite residues. A convenient way to say this is that $\omega|_0$ is meromorphic on Σ_0, *seen as a Riemann surface without double points*, and has two simples poles with opposite residues at the two points corresponding to the double point.

We may now state the

Lemma 1 *For any $y \in D(\varepsilon)$, there exists a neighborhood U of y and a meromorphic 2-form ω on $\pi^{-1}(U) \subset X$ such that $\omega|_y = \eta_y$ for any y in U, $y \neq 0$. Moreover, ω has a simple pole on $D(\varepsilon) \times \{q_j\} \subset X'$, $j = n_e + 1, \cdots n_e + n_\infty$, and is holomorphic everywhere else*

The proof of this lemma uses the language of sheaves, so we prove it in section 4.

2.4 Proof of proposition 2

In this section we see Σ_0 as a Riemann surface without double points. Let $\eta_0 = \omega|_0$.

Since ω has a simple pole on $D(\varepsilon) \times \{q_j\}$, we may write locally $\omega = \frac{f(t,z)}{z - z(q_j)} dt \wedge dz$ with f holomorphic. Hence $\eta_0 = \frac{f(0,z)}{z - z(q_j)} dz$ has a simple pole at q_j.

As seen in the previous section, η_0 has two more simple poles at the two points corresponding to the double point.

Since $\int_{\gamma_j} \omega|_y$ depends continuously on y and is equal to 1 if $y \neq 0$, we see that the periods and residues of η_0 are as in proposition 2.

From the fact that ω is meromorphic we see that η_y depends holomorphically on y, and in particular converges to η_0 when $y \to 0$. Recalling that $y = y_i$, we have proven that $y_i \mapsto \eta_{\mathbf{x},\mathbf{y}}$ is holomorphic when all other variables x_j, y_k, $k \neq i$, have fixed arbitrary values, with $y_k \neq 0$. The same is true when some y_k is zero by the same argument. The only difference is that $\eta_{\mathbf{x},\mathbf{y}}$ will have two more simples poles per y_k which is zero. The fact that $x_i \mapsto \eta_{\mathbf{x},\mathbf{y}}$ is holomorphic is standard. Hence the map $(\mathbf{x},\mathbf{y}) \mapsto \eta_{\mathbf{x},\mathbf{y}}$ is holomorphic, since a map which is holomorphic with respect to each variable when the other variables have arbitrary fixed values, is holomorphic (see [7] Theorem 2.2.8 page 28).

2.5 A formula for $\int_{\Gamma_i} z^k \eta_{\mathbf{x},\mathbf{y}}$

Recall that for each bounded each edge E_i we defined a path Γ_i which goes from the point $2 \in \mathbb{C}_{i_1}$ to the point $2 \in \mathbb{C}_{i_2}$. In fact there is no way to choose Γ_i so that it depends continuously on y_i, there is a multi-valuation problem. The reader may try to convince himself that when y_i makes on turn around 0 and we follow Γ_i continuously, we end up with a path homotopic to $\Gamma_i + \gamma_i$.

Proposition 3 *The function* $f_i(\mathbf{x},\mathbf{y}) = \int_{\Gamma_i} z^k \eta_{\mathbf{x},\mathbf{y}} - \frac{\log y_i}{2\pi_i} \int_{\gamma_i} z^k \eta_{\mathbf{x},\mathbf{y}}$ *where* $k \in \mathbb{Z}$ *is a fixed integer, is well defined when* $y_i \neq 0$*, and extends holomorphically to* $y_i = 0$*.*

Proof: We continue with the notations of the previous sections. First note that the function $p \mapsto z(p)$ is meromorphic on X, as may be seen by writing

z as a function of (V, W). By lemma 1 we may write $z^k \eta_y = (z^k \omega)|_y$. Let $z^k \omega = f(V, W) dV \wedge dW$ where f is holomorphic. We may write $f(V, W) = \sum a_{nm} V^n W^m$. Let $2R > 0$ be the radius of convergence of this series. We get

$$z^k \eta_y = -f(V, \frac{t}{V}) \frac{dV}{V} = - \sum_{n,m \geq 0} a_{nm} V^{n-1-m} t^m dV$$

We now choose representatives for the homotopy classes of γ_i and Γ_i in Σ_y. Note that when y is small, the point in Σ_y such that $V = R$ satisfies $|W| = |t/R| \ll R$, hence $v \sim w \sim R/2$. So this point is in $\overline{\mathbb{C}}_{i_1}$ and satisfies $z \sim e^{ix_i} + R/2$, so does not depend very much on y.

In the same way, the point such that $V = t/R$ satisfies $v \sim -w \sim R/2$, so this point is in $\overline{\mathbb{C}}_{i_2}$ and also does not depend much on y.

Let γ_i be the path $V(s) = Re^{is}$, $s \in [0, 2\pi]$. We define Γ_i as the product of the following three paths:

- A curve from $2 \in \overline{\mathbb{C}}_{i_1}$ to the point $V = R$,
- A curve which goes from the point $V = R$ to the point $V = t/R$ and stays inside the annulus $\frac{|t|}{R} \leq |V| \leq R$. For example we may take the spiral $V(s) = ((1-s)R + s|t|/R) e^{is \arg(t)}$, $s \in [0, 1]$.
- A curve from the point $V = t/R$ to the point $2 \in \overline{\mathbb{C}}_{i_2}$.

The first and third paths may be chosen to depend continuously on y. On the other hand, the second path is not well defined because of the multi-valuation of $\arg(t)$. We compute

$$\int_{\gamma_i} z^k \eta_y = - \int_{s=0}^{2\pi} \sum a_{nm} (Re^{is})^{n-m-1} t^m i Re^{is} ds = -2\pi i \sum a_{nn} t^n$$

The integral of $z^k \eta_y$ on the first path defining Γ_i is a well defined holomorphic function of y which extends holomorphically to $y = 0$ because this path is contained in a domain where we have seen that η_y converges to η_0. Same thing for the third path. We compute the integral on the second path defining Γ_i.

$$\begin{aligned}
\int_{V=R}^{t/R} z^k \eta_y &= -\sum a_{nm} t^m \int_{V=R}^{t/R} V^{n-1-m} dV \\
&= -\sum_{n \neq m} \frac{a_{nm}(t^n - t^m)}{R^{n-m}(n-m)} - \sum_n a_{nn} t^n \log \frac{t}{R^2} \\
&= \text{holomorphic}(t) + \frac{1}{2\pi i} \left(\log \frac{t}{R^2} \right) \int_{\gamma_i} z^k \eta_y
\end{aligned}$$

Recalling that $y \mapsto t$ is holomorphic one to one in a neighborhood of 0, we see that we have proven that the function f_i of the proposition extends holomorphically to $y_i = 0$ when all other variables have fixed values.

When y_i is given a fixed, nonzero value, the function f_i depends holomorphically on all other variables because Γ_i is a fixed path included in a domain where $\eta_{\mathbf{x},\mathbf{y}}$ depends holomorphically on $(\mathbf{x},\mathbf{y})$. Hence $(\mathbf{x},\mathbf{y}) \mapsto f_i(\mathbf{x},\mathbf{y})$ is holomorphic on the domain $y_i \neq 0$. We now prove it is bounded. For any $\varepsilon' < \varepsilon$, f_i is holomorphic on the compact $|y_i| = \varepsilon'$, $|y_k| \leq \varepsilon'$, $|x_j - \theta_j| \leq \varepsilon'$. Hence it is bounded by a constant M. Now for fixed x_j and y_k, $k \neq i$, $y_i \mapsto f_i(\mathbf{x},\mathbf{y})$ is holomorphic on $D(\varepsilon')$ so its maximum is on the boundary, so it is less than M. This proves that f_i is bounded by M on the polydisk of radius ε'. By the Riemann extension theorem, f_i is holomorphic. This concludes the proof.

3 Using the implicit function theorem

3.1 Symmetry

We now restrict to the case where all variables x_i and y_i are real. When all variables are real, the surface will have a symmetry which makes the problem simpler.

Proposition 4 *Assume that all x_i and y_i are real, $y_i \neq 0$. Then there exists an antiholomorphic involution $\sigma : \Sigma_{\mathbf{x},\mathbf{y}} \to \Sigma_{\mathbf{x},\mathbf{y}}$ such that $z \circ \sigma = \dfrac{1}{\overline{z}}$ and $\sigma^* \eta_{\mathbf{x},\mathbf{y}} = -\overline{\eta_{\mathbf{x},\mathbf{y}}}$. The equations 3, 4, 5, 6 reduce to the following equations:*

$$\eta_{\mathbf{x},\mathbf{y}}(0_i) = 0 \qquad (i = 1, \cdots n_v)$$

$$\int_{B_i} \phi_1 = \int_{B_i} \phi_2 = 0 \text{ and } \int_{B_i} \phi_3 = 0 \bmod 1 \quad (i = 1, \cdots g)$$

Proof. For each bounded edge, consider a small disk in $\mathbb{C}$ containing the cut and symmetric with respect to the unit circle (i.e. invariant by the inversion $z \mapsto \dfrac{1}{\overline{z}}$). We define σ on each $\overline{\mathbb{C}}_j$ minus these disks as the inversion $z \mapsto \dfrac{1}{\overline{z}}$. This defines σ on $\Sigma_{\mathbf{x},\mathbf{y}}$ away from the branch points.

It remains to define σ in a neighborhood of the branch points. To do this we use the function w introduced in section 2.1. Recall that w^2 only depends on z. A straightforward computation shows that when both x_i and y_i are real,

$$w^2\left(\frac{1}{\overline{z}}\right) = \frac{e^{2ix_i}\overline{w}(z)^2}{\overline{z}^2}$$

We define σ in a neighborhood of the branch points by

$$z\circ\sigma = \frac{1}{\overline{z}} \qquad w\circ\sigma = -\frac{e^{ix_i}}{\overline{z}}\overline{w}$$

We have to prove that the two definitions of σ are compatible. Recall that away from the branch points we have $w \sim \alpha(z - e^{ix_i})$ where $\alpha = +1$ on $\overline{\mathbb{C}}_{i_1}$ and $\alpha = -1$ on $\overline{\mathbb{C}}_{i_2}$. Hence

$$w\circ\sigma \sim -\frac{e^{ix_i}}{\overline{z}}\alpha(\overline{z} - e^{-ix_i}) = \alpha(\frac{1}{\overline{z}} - e^{ix_i})$$

This means that away from the branch points, σ maps $\overline{\mathbb{C}}_{i_1}$ to $\overline{\mathbb{C}}_{i_1}$ and $\overline{\mathbb{C}}_{i_2}$ to $\overline{\mathbb{C}}_{i_2}$. Hence the definition of σ in a neighborhood of the branch points agrees with the definition on each $\overline{\mathbb{C}}_j$, so σ is well defined on $\Sigma_{\mathbf{x},\mathbf{y}}$.

The poles of $\eta_{\mathbf{x},\mathbf{y}}$ are invariant by σ so $\sigma^*\eta_{\mathbf{x},\mathbf{y}}$ has the same poles as $\eta_{\mathbf{x},\mathbf{y}}$. For any edge E_i (bounded or not), the curve $\sigma(\gamma_i)$ is homologous to $-\gamma_i$. Hence

$$\int_{\gamma_i}\eta_{\mathbf{x},\mathbf{y}} = \overline{\int_{\gamma_i}\eta_{\mathbf{x},\mathbf{y}}} = \overline{\int_{-\gamma_i}\sigma^*\eta_{\mathbf{x},\mathbf{y}}} = -\int_{\gamma_i}\overline{\sigma^*\eta_{\mathbf{x},\mathbf{y}}}$$

Hence the residues of $\eta_{\mathbf{x},\mathbf{y}} + \overline{\sigma^*\eta_{\mathbf{x},\mathbf{y}}}$ are all zero, so it is holomorphic, and all its A-periods vanish, so it is zero.

We compute

$$\int_{\gamma_i}(\frac{1}{z} - z)\eta_{\mathbf{x},\mathbf{y}} = \int_{-\gamma_i}(\overline{z} - \frac{1}{\overline{z}})(-\overline{\eta_{\mathbf{x},\mathbf{y}}}) = -\overline{\int_{\gamma_i}(\frac{1}{z} - z)\eta_{\mathbf{x},\mathbf{y}}}$$

Hence $\mathrm{Re}\int_{\gamma_i}\phi_1 = 0$. A similar computation shows that $\mathrm{Re}\int_{\gamma_i}\phi_2 = 0$ and $\mathrm{Im}\int_{\gamma_i}\phi_3 = 0$. This proves the proposition.

Remark 5 We already know that $\mathrm{Im}\int_{\gamma_i}\phi_3 = 0$ by proposition 1! In fact when the variables x_i and y_i are complex, one should ask that $\int_{A_i}\eta_{\mathbf{x},\mathbf{y}} = 1 + i\alpha_i$, where $(\alpha_1, \cdots \alpha_g)$ are g real parameters.

3.2 The equation $\eta_{\mathbf{x},\mathbf{y}}(0_i) = 0$

We consider the map $F : \mathbb{R}^{n_e+n_\infty} \times \mathbb{R}^{n_e} \to \mathbb{C}^{n_v}$ defined by $F_i(\mathbf{x},\mathbf{y}) = 2\pi \frac{\eta_{\mathbf{x},\mathbf{y}}}{dz}(0_i)$. By proposition 2, F is defined in a neighborhood of $(\theta, 0)$ and analytic.

Proposition 5 *$F(\theta,0) = 0$ and $\frac{\partial F}{\partial \mathbf{x}}(\theta,0)$ is onto. We use the notation $\frac{\partial F}{\partial \mathbf{x}}(\theta,0)$ for the partial differential of F with respect to the variable $\mathbf{x}$.*

Proof: here we see $\Sigma_{\mathbf{x},0}$ as a Riemann surface without nodes. $\Sigma_{\mathbf{x},0}$ is the disjoint union of n_v Riemann spheres $\overline{\mathbb{C}}_i$. For each vertex V_i and each edge E_j with endpoint V_i, $\eta_{\mathbf{x},\mathbf{y}}$ has a simple pole in $\overline{\mathbb{C}}_i$ at the point e^{ix_j} with residue $\frac{\varepsilon_j}{2\pi i}$, where $\varepsilon_j = 1$ if the edge is oriented away from V_i and $\varepsilon_j = -1$ if it is oriented toward V_i. Thus $\eta_{\mathbf{x},\mathbf{y}}$ has four poles on each $\mathbb{C}_i$ and

$$\eta_{\mathbf{x},0} = \frac{1}{2\pi i} \sum_{V_i \in \partial E_j} \frac{\varepsilon_j dz}{z - e^{ix_j}}$$

where $V_i \in \partial E_j$ means that the summation is taken on the four indices j such that E_j is an edge with endpoint V_i. Hence

$$F_i(\mathbf{x},0) = i \sum_{V_i \in \partial E_j} \varepsilon_j e^{-ix_j}$$

and $F(\theta,0) = 0$. Moreover

$$\frac{\partial F_i}{\partial \mathbf{x}}(\theta,0) = \sum_{V_i \in \partial E_j} \varepsilon_j e^{-i\theta_j} dx_j$$

As usual we may assume that all vertices have distinct abscissa and order them by their abscissa. For each vertex V_i, let $E_{k_{i,1}}$ and $E_{k_{i,2}}$ be the two edges with endpoint V_i that are on the right of V_i. Let $n = n_e$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. The matrix of $\frac{\partial F}{\partial \mathbf{x}}(\theta,0)$ restricted to the $2n$ variables $x_{k_{1,1}}, x_{k_{1,2}}, \cdots x_{k_{n,1}}, x_{k_{n,2}}$ is a rectangular block matrix with $n \times n$ blocks of size 2×2. All the blocks above the diagonal are zero because if $i < j$ then V_i is not an endpoint of $E_{k_{j,1}}$ nor $E_{k_{j,2}}$ so F_i does not depend on the corresponding x variables. The i^{th} block on the diagonal is the matrix of $(x_{k_{j,1}}, x_{k_{j,2}}) \mapsto \varepsilon_{k_{i,1}} x_{k_{i,1}} e^{-i\theta_{k_{i,1}}} + \varepsilon_{k_{i,2}} x_{k_{i,2}} e^{-i\theta_{k_{i,2}}}$. It is invertible because $e^{-i\theta_{k_{i,1}}}$ and $e^{-i\theta_{k_{i,2}}}$ are independent over $\mathbb{R}$. This proves that the matrix has real rank $2n$. Hence the proposition is proven.

3.3 The B-periods of ϕ_1 and ϕ_2

Let F_i be a bounded face. For each edge E_j we defined a path Γ_j in section 1.2. Then

$$B_i = \sum_{E_j \in \partial F_i} \Gamma_j$$

where the $+$ means the product of paths and $E_j \in \partial F_i$ means that the sum is taken on all indices j such that E_j is on the boundary of F_i. By proposition 3 we have

$$\int_{\Gamma_j} z^k \eta_{\mathbf{x},\mathbf{y}} = \left(\frac{\log y_j}{2\pi i} \int_{\gamma_j} z^k \eta_{\mathbf{x},\mathbf{y}} \right) + f(\mathbf{x},\mathbf{y})$$

where in this section, $f(\mathbf{x},\mathbf{y})$ means an analytic function of $(\mathbf{x},\mathbf{y})$ in a neighborhood of $(\theta, 0)$. Hence for $k = 1, 2, 3$

$$\mathrm{Re}\left(\int_{B_i} \phi_k \right) = \sum_{E_j \in \partial F_i} \mathrm{Re}\left(\frac{\log y_j}{2\pi i} \int_{\gamma_j} \phi_k \right) + f(\mathbf{x},\mathbf{y}) \qquad (7)$$

We have seen that because of the symmetry $\int_{\gamma_j} \phi_k$ is imaginary when $k = 1, 2$, so

$$\mathrm{Re}\left(\int_{B_i} \phi_k \right) = \sum_{E_j \in \partial F_i} \frac{\log |y_j|}{2\pi} \mathrm{Im}\left(\int_{\gamma_j} \phi_k \right) + f(\mathbf{x},\mathbf{y}) \qquad (k = 1, 2)$$

We see that each term in the above sum diverges when $\mathbf{y} = 0$. What we would like to do is normalise the period by dividing by $\log |y_j|$. There are two problems: the y_j are not all the same, and also $y \mapsto \dfrac{1}{\log |y|}$ is not differentiable at 0. We solve both problems as follows, but doing this, we leave the realm of analytic functions.

For each bounded edge E_j we fix an $\varepsilon_j = \pm 1$. (These ε_j have nothing to do with the ε_j of the previous section). For any $\mathbf{r} = (r_1, \cdots r_{n_e}) \in (0, \infty)^{n_e}$ such that $\sum r_j = 1$, and for any $\tau \in \mathbb{R}$, let

$$y_j(\mathbf{r}, \tau) = \varepsilon_j \exp\left(\frac{-r_j}{\tau^2} \right) \qquad y_j(\mathbf{r}, 0) = 0$$

The map $(\mathbf{r}, \tau) \mapsto \mathbf{y}$ is smooth in a neighborhood of any $(\mathbf{r}, 0)$, and is one to one if $\tau > 0$ thanks to the normalisation $\sum r_j = 1$. From now on our variables are $(\mathbf{x}, \mathbf{r}, \tau)$ instead of $(\mathbf{x}, \mathbf{y})$.

Multiplying the above formula by τ^2 we get

$$\tau^2 \mathrm{Re}\left(\int_{B_i} \phi_k\right) = -\sum_{E_j \in \partial F_i} \frac{r_j}{2\pi} \mathrm{Im}\left(\int_{\gamma_j} \phi_k\right) + \tau^2 f(\mathbf{x},\mathbf{y}) \quad (k=1,2)$$

We define a complex-valued function

$$G_i(\mathbf{x},\mathbf{r},\tau) = 2\pi\tau^2\left(\mathrm{Re}\int_{B_i}\phi_1 + i\mathrm{Re}\int_{B_i}\phi_2\right) \text{ if } \tau \neq 0$$

By the above formula we see that G_i extends smoothly to $\tau = 0$. Also when $\tau = 0$, we have $\mathbf{y} = 0$ so we may compute the integral on γ_j as a residue at e^{ix_j}. A straightforward computation gives

$$\int_{\gamma_j} \phi_1 = -i \sin x_j \qquad \int_{\gamma_j} \phi_2 = i \cos x_j$$

Hence

$$G_i(\mathbf{x},\mathbf{r},0) = -i \sum_{E_j \in \partial F_i} r_j e^{ix_j}$$

Proposition 6 *Let* $G = (G_1, \cdots G_g)$. *Let* ℓ_j *be the length of the edge* E_j, *and* $\ell = (\ell_1, \cdots \ell_{n_e})$. *Note that by scaling the graph we may assume that* $\sum \ell_j = 1$. *Then* $G(\theta,\ell,0) = 0$ *and* $\frac{\partial G}{\partial \mathbf{r}}(\theta,\ell,0)$ *is onto.*

Proof: The first point is clear since $-ie^{i\theta_j}$ is the tangent vector to the oriented edge E_j. To prove the second point, note that $G(\theta,\mathbf{r},0)$ is linear in $\mathbf{r}$ so we must prove that $\mathbf{r} \mapsto G(\theta,\mathbf{r},0)$ is onto.

We order the bounded faces by the abscissa of their leftmost point. For each face F_i, let $E_{k_{i,1}}$ and $E_{k_{i,2}}$ be the two edges on the boundary of F_i whose endpoint is the leftmost vertex of F_i. We identify $\mathbb{C}^g$ with $\mathbb{R}^{2g}$. The matrix of $\mathbf{r} \mapsto G(\theta,\mathbf{r},0)$ restricted to the $2g$ variables $r_{k_{1,1}}, r_{k_{1,2}}, \cdots r_{k_{g,1}}, r_{k_{g,2}}$ is a rectangular block matrix with $g \times g$ blocks of size 2×2. All the blocks below the diagonal are zero because if $i > j$, $E_{k_{j,1}}$ and $E_{k_{j,2}}$ are not on the boundary of F_i so G_i does not depend on $r_{k_{j,1}}$ nor $r_{k_{j,2}}$. The i^{th} block on the diagonal is the matrix of $(r_{k_{j,1}}, r_{k_{j,2}}) \mapsto -i(r_{k_{i,1}} e^{i\theta_{k_{i,1}}} + r_{k_{i,2}} e^{i\theta_{k_{i,2}}})$ which is invertible as in the previous proposition. Hence $\mathbf{r} \mapsto G(\theta,\mathbf{r},0)$ is onto, from $\mathbb{R}^{n_e}$ to $\mathbb{C}^g$.

It remains to prove that it is still onto when restricted to $\sum r_i = 0$. Now for any $\mathbf{r}$, the vector $\mathbf{s} = \mathbf{r} - (\sum r_i)\ell$ satisfies $\sum s_i = 0$ and $G(\theta,\mathbf{s},0) = G(\theta,\mathbf{r},0)$. This proves the proposition.

Consider the map $H(\mathbf{x},\mathbf{r},\tau) = (F(\mathbf{x},\mathbf{y}(\mathbf{r},\tau)), G(\mathbf{x},\mathbf{r},\tau))$ where F is the map defined in the previous section. H is smooth in a neighborhood of $(\theta,\ell,0)$. It takes values in $\mathbb{C}^{n_v+n_f}$. The differential of H with respect to the variables $\mathbf{x}$ and $\mathbf{r}$ at $(\theta,\ell,0)$ has the matrix

$$\begin{pmatrix} \frac{\partial F}{\partial \mathbf{x}}(\theta,0) & 0 \\ \\ \frac{\partial G}{\partial \mathbf{x}}(\theta,\ell,0) & \frac{\partial G}{\partial \mathbf{r}}(\theta,\ell,0) \end{pmatrix}$$

The upper right block is zero because $\frac{\partial \mathbf{y}}{\partial \mathbf{r}}(\mathbf{r},0) = 0$. By proposition 5 and 6, the matrix has real rank $2n_v + 2n_f$. By the implicit function theorem we get

Corollary 1 *In a neighborhood of $(\theta,\ell,0)$, $H^{-1}(0)$ is a smooth submanifold of $\mathbb{R}^{n_e+n_\infty} \times \mathbb{R}^{n_e-1} \times \mathbb{R}$ of dimension $(n_e+n_\infty)+(n_e-1)+1-2n_v-2n_f = n_\infty - 2$. Here we identify $\mathbb{R}^{n_e-1}$ with the subspace $\sum r_j = 1$ of $\mathbb{R}^{n_e}$.*

3.4 The B-periods of ϕ_3

Recall that there is no canonical way to choose the closed curve B_i but all choices are homotopic modulo the curves γ_j for $E_j \in \partial F_i$. Since $\int_{\gamma_j} \phi_3 = 1$, the integral of ϕ_3 on B_i is well defined modulo 1.

When all y_j are positive, we may choose the path B_i such that $\sigma(B_i) = B_i$. Then $\sigma^*\phi_3 = -\overline{\phi_3}$ implies that $\operatorname{Re}\int_{B_i} \phi_3 = 0$.

When some of the y_j are negative, all one can say is that $\sigma(B_i)$ is homotopic to B_i modulo the curves γ_j for $E_j \in \partial F_i$. Then $\int_{B_i} -\overline{\phi_3} = \int_{B_i} \phi_3 \bmod 1$ hence

$$\operatorname{Re}\int_{B_i} \phi_3 = 0 \bmod \frac{1}{2}$$

On the other hand, from formula 7 of the previous section,

$$\operatorname{Re}\int_{B_i} \phi_3 = \sum_{E_j \in \partial F_i} \operatorname{Re}\left(\frac{\log y_j}{2\pi i}\right) + f(\mathbf{x},\mathbf{y})$$

Since $\frac{\log y_j}{2\pi i} = 0 \bmod \frac{1}{2}$, the function f is equal to zero modulo $\frac{1}{2}$. Since it is continuous, it is constant. Evaluating when all y_j are positive, we find that f is zero. Hence

Proposition 7 *The period* $\mathrm{Re}\int_{B_i}\eta_{\mathbf{x},\mathbf{y}}$ *is equal modulo* 1 *to half the number of edges* E_j *on the boundary of the face* F_i *such that* $y_j < 0$.

In section 3.5 we will see a geometric explanation of this proposition. From now on we fix the values of $\varepsilon_j = \pm 1$ so that for each face, the number of edges E_j on its boundary with $\varepsilon_j = -1$ is even.

3.5 The surfaces $M_{D,\tau}$

Propositions 6 and 7 give the existence of a smooth family of minimal surfaces depending on $n_\infty - 2$ parameters. We shall see that when τ is fixed, the remaining $n_\infty - 3$ parameters correspond to the deformations of the set of lines D.

Consider an integer $n \geq 3$. Let $\mathcal{D}$ be the set of all n-uples of lines $D = (D_1, \cdots D_n)$ such that:

i) The intersection of any three lines of D is empty.

ii) Any two lines of D are non parallel.

iii) D_1 and D_2 intersect at the origin.

iv) The sum of the lengths of the bounded edges of the graph $D_1 \cup \cdots D_n$ is one.

$\mathcal{D}$ is in a natural way a smooth (non connected) manifold of dimension $2n-3$.

Theorem 1 *There exists a neighborhood* U *of* $\mathcal{D}\times\{0\}$ *in* $\mathcal{D}\times\mathbb{R}$ *and a family of minimal surfaces* $M_{D,\tau}$ *where* $(D,\tau) \in U \setminus (\mathcal{D}\times\{0\})$ *which satisfies the assertions i) to v) of the introduction.*

Proof: It suffices to prove the theorem for each connected component of $\mathcal{D}$. We consider one component and still call it $\mathcal{D}$. Each element D of $\mathcal{D}$ defines a planar graph. From hypothesis i and ii, all these graphs are isomorphic so we may orient and label the edges $E_i(D)$ so that $E_i(D)$ depends continuously on D.

Let $e^{i\theta_i(D)}$ be the normal to the oriented edge $E_i(D)$. Since there is a multi-valuation problem we see $\theta_i(D)$ as a number in $\mathbb{R}/2\pi$. Let $\ell_i(D)$ be the length of the edge $E_i(D)$.

Let $E = (\mathbb{R}/2\pi)^{n_e+n_\infty}\times\mathbb{R}^{n_e-1}\times\mathbb{R}$. Recall that we identified $\mathbb{R}^{n_e-1}$ with the subspace $\sum r_i = 1$ of $\mathbb{R}^{n_e}$. We define $\psi : \mathcal{D} \to E$ by

$$\psi(D) = (\theta(D), \ell(D), 0)$$

It is easy to see that ψ is an injective immersion. The map H defined in section 3.3 is well defined in a neighborhood of $\psi(\mathcal{D})$ in E. By proposition 6, for any $D \in \mathcal{D}$, $H(\psi(D)) = 0$ and H is a submersion at $\psi(D)$. Hence $H^{-1}(0)$ is a smooth submanifold of E in a neighborhood of $\psi(\mathcal{D})$. Note that $H^{-1}(0) \cap \{\tau = 0\}$ is a codimension one submanifold of $H^{-1}(0)$. Since $\mathcal{D}$ and $H^{-1}(0) \cap \{\tau = 0\}$ have the same dimension, ψ is a diffeomorphism from $\mathcal{D}$ to an open subset of $H^{-1}(0) \cap \{\tau = 0\}$.

We define $h : H^{-1}(0) \to \mathbb{R}$ by $h(\mathbf{x}, \mathbf{r}, \tau) = \tau$. To prove theorem 1 we parametrise in a natural way $H^{-1}(0)$ by $h^{-1}(0) \times \mathbb{R}$. We define a smooth vector field on $H^{-1}(0)$ by $\chi = \dfrac{\nabla h}{||\nabla h||^2}$. We define $\varphi : h^{-1}(0) \times \mathbb{R} \to H^{-1}(0)$ by

$$\varphi(v, 0) = v \qquad \frac{\partial}{\partial s}\varphi(v, s) = \chi(\varphi(v, s))$$

Then φ is defined and smooth in a neighborhood of $h^{-1}(0) \times \{0\}$ and $\dfrac{d}{ds} h \circ \varphi(v, s) = \langle \nabla h, \chi \rangle = 1$. Hence φ maps $h^{-1}(0) \times \{\tau\}$ to $h^{-1}(\tau)$.

Given $D \in \mathcal{D}$ and $\tau > 0$ small enough, let $(\mathbf{x}, \mathbf{r}, \tau) = \varphi(\psi(D), \tau)$ and $\mathbf{y} = \mathbf{y}(\mathbf{r}, \tau)$. Then $(\Sigma_{\mathbf{x},\mathbf{y}}, z, \eta_{\mathbf{x},\mathbf{y}})$ is the Weierstrass data for a complete minimal immersion $X_{D,\tau}$ into $\mathbb{R}^3/(0, 0, 1)$. It is defined up to a translation. We choose the translation so that $X_{D,\tau}(0_1) = (0, 0, 0)$ where V_1 is the intersection of D_1 and D_2 and 0_1 is the corresponding zero of z in $\overline{\mathbb{C}}_1$. Let $M_{D,\tau}$ be the image of $X_{D,\tau}$ scaled by τ^2.

In the remaining of the section we briefly discuss the geometry of $M_{D,\tau}$. We start with embeddedness. We fix a $D \in \mathcal{D}$.

On each $\overline{\mathbb{C}}_i$ minus four small disks around the ends and cuts, the Weierstrass data converges when $\tau \to 0$ to the Weierstrass data of a Scherk surface whose ends have asymptotic normal $e^{i\theta_j(D)}$. Since the Scherk surfaces are embedded, the image of the above domain by $X_{D,\tau}$ is embedded for τ small enough.

The image of a neighborhood $|z - e^{i\theta_j}| < r$ of a branch point or an end is also embedded because it is a graph.

For any edge E_i with endpoints V_{i_1} and V_{i_2}, $\tau^2 \left(X_{D,\tau}(0_{i_2}) - X_{D,\tau}(0_{i_1})\right)$ converges when $\tau \to 0$ to $V_{i_2} - V_{i_1}$. Hence $\tau^2 X_{D,\tau}(0_j)$ converges to V_j when $\tau \to 0$.

The normal at the end q_j is $e^{ix_j(D,\tau)}$ and $x_j(D, \tau)$ converges to $\theta_j(D)$ when $\tau \to 0$. Hence the asymptotic half-plane of the end q_j of $M_{D,\tau}$ converges when $\tau \to 0$ to the vertical half-plane $E_j \times \mathbb{R}$. Since the lines of D are non parallel, the asymptotic half-planes of the ends will not intersect if

τ is small enough. This implies that $M_{D,\tau}$ is embedded.

The involution σ corresponds to a symmetry of $M_{D,\tau}$ with respect to a horizontal plane. $M_{D,\tau}$ has in fact two horizontal planes of symmetry at distance $\frac{\tau^2}{2}$ from each other.

If $y_i < 0$, then the two branch points corresponding to the edge E_i are fixed by σ. It is not hard to see that they are not on the same horizontal plane of symmetry. The intersection of $M_{D,\tau}$ with each plane of symmetry has a flex point at the branch point.

If $y_i > 0$, then the branch points are not on the planes of symmetry. The intersection of $M_{D,\tau}$ with each plane of symmetry is locally convex.

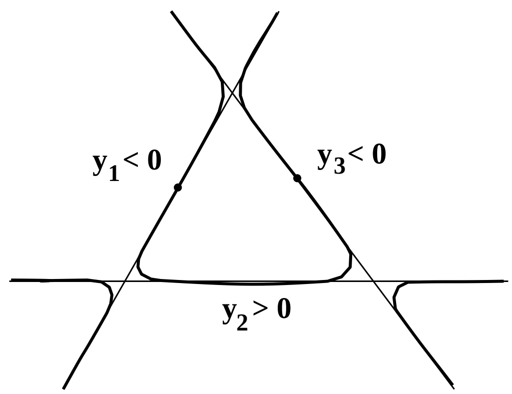

Fig. 6: The intersection of $M_{D,\tau}$ with one of its planes of symmetry. We see that the number of flex points per face has to be even, which gives a geometrical interpretation of proposition 7.

4 A short detour in coherent sheaves

In this section we prove lemma 1. We assume the reader is familiar with elementary sheaf theory, such as presented in [3], chapter 6.

The notations are those of section 2. Let $n = n_\infty$ be the number of poles of η_y when $y \neq 0$ and δ be the set of poles of η_y. Recall that δ does not depend on y. Let Δ be the set $D(\varepsilon) \times \delta$ on X. Let $\mathcal{F}$ be the sheaf of meromorphic 2-forms on X who have at most simple poles on Δ and are otherwise holomorphic.

We state the following theorem of Grauert.

Theorem 2 (Grauert) *Let $\pi : X \to Y$ be a proper morphism of complex analytic manifolds. Let X_y be the fiber $\pi^{-1}(\{y\})$. Let $\mathcal{F}$ be a coherent analytic sheaf on X which is flat with respect to Y. Suppose that* $\dim H^0(X_y, \mathcal{F}_y)$ *does not depend on $y \in Y$. Then the sheaf $\pi_*\mathcal{F}$ is locally free of rank* $\dim H^0(X_y, \mathcal{F}_y)$.

See [4], page 291 for the original statement (which is more general) or [1], Theorem 4.12, page 134. See also the classical book [6], Corollary 12.9, page 288 for the corresponding statement in the algebraic (instead of analytic) case.

We now prove that the hypotheses of the theorem are satisfied in our case. $\mathcal{F}$ coherent means that locally, i.e. for small open sets $U \subset X$, $\mathcal{F}(U)$ is a module on the ring $\mathcal{O}_X(U)$ which is finitely generated, and moreover the relations between the generators are finitely generated. $\mathcal{O}_X$ is the sheaf of holomorphic functions on X. In our case, $\mathcal{F}$ is locally free or rank one hence coherent.

The word "flat" has an algebraic meaning here. $\pi : X \to Y = \mathbb{C}$ is open because it is a non constant holomorphic function. Then $\mathcal{O}_X$ is flat with respect to Y by [1], Theorem 2.13 page 181. Since $\mathcal{F}$ is locally free of rank one, $\mathcal{F}$ is flat over Y.

$\mathcal{F}_y$ is the quotient sheaf $\mathcal{F}/\mathcal{I}_y$, where $\mathcal{I}_y$ is the subsheaf of $\mathcal{F}$ of 2-forms which vanish on X_y. We first show that $\mathcal{F}_y$ is isomorphic to a more familiar sheaf.

If $y \neq 0$, let Ω_y be the sheaf of meromorphic 1-forms on X_y who have at most a pole on δ and are otherwise holomorphic. If $y = 0$, let Ω_0 be the sheaf of meromorphic 1-forms on X_0 minus the double point, which have at most a pole on δ, and which, in a neighborhood of the double point, may be written $f(W)dW/W$ in the component $V = 0$ and $g(V)dV/V$ in the component $W = 0$, with $f(0) + g(0) = 0$.

The restriction operator $\omega \mapsto \omega|_y$ defined in this section induces a sheaf homomorphism $R_y : \mathcal{F} \to \Omega_y$, where we see Ω_y as a sheaf on X. The following facts are not hard to check:
- The kernel of this homomorphism is $\mathcal{I}_y$,
- For small open sets $U \subset X$, $R_y(\mathcal{F}(U)) = \Omega_y(U)$.
Hence we have an exact sequence of sheaves

$$0 \to \mathcal{I}_y \to \mathcal{F} \to \Omega_y \to 0$$

from which we get

$$\dim H^0(X_y, \mathcal{F}_y) = \dim H^0(X_y, \Omega_y)$$

When $y \neq 0$, we have $\dim H^0(X_y, \Omega_y) = n-1+g$, because a meromorphic 1-form with simple poles is given by its n residues and its g A-periods, with the only condition that the sum of the residues is zero.

When $y = 0$ there are two cases:

- If X_0 minus the double point is connected, then its genus is $g-1$. A 1-form in $H^0(X_0, \Omega_0)$ has $n+2$ poles, with the condition that the two poles corresponding to the double point have opposite residues, and the sum of all residues is zero. Hence the dimension of H^0 is $n+2-1-1+g-1$.
- If X_0 minus the double point is not connected, then it has two components of genus g' and g'' with $g'+g''=g$. Let n' (resp. n'') be the number of points of δ which are in the first (resp. second) component, so that $n'+n''=n$. Then the dimension of H^0 is $(n'+1-1+g')+(n''+1-1+g'')-1$. (We have a -1 in each parenthesis because the sum of the residues in each component has to be zero, and the last -1 is there because the two poles corresponding to the node must have opposite residues).

In all cases we find $\dim H^0 = n+g-1$ so by the theorem, $\pi_*\mathcal{F}$ is locally free of rank $n+g-1$. This means that for any $y_0 \in Y$, there exists a neighborhood U of y_0 such that $\pi_*\mathcal{F}(U)$ is a free module of rank $n+g-1$ over $\mathcal{O}_Y(U)$, where by definition $\pi_*\mathcal{F}(U) = \mathcal{F}(\pi^{-1}(U))$. Let $\omega_1, \cdots \omega_{n+g-1}$ be a basis of $\mathcal{F}(\pi^{-1}(U))$ over $\mathcal{O}_Y(U)$. We define

$$F(y,\omega) = \left(\int_{A_1} \omega|_y, \cdots \int_{A_g} \omega|_y,\ \text{Res}_{\,q_1}\omega|_y, \cdots\ \text{Res}_{\,q_{n-1}}\omega|_y\right)$$

where $q_1, \cdots q_n$ are the poles of η_y.

Let $M_{ij}(y) = F_i(y, \omega_j)$. For any y in U, the square matrix $M(y)$ is invertible. Indeed, if there exists $\xi \in \mathbb{C}^{n+g-1}$ such that $M(y)\xi = 0$, let $\omega = \sum \xi_j \omega_j$. Then $F(y,\omega) = 0$ implies that $\omega|_y = 0$. Hence $\omega = (\pi - y)\omega'$ where $\omega' \in \mathcal{F}(\pi^{-1}(U))$. Write $\omega' = \sum \xi'_j \omega_j$ where the ξ'_j are functions in $\mathcal{O}_Y(U)$. Then by uniqueness of the decomposition, $\xi_j = (\pi - y)\xi'_j$. Evaluating on X_y where $\pi = y$ we see that $\xi = 0$ so $M(y)$ is invertible.

Let $\xi(y) = M(y)^{-1}(1, \cdots 1, \pm\frac{1}{2\pi i}, \ldots \pm\frac{1}{2\pi i})$ where the signs are those of the residues of η_y. Then $\xi_i \in \mathcal{O}_Y(U)$ and $\omega = \sum \xi_i(y)\omega_i$ satisfies $\omega|_y = \eta_y$ for any $y \in U$, $y \neq 0$. This proves lemma 1.

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