A simple method for singularity avoidance and some consequences

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Quantum cosmology

Hamiltonian GR (3+1)

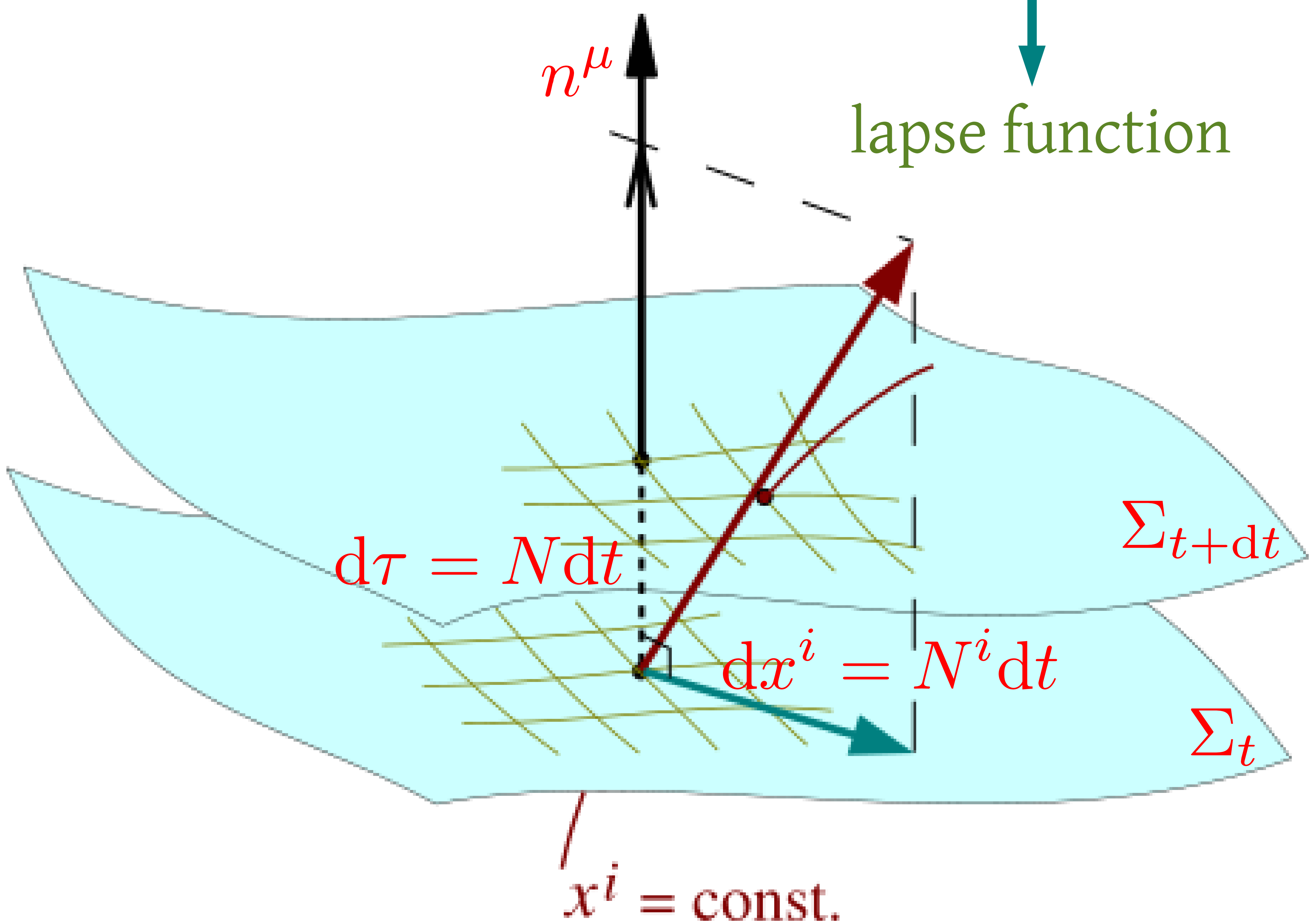
$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right)$$

lapse function

intrinsic metric =
first fundamental form

shift vector

intrinsic curvature tensor
$^3R^i{}_{jkl}(h)$



extrinsic curvature =
second fundamental form:

$$K_{ij} = -\nabla_j^{(h)} n_i = \frac{1}{2N}\left(\nabla_j^{(h)} N_i + \nabla_i^{(h)} N_j - \frac{\partial h_{ij}}{\partial t}\right)$$

Action (Einstein-Hilbert, compact space):

$$\mathcal{S} = \frac{1}{16\pi G_{\rm N}} \left[\int_{\mathcal{M}} \sqrt{-g}\,\,(R - 2\Lambda)\, \mathrm{d}^4 x + 2\int_{\partial\mathcal{M}} \sqrt{h} K^i{}_i \mathrm{d}^3 x\right] + \mathcal{S}_{\rm matter}\left[\Phi\left(x\right)\right]$$

$$\longrightarrow \mathcal{S} = \int L \mathrm{d}t = \frac{1}{16\pi G_{\rm N}} \int \mathrm{d}t \left[\int \mathrm{d}^3 x\, N\sqrt{h}\left(K_{ij}K^{ij} - K^2 + {}^3R - 2\Lambda\right) + L_{\rm matter}\right]$$

Canonical momenta

$$\pi^{ij} \equiv \frac{\delta L}{\delta \dot{h}_{ij}} = -\frac{\sqrt{h}}{16\pi G_{\rm N}}\left(K^{ij} - h^{ij}K\right)$$

$$\pi^{\Phi} \equiv \frac{\delta L}{\delta\dot{\Phi}} = -\frac{\sqrt{h}}{N}\left(\dot{\Phi} - N\frac{\partial\Phi}{\partial x^i}\right)$$

$$\left.\begin{aligned} \pi^0 &\equiv \frac{\delta L}{\delta \dot{N}} = 0 \\ \pi^i &\equiv \frac{\delta L}{\delta \dot{N}^i} = 0 \end{aligned}\right\} \text{primary constraints}$$

Hamiltonian

$$H \equiv \int \mathrm{d}^3x \left(\pi^0 \dot{N} + \pi^i \dot{N}_i + \pi^{ij} \dot{h}_{ij} + \pi^\Phi \dot{\Phi}\right) - L$$

$$= \int \mathrm{d}^3x \left(\pi^0 \dot{N} + \pi^i \dot{N}_i + N\mathcal{H} + N_i \mathcal{H}^i\right)$$

variation wrt lapse: $\mathcal{H} = 0 \rightarrow$ Hamiltonian constraint

variation wrt shift: $\mathcal{H}^i = 0 \rightarrow$ momentum constraint

$\Longrightarrow$ classical description complete

Superspace & canonical quantization

relevant configuration space $\mathrm{Riem}\left(\Sigma\right) \equiv \left\{h_{ij}\left(x^{\mu}\right), \Phi\left(x^{\mu}\right) | x \in \Sigma\right\}$

→ parameters

GR $\Longrightarrow$ invariance/diffeomorphisms $\Longrightarrow$ $\mathrm{Conf} = \dfrac{\mathrm{Riem}\left(\Sigma\right)}{\mathrm{Diff}\left(\Sigma\right)}$: superspace

Wave functional $\Psi\left[h_{ij}\left(x\right), \Phi\left(x\right)\right]$

+ Dirac canonical quantization procedure

$$\pi^{ij} \to -i\frac{\delta}{\delta h_{ij}} \qquad \pi^{\Phi} \to -i\frac{\delta}{\delta \Phi} \qquad \pi^{0} \to -i\frac{\delta}{\delta N} \qquad \pi^{i} \to -i\frac{\delta}{\delta N_i}$$

primary constraints
$$\begin{cases} \hat{\pi}^0 = -i\dfrac{\delta\Psi}{\delta N} = 0 \\ \hat{\pi}^i = -i\dfrac{\delta\Psi}{\delta N_i} = 0 \end{cases}$$

momentum $\hat{\mathcal{H}}^i\Psi = 0 \quad \Longrightarrow \quad i\nabla_j^{(h)}\left(\dfrac{\delta\Psi}{\delta h_{ij}}\right) = 8\pi G_{\mathrm{N}}\hat{T}^{0i}\Psi$

same Ψ for configurations related by a coordinate transformation

Hamiltonian

$$\hat{\mathcal{H}}\Psi = \left[-16\pi G_{\mathrm{N}}\mathcal{G}_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} + \frac{\sqrt{h}}{16\pi G_{\mathrm{N}}}\left(-{}^3R + 2\Lambda + 16\pi G_{\mathrm{N}}\hat{T}^{00}\right)\right]\Psi = 0$$

Wheeler - De Witt equation

$$\mathcal{G}_{ijkl} = \frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right)$$

De Witt metric

primary constraints
$$\begin{cases} \hat{\pi}^0 = -i\dfrac{\delta\Psi}{\delta N} = 0 \\ \hat{\pi}^i = -i\dfrac{\delta\Psi}{\delta N_i} = 0 \end{cases}$$

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same Ψ for configurations related by a coordinate transformation

Hamiltonian

$$\hat{\mathcal{H}}\Psi = 0$$

Wheeler - De Witt equation

mini-superspace

restrict attention from an infinite dimensional configuration space to a 2 dimensional space = mini-superspace

$$h_{ij}\mathrm{d}x^i\mathrm{d}x^j \quad \mapsto \quad a^2(t)\left[\frac{\mathrm{d}r^2}{1-\mathcal{K}r^2}+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)\right]$$

WDW equation becomes Schrödinger like for $\Psi\left[a(t),\phi(t)\right]$

Conceptual & technical issues

infinite # d.o.f. to a few: mathematical consistency?

freeze momenta... Heisenberg uncertainties?

mini-superspace

restrict attention from an infinite dimensional configuration space to a 2 dimensional space = mini-superspace

$$h_{ij}\mathrm{d}x^i\mathrm{d}x^j \quad \mapsto \quad a^2(t)\left[\frac{\mathrm{d}r^2}{1-\mathcal{K}r^2}+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)\right]$$

WDW equation becomes Schrödinger like for $\Psi\left[a(t),\phi(t)\right]$

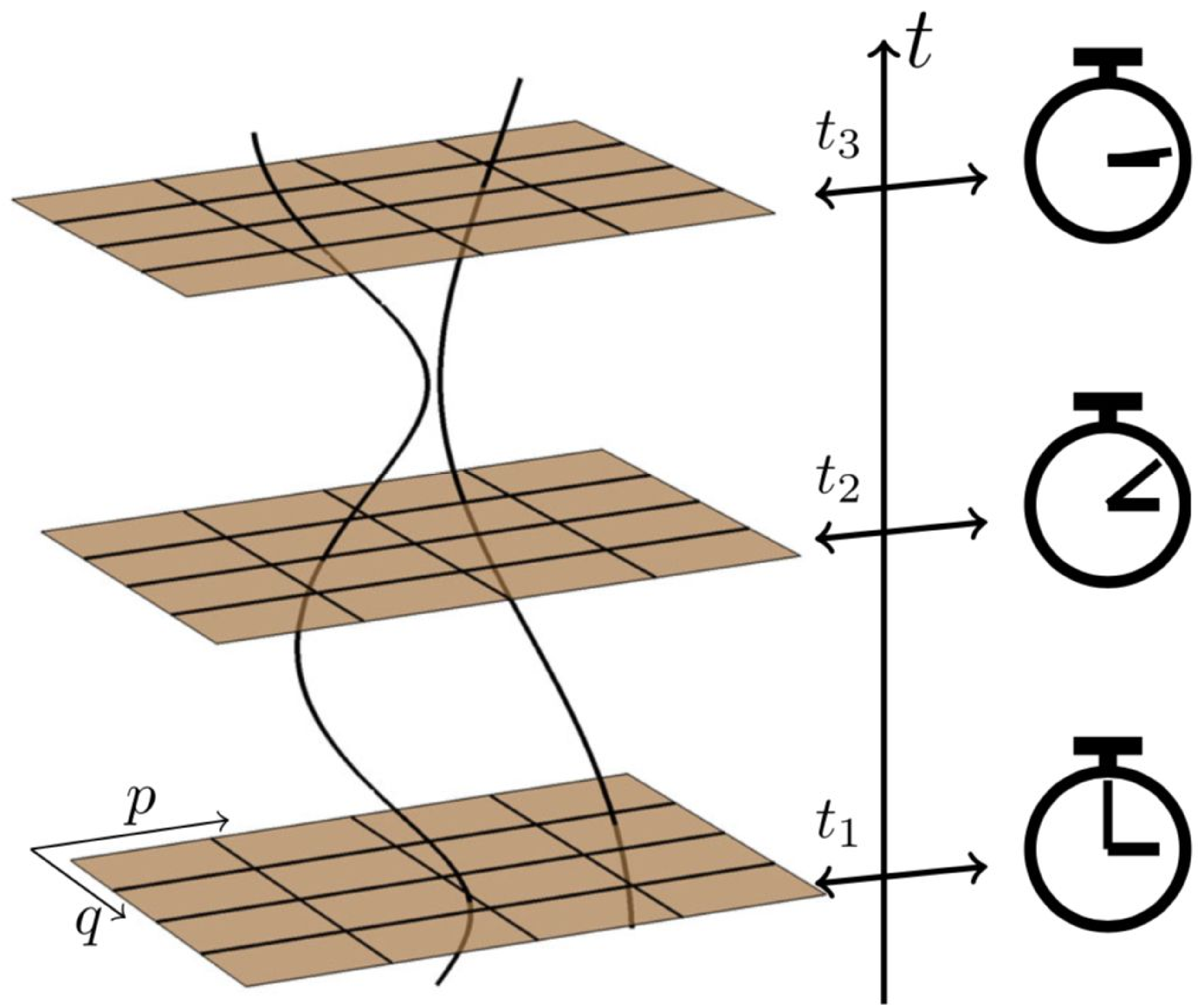
Conceptual & technical issues

ACTUALLY MAKE CALCULATIONS!

The clock issue in quantum cosmology

→ GR = constrained system: lack of external time

→ arbitrary degree of freedom: internal clock



Classical system q_i & p_i

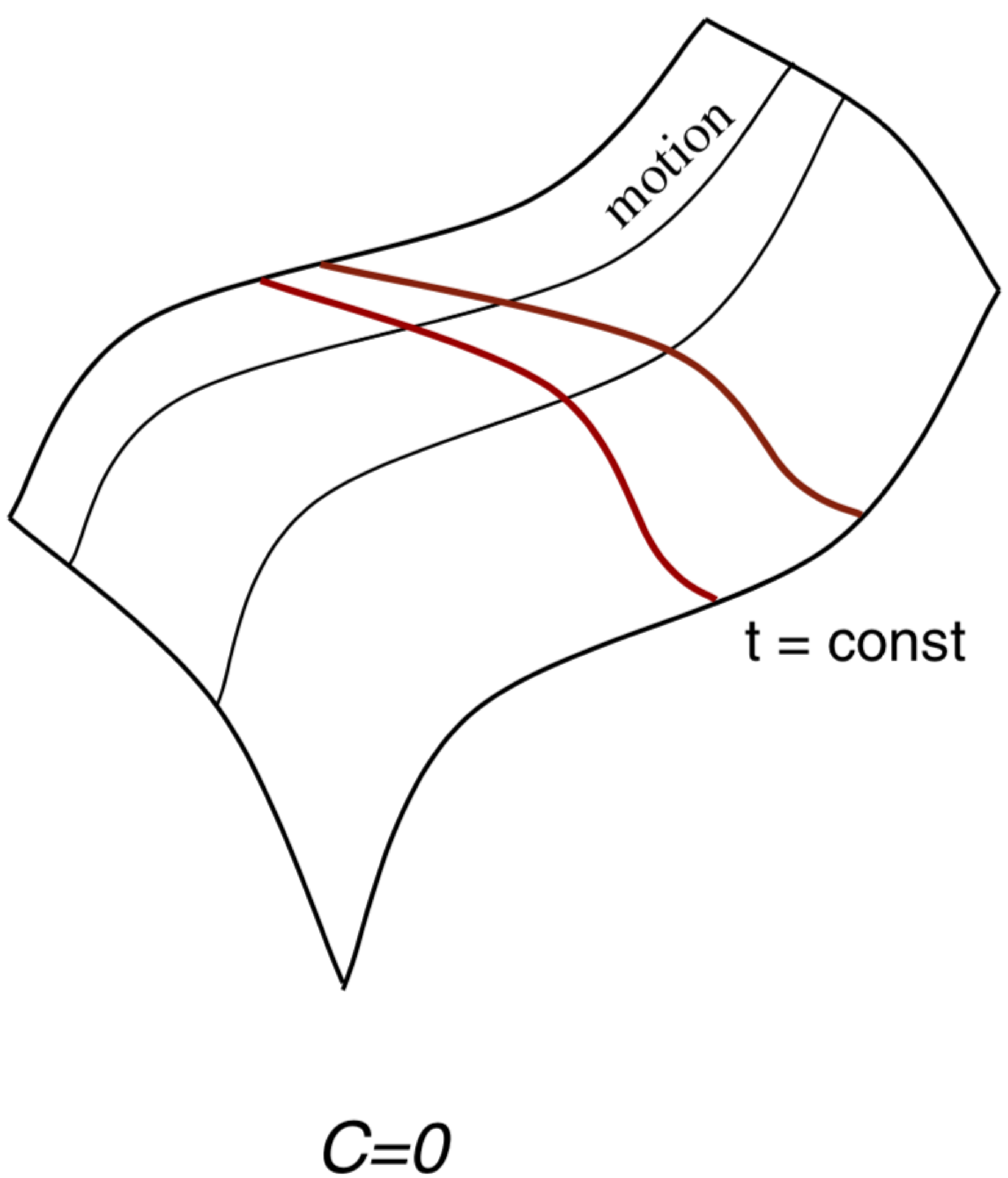
Constraint $C(q_i, p_i) = 0$ & $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{O}(q_i, p_i) = \{\mathcal{O}, C\}_{\mathrm{P.B}}$

evolution parameter (time)

observable

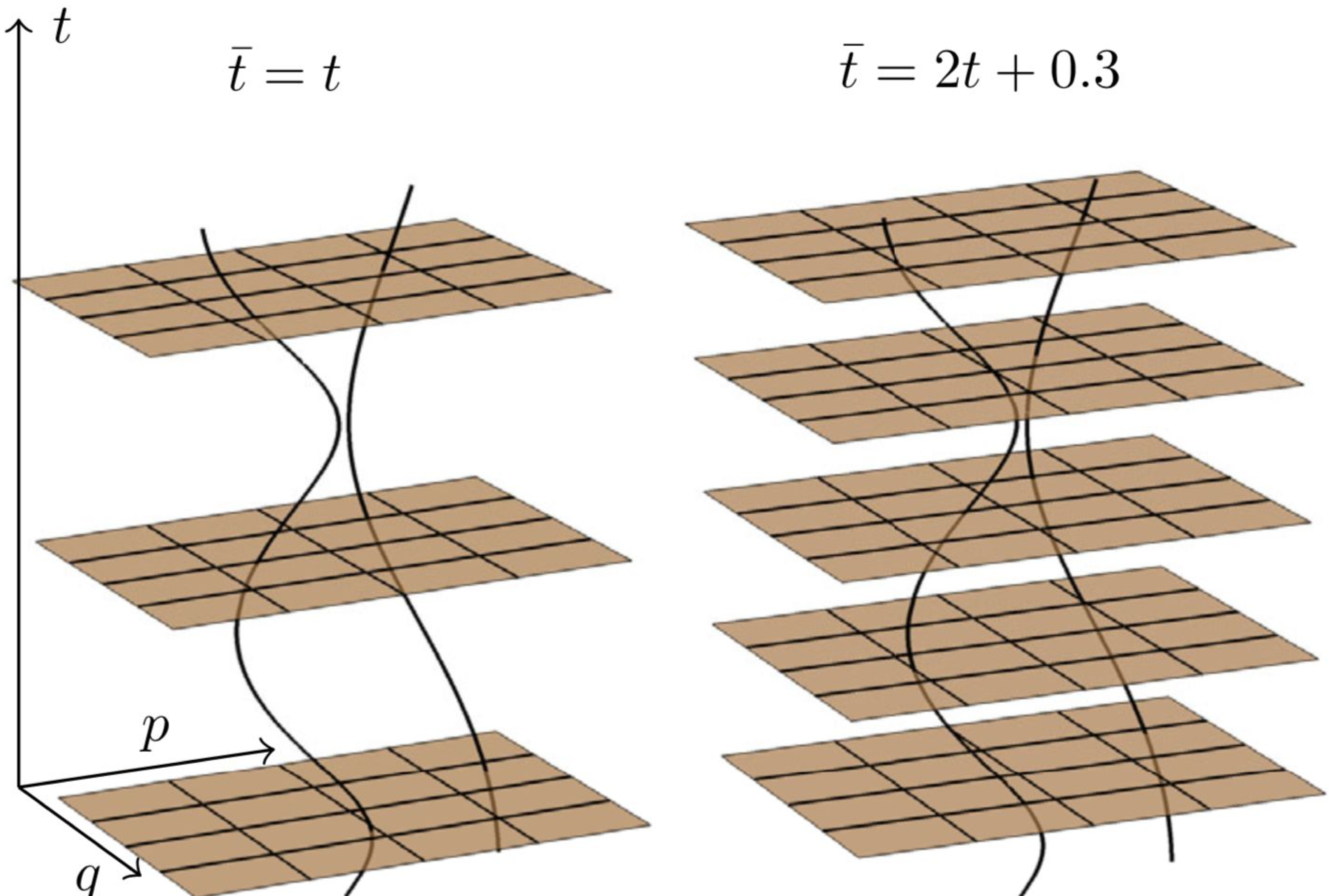
Time parametrization invariance $\tau \to \tau'$ → $N(q_i, p_i, \tau)$

arbitrary non vanishing lapse function



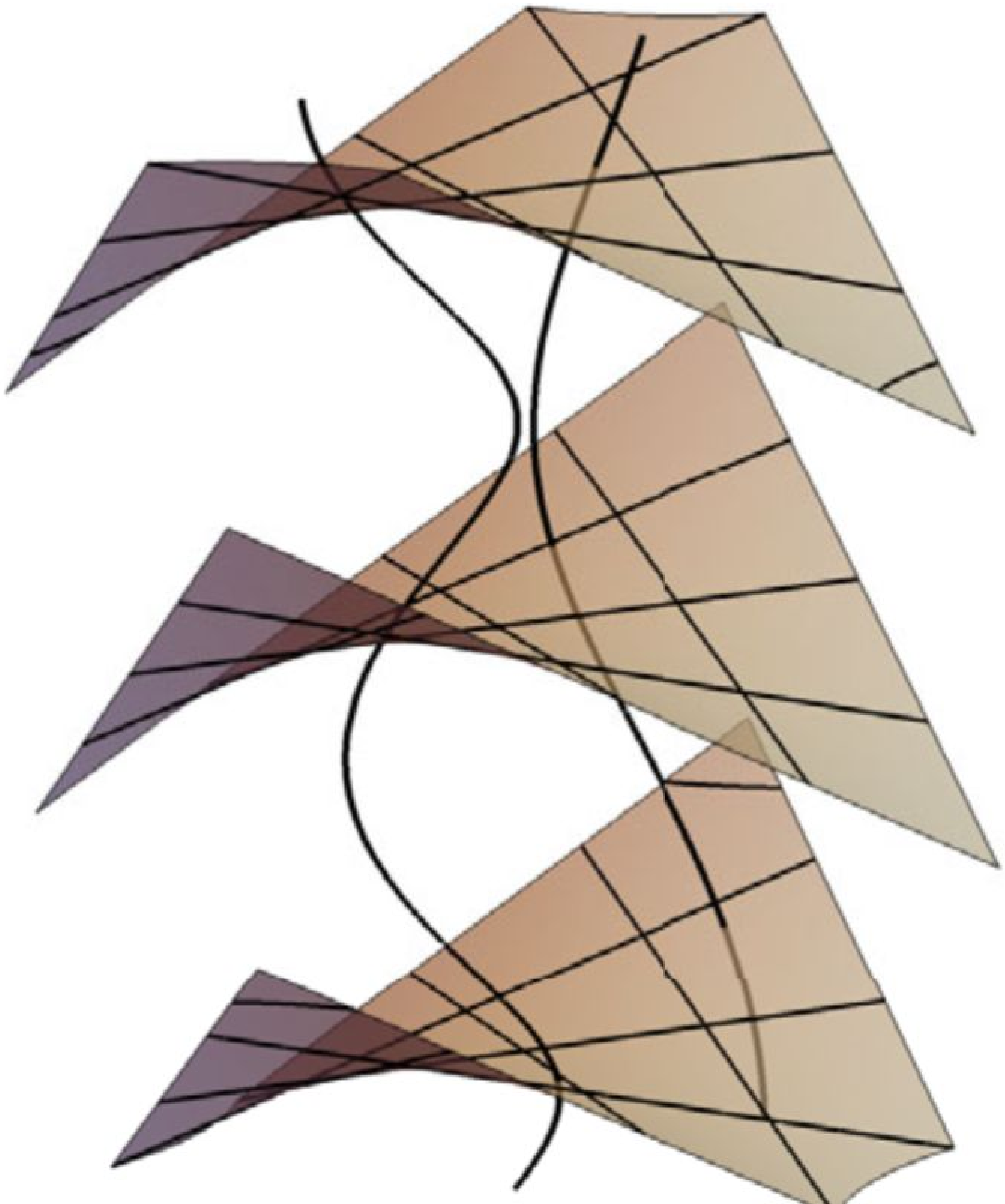
$$\mathrm{d}\tau = N\mathrm{d}\tau' \implies \frac{\mathrm{d}}{\mathrm{d}\tau'}\mathcal{O}(q_i, p_i) = \{\mathcal{O}, NC\}_{\mathrm{P.B}}$$

hamiltonian $H = NC$

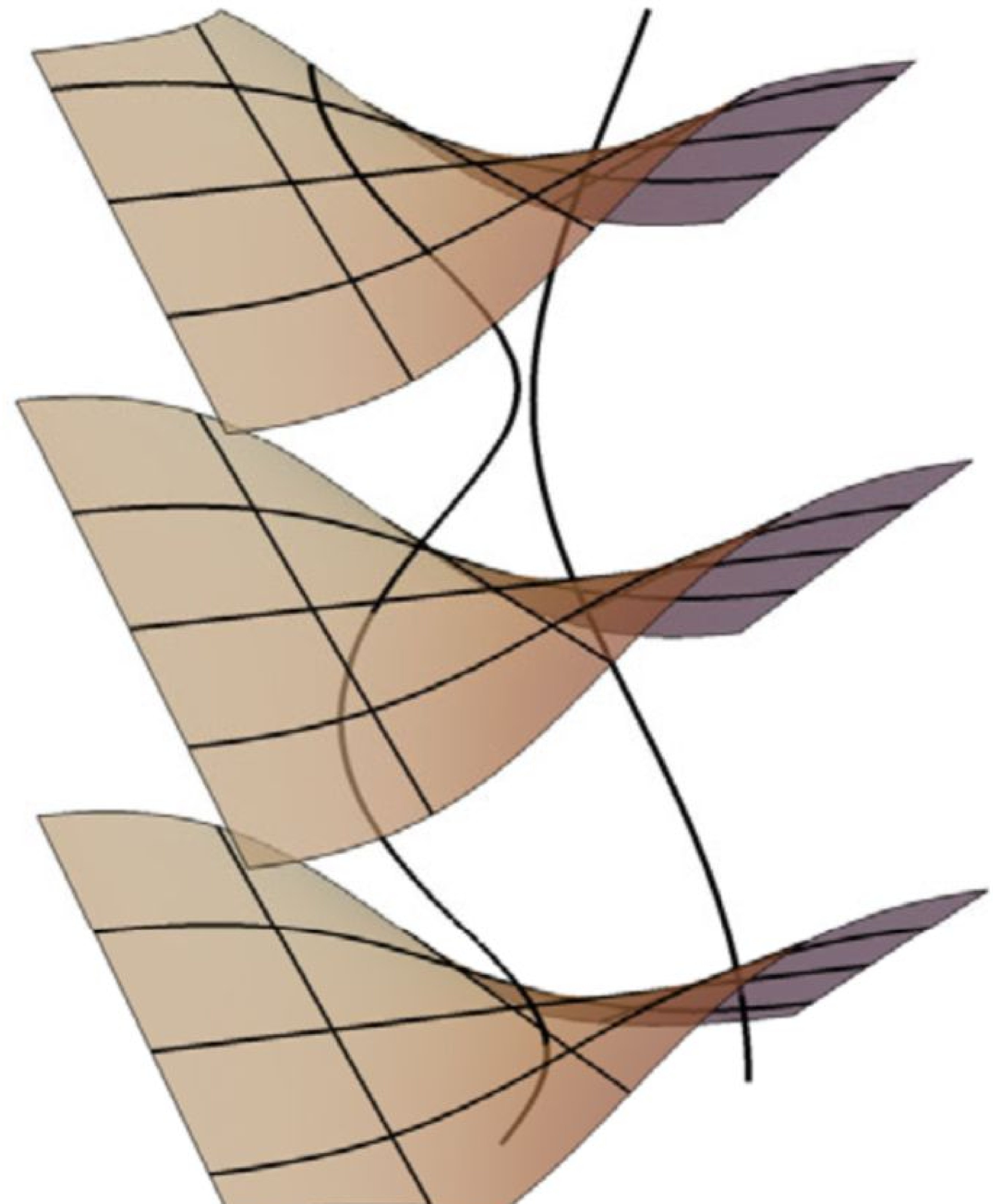


t
$\bar{t} = t$
$\bar{t} = 2t + 0.3$
p
q

$$\bar{t} = t + qp$$



$$\bar{t} = t - \frac{3qp}{3p^2+1}$$



Quantum system $\hat{C}\Psi(q_i) = 0$

if $\hat{C}$ linear in $\hat{p}_1 = -i\dfrac{\partial}{\partial q_1}$

will become time

$\longrightarrow$ $\hat{C} = \hat{p}_1 + \hat{H}(p_2, \cdots, p_n, \{q_i\})$

$\hat{C}\Psi(q_i) = 0 \Longrightarrow$ time dependent Schrödinger equation

Bianchi I case $$\mathrm{d}s^2 = -N\mathrm{d}\tau^2 + \sum_{i=1}^{3} a_i^2 \left(\mathrm{d}x^i\right)^2$$

Scale factors
$$\begin{cases} a_1 &= \mathrm{e}^{\beta_0+\beta_+ + \sqrt{2}\beta_-} \\ a_2 &= \mathrm{e}^{\beta_0+\beta_+ - \sqrt{2}\beta_-} \\ a_3 &= \mathrm{e}^{\beta_0 - 2\beta_+} \end{cases}$$

Volume $V \equiv a_1 a_2 a_3 = \mathrm{e}^{3\beta_0}$

$$\mathrm{d}\beta_0 = \frac{1}{3}\mathrm{e}^{-3\beta_0}\mathrm{d}V$$

Action $$\mathcal{S} = \int \mathrm{d}\tau \Big(\underbrace{p_0\dot{\beta}_0 + p_+\dot{\beta}_+ + p_-\dot{\beta}_-}_{\frac{\mathrm{d}\theta}{\mathrm{d}\tau}} - NC \Big)$$

$\frac{\mathrm{d}\theta}{\mathrm{d}\tau}$ → canonical one-form

NC → H

constraint $$C = \frac{\mathrm{e}^{-3\beta_0}}{24}\left(-p_0^2 + p_+^2 + p_-^2\right)$$

ensure canonical one-form remains canonical $p_V \equiv \frac{e^{-3\beta_0}}{3} p_0$

$$d\theta = p_V dV + p_+ d\beta_+ + p_- d\beta_-$$

constraint $C = \frac{3V}{8}\left(-p_V^2 + \frac{p_+^2 + p_-^2}{9V^2}\right)$

cyclic variable $\dot{p}_\pm = 0$ set $p_+ = k\cos\alpha$ and $p_- = k\sin\alpha$

$$\longrightarrow d\theta = p_V dV + p_k dk + p_\alpha d\alpha + \underbrace{d\left(k\cos\alpha\beta_+ + k\sin\alpha\beta_-\right)}_{\to \text{ exact... ignore!}}$$

$$\begin{aligned} p_k &\equiv -\left(\cos\alpha\beta_+ + \sin\alpha\beta_-\right), \\ p_\alpha &\equiv \left(k\sin\alpha\beta_+ - k\cos\alpha\beta_-\right) \end{aligned}$$

neither α nor P_α in $H = NC$

the system reduces to

$$\begin{cases} \mathrm{d}\theta &= p_V \mathrm{d}V + p_k \mathrm{d}k \\ C &= \frac{3V}{8}\left(-p_V^2 + \frac{k^2}{9V^2}\right) \end{cases}$$

Hamilton equations

$$\begin{aligned} \dot{k} &= 0 \\ \dot{p}_k &= -N\frac{k}{12V} \\ \dot{V} &= -N\frac{3Vp_V}{4} \\ \dot{p}_V &= -N\left[\frac{3}{8}\left(-p_V^2 + \frac{k^2}{9V^2}\right) - \frac{k^2}{12V^2}\right] \end{aligned}$$

+ constraint

$$\frac{3V}{8}\left(-p_V^2 + \frac{k^2}{9V^2}\right) = 0$$

→ closed for V and p_V

Choosing a time

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(9\frac{p_k}{k}\right) = -\frac{3}{4}\frac{N}{V}$$ monotonically increasing function

valid time choice $\tau = \frac{9p_k}{k} \implies N = -\frac{4}{3}V$

Solving directly in the action $\mathcal{S} = \int \mathrm{d}\theta = \int \mathrm{d}\tau \left(p_V\dot{V} - \frac{V^2 p_V^2}{2}\right)$

H

classical unconstrained one dimensional system

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(Vp_V) = 0 \implies Vp_V = V_0 p_{V0}$$

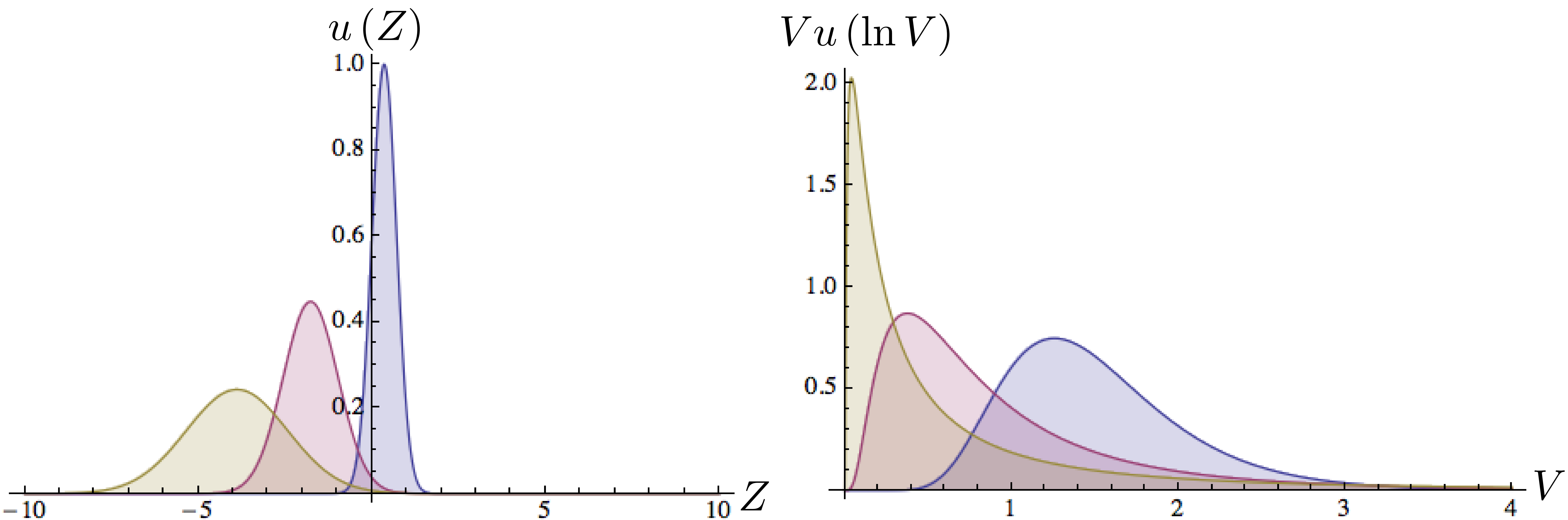
$V = V_0 \mathrm{e}^{(Vp_V)\tau}$ and $p_V = p_{V0}\mathrm{e}^{-(Vp_V)\cdot\tau}$

symmetric ordering choice

$$H = V^2 p_V^2 \quad \mapsto \quad \hat{H} = \sqrt{V}\frac{1}{i}\partial_V\sqrt{V}\cdot\sqrt{V}\frac{1}{i}\partial_V\sqrt{V}$$

coordinate transformation $V \mapsto Z = \ln V$

$$\longrightarrow \quad U\hat{H}U^{-1} = -\partial_Z^2, \quad \text{and} \quad Z \in \mathbb{R}$$



slow-gauge time
$$d\theta = (Vp_V)dV - \left(\frac{V^2p_V^2}{2}\right) d\left(\frac{9p_k}{k} + \frac{V - \ln V}{Vp_V}\right)$$
$$+d\left(\frac{9p_k}{2k}V^2p_V^2 + \frac{1}{2}V\ln Vp_V - \frac{1}{2}V^2p_V\right)$$

→ Action
$$\mathcal{S} = \int d\theta = \int d\eta \left(Vp_V\acute{V} - \frac{V^2p_V^2}{2}\right)$$

$$\acute{V} \equiv dV/d\eta$$

new time variable $\eta \equiv \dfrac{9p_k}{k} + \dfrac{V - \ln V}{Vp_V}$

canonical if
$$\pi_V = p_V V$$

↓

$$H = \frac{1}{2}V^2p_V^2 = \frac{1}{2}\pi_V^2$$

freely moving particle...
$$(V, \pi_V) \in \mathbb{R}_+ \times \mathbb{R}$$
on the half line

Quantization: a gaussian wave packet

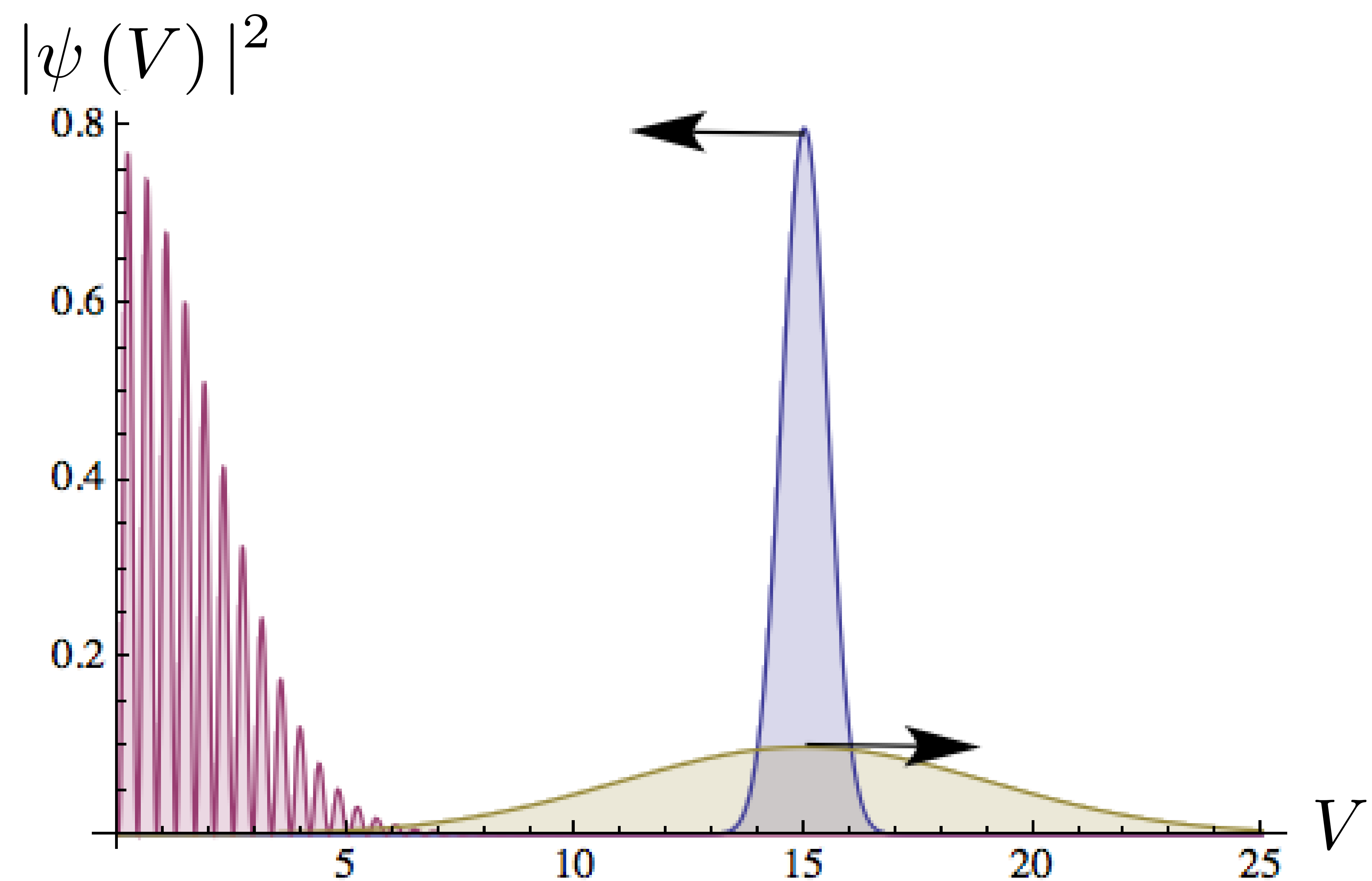
$$u(V,\eta) = \frac{\mathrm{e}^{-k^2/4}}{\sqrt{1+4i\eta}} \exp\left[-\frac{(V-ik/2)^2}{1+4i\eta}\right]$$

implement Dirichlet boundary conditions to ensure self-adjointness

$$\psi(V,\eta) = \frac{u(V+V_0,\eta) - u(-V+V_0,\eta)}{\left[\sqrt{\pi/2}\left(1-\mathrm{e}^{-V_0^2-k^2/2}\right)\right]^{1/2}}$$

→ ***solves the Schrödinger equation***

$$i\frac{\partial}{\partial\eta}\psi = -\bigtriangleup_D\,\psi$$


$|\psi(V)|^2$
0.8
0.6
0.4
0.2
5
10
15
20
25
V

Operator ordering ambiguity $\pi_V^2 \mapsto \hat{V}^s \hat{\pi}_V \hat{V}^{-2s} \hat{\pi}_V \hat{V}^s$

→ $\pi_V^2 \mapsto \hat{\pi}_V^2 + s\hat{V}^{-2}$

↓

self-adjoint hamiltonian on the half-line $s > 3/4$

Closed algebra of operators
$$\begin{cases} [\hat{V}^2, \hat{H}] &= 4i\hat{D}, \\ [\hat{D}, \hat{H}] &= 2i\hat{H}, \\ [\hat{V}^2, \hat{D}] &= 2i\hat{V}^2, \end{cases}$$

$$\hat{D} \equiv \frac{1}{2}\left(\hat{V}\hat{\pi}_V + \hat{\pi}_V\hat{V}\right)$$

→ Heisenberg equations of motion

→ Heisenberg equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\hat{V}^2 = -i[\hat{V}^2, \hat{H}] = 4\hat{D}$$

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\hat{D} = -i[\hat{D}, \hat{H}] = 2\hat{H}$$

→ solution as time-dependent operators

$$\hat{D}(\eta) = 2\hat{H}\eta + \hat{D}(0) \quad \Longrightarrow \quad \hat{V}^2 = 4\hat{\eta}^2 + 4\hat{D}(0)\eta + \hat{V}^2(0)$$

expectation values follows similar equations…

→ ***semi-classical variables***

$$v(t) = \sqrt{\langle \hat{V}^2(t) \rangle}$$

$$p(t) = \frac{\langle \hat{D}(t) \rangle}{v(t)}$$

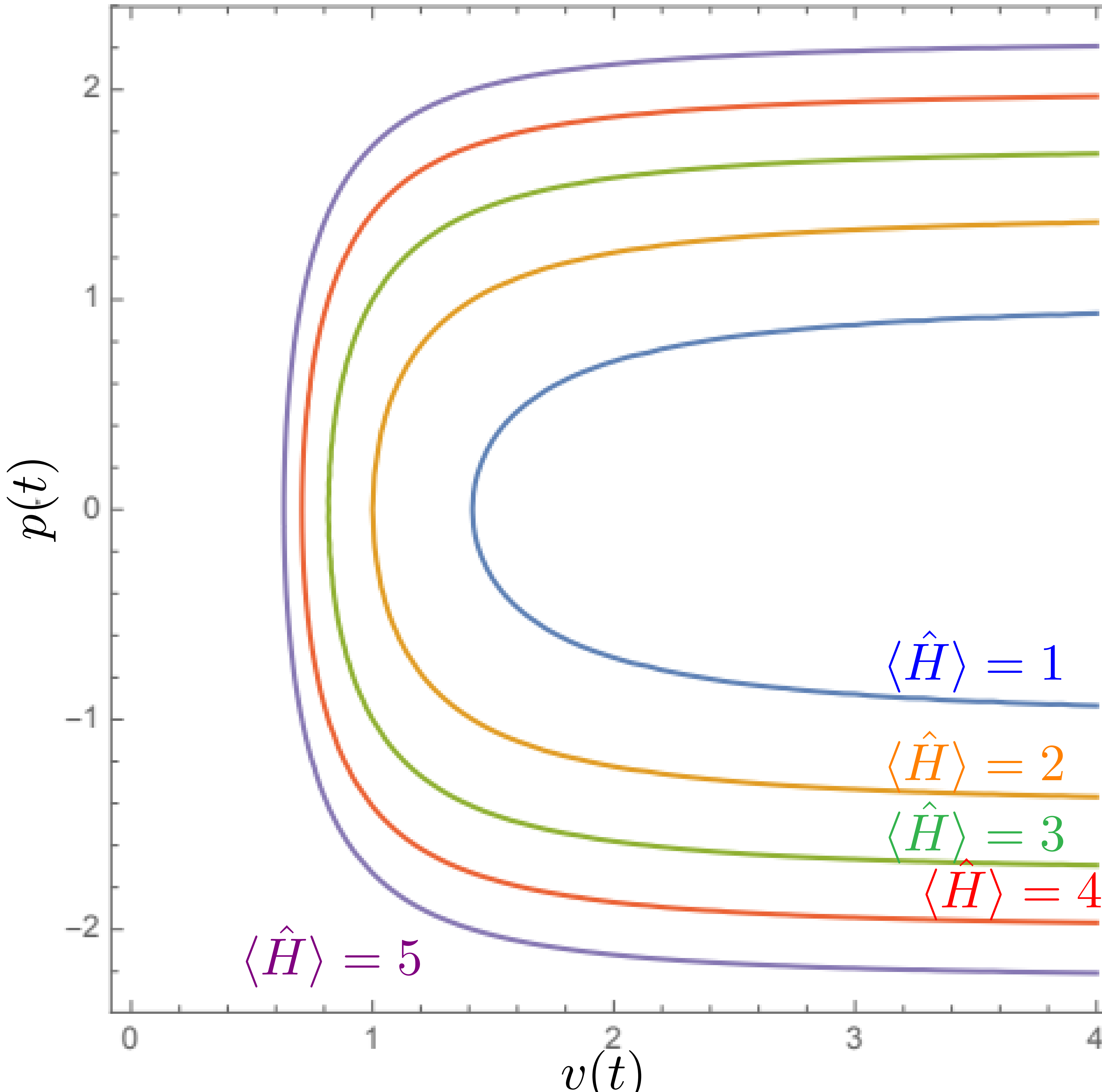
→ *phase space solution*

$$v(t) = \sqrt{4\langle\hat{H}\rangle t^2 + V_0^2}$$

$$p(t) = \frac{2\langle\hat{H}\rangle t}{\sqrt{4\langle\hat{H}\rangle t^2 + V_0^2}}$$

↓

NO SINGULARITY



Changing the time variable $\eta' = \eta'(\eta, V, \pi_V)$

redefining the dynamical variables in the process

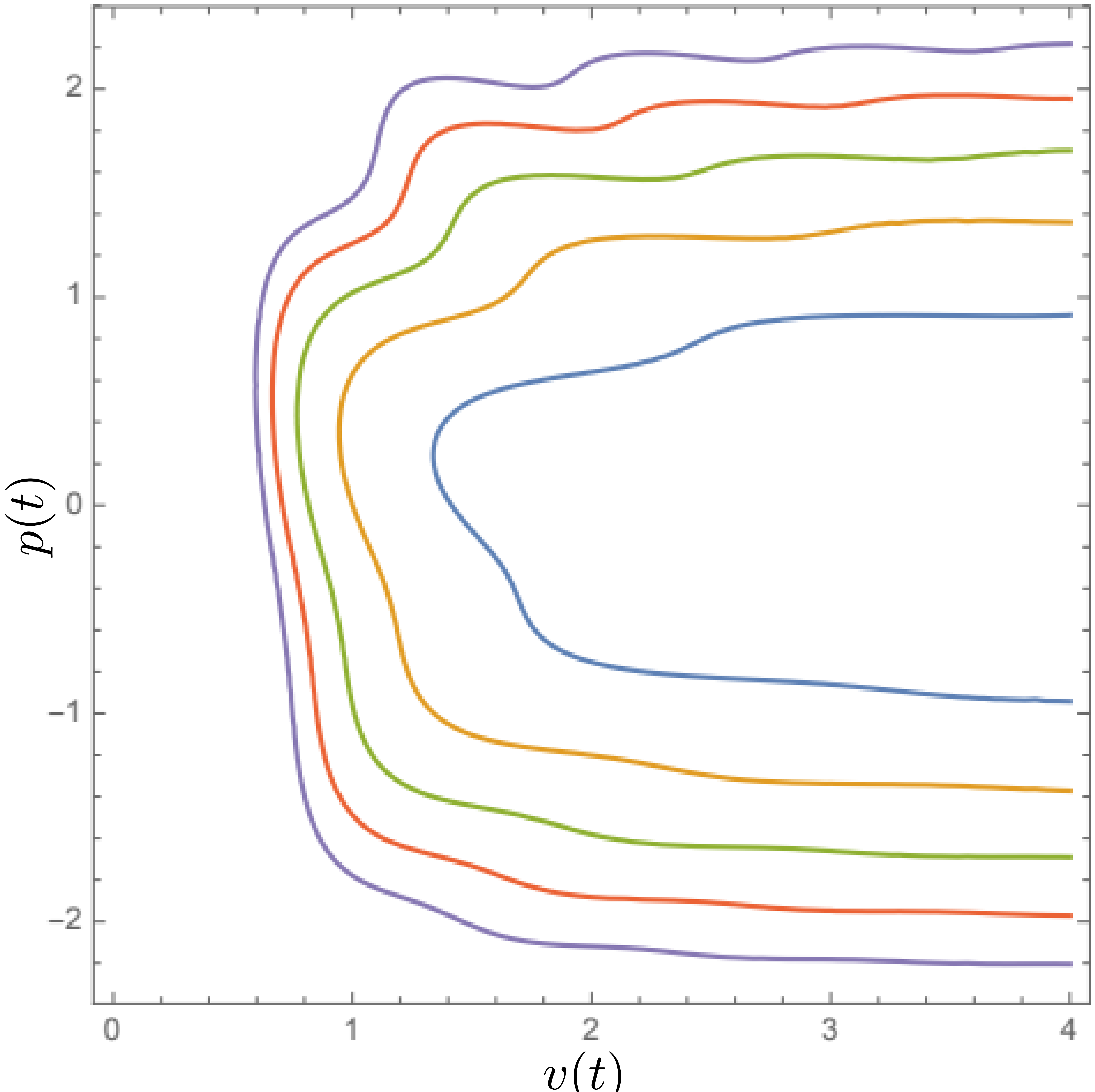
$\pi'_V = \pi_V$ & $V' = V + \pi_V(\eta' - \eta)$ no change of range...

change the canonical one-form

$$\mathrm{d}\theta = \pi_V \mathrm{d}V - \frac{\pi_V^2}{2}\mathrm{d}\eta = \pi'_V \mathrm{d}V' - \frac{\pi_V'^2}{2}\mathrm{d}\eta' + \mathrm{d}\left[(\eta - \eta')\frac{\pi_V'^2}{2}\right]$$

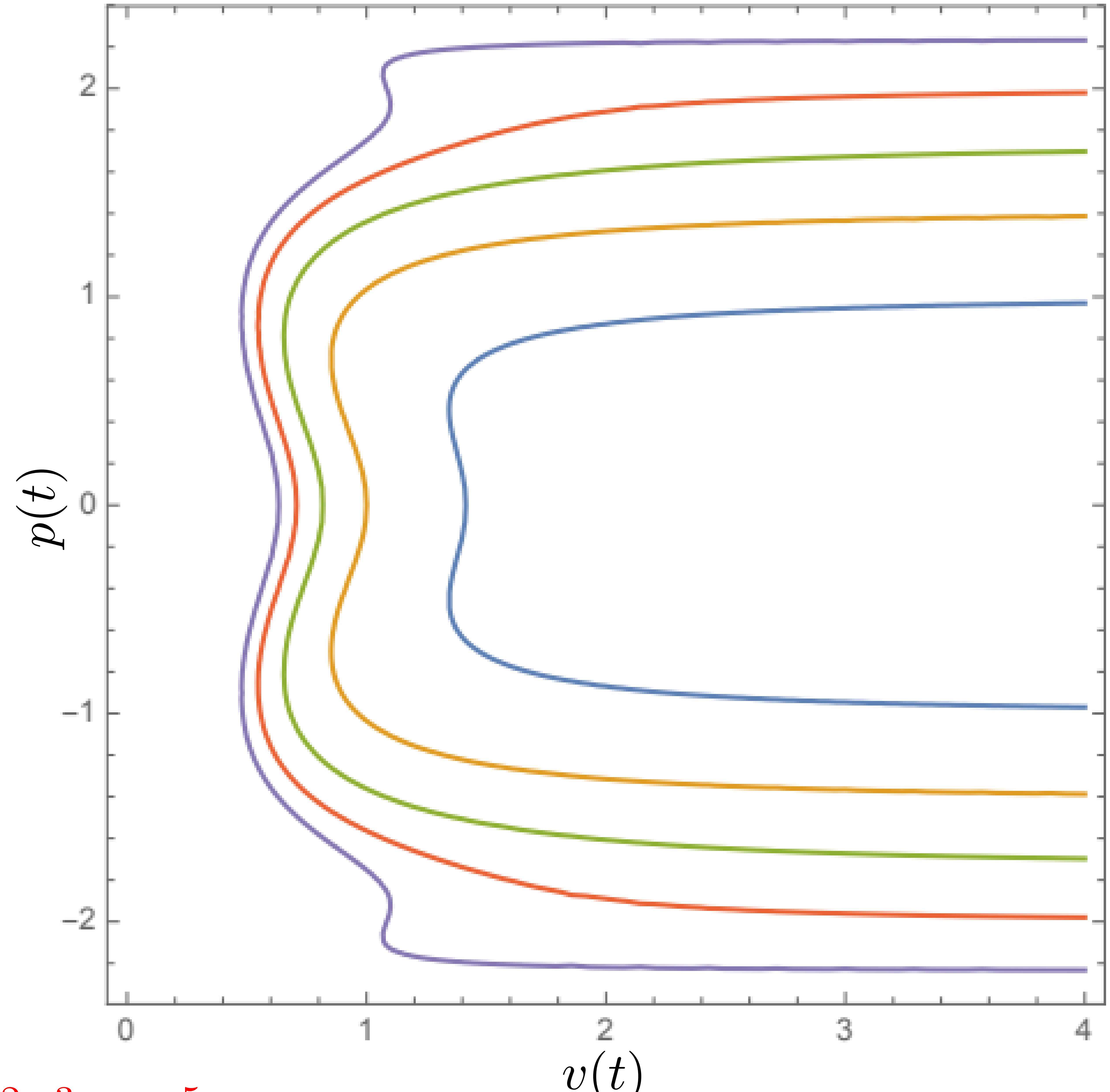
→ ***same system!***

delay function $\Delta(V, \pi_V) = \eta' - \eta$ no dependency on time



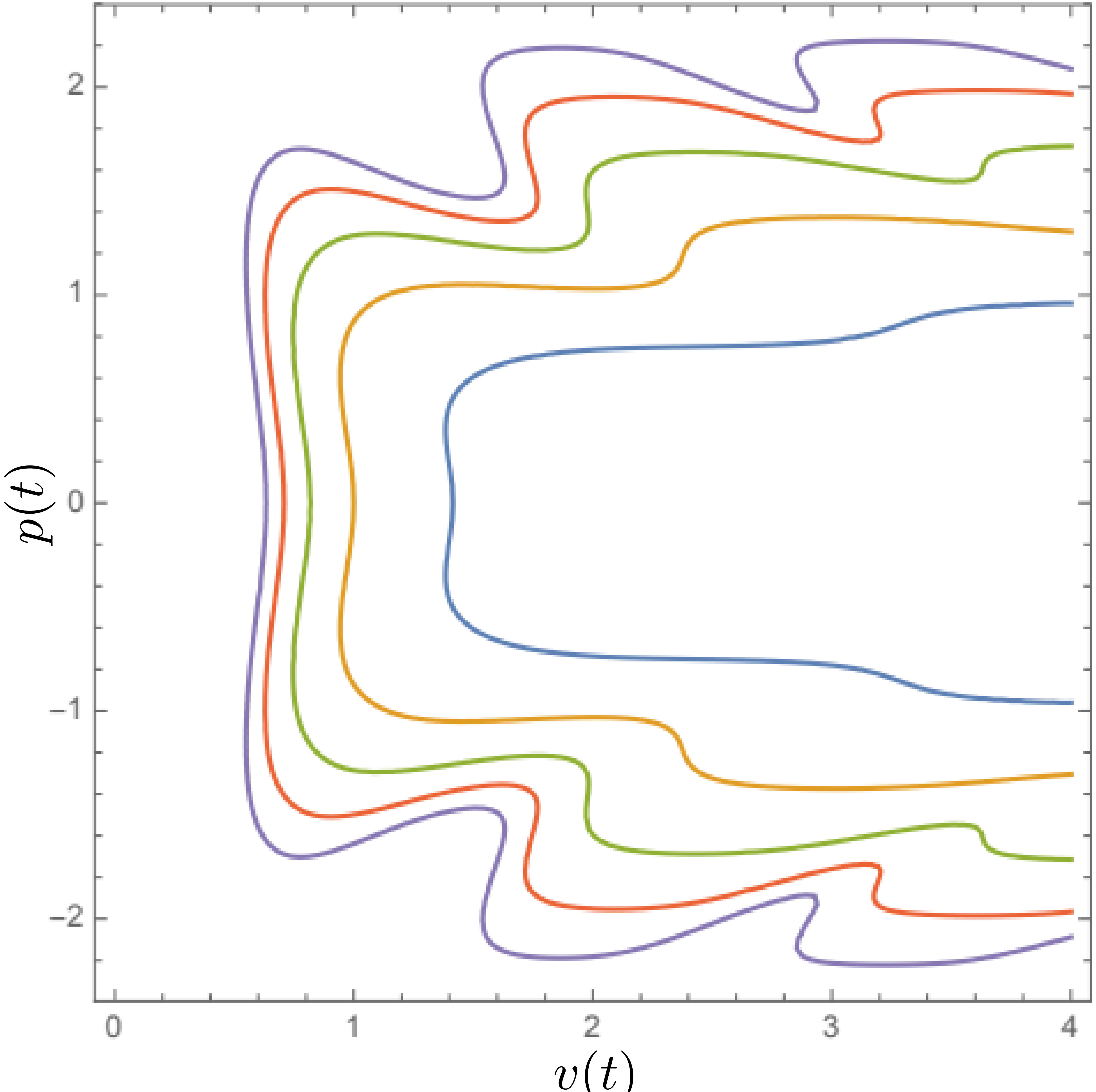
Delay function

$\Delta = V e^{\pi_V/3} \sin(3V\pi_V)/(10\pi_V)$



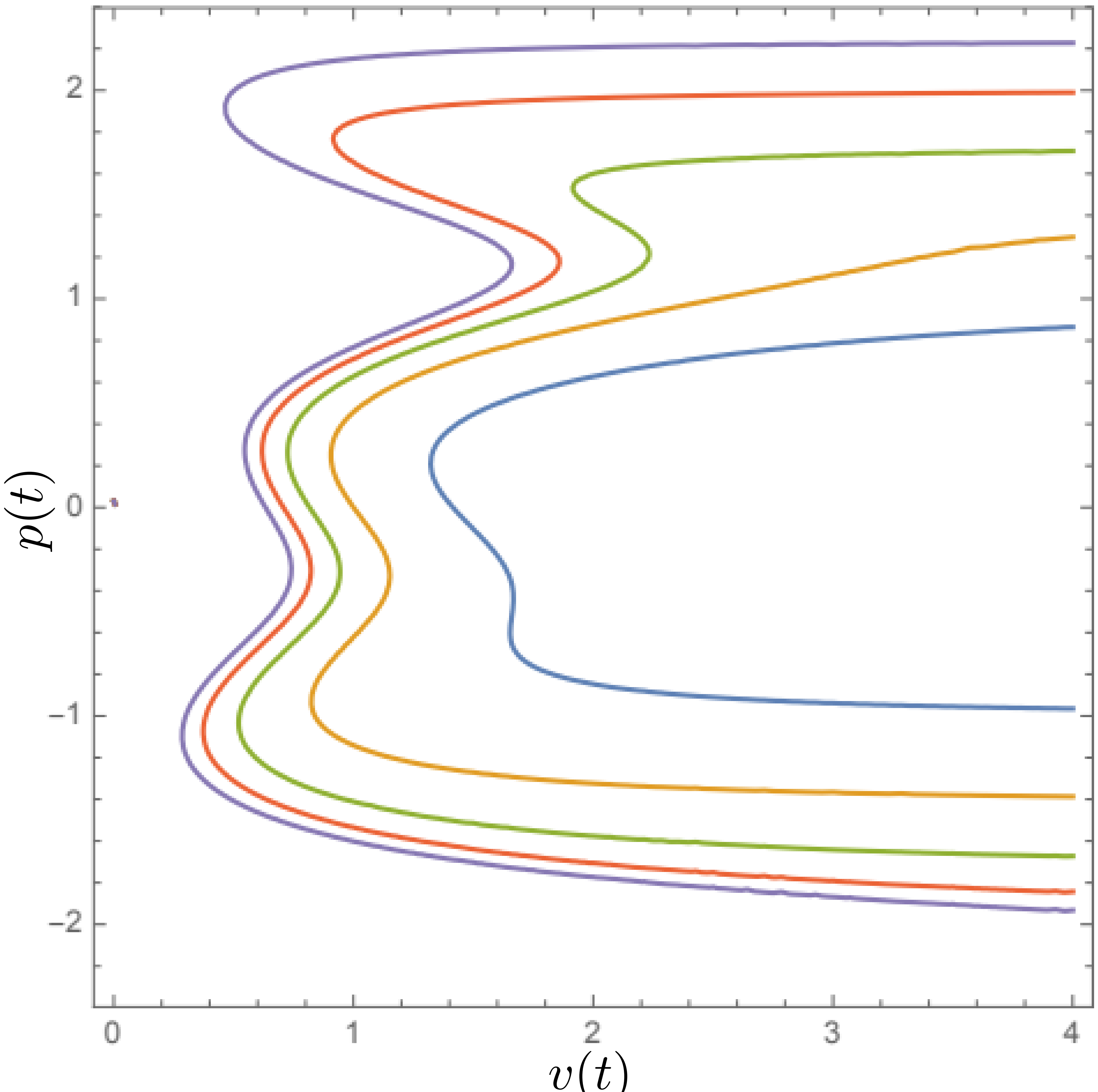
Delay function

$$\Delta = V(\pi_V - 10^{-0.2}\pi_V^3 + \pi_V^5/10)$$



Delay function

$\Delta = 10^{-0.5} V \sin(2V\pi_V)$

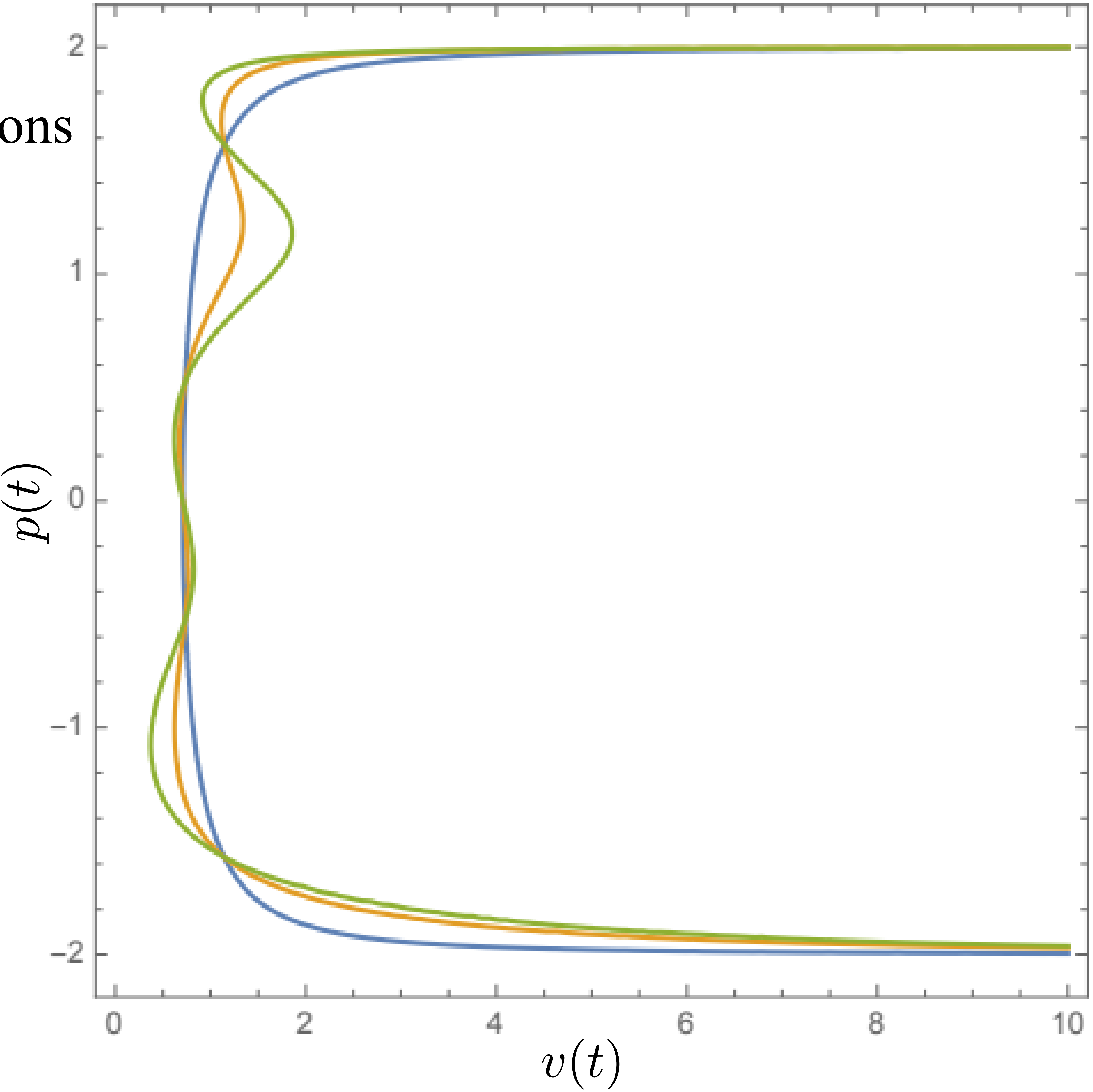


Delay function

$\Delta = 10^{-0.5}(V+1)\cos(3\pi_V)$

Comparison between different delay functions

Same asymptotics



Another way to obtain trajectories

Trajectory *formulation* to QM

(1) ordinary QM

Schrödinger equation $$i\hbar\frac{\partial\psi\left(\boldsymbol{x},t\right)}{\partial t}=\left[-\frac{\hbar^{2}\boldsymbol{\nabla}^{2}}{2m}+V\left(\boldsymbol{x}\right)\right]\psi\left(\boldsymbol{x},t\right)$$

polar form of the wave function $\psi\left(\boldsymbol{x},t\right)=A\left(\boldsymbol{x},t\right)\mathrm{e}^{iS(\boldsymbol{x},t)/\hbar}$

from now on, $\hbar\to 1$

modified Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}+\frac{\left(\boldsymbol{\nabla}S\right)^{2}}{2m}+V\left(\boldsymbol{x}\right)+Q(\boldsymbol{x},t)=0$$

Quantum potential $\equiv-\frac{1}{2m}\frac{\boldsymbol{\nabla}^{2}A}{A}$

Trajectory *formulation* to QM

$\exists\, \boldsymbol{x}(t)$ trajectory satisfying $$m \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \Im\mathrm{m}\, \frac{\Psi^* \boldsymbol{\nabla} \Psi}{|\Psi(\boldsymbol{x}, t)|^2} = \boldsymbol{\nabla} S$$

The trajectory *formulation* to QM

$\exists\, \boldsymbol{x}(t)$ trajectory satisfying

$$m\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = -\boldsymbol{\nabla}(V+Q)$$

$$Q = -\frac{1}{2m}\frac{\boldsymbol{\nabla}^2 A}{A}$$

Trajectory *formulation* to QM

$\exists\, \boldsymbol{x}(t)$ trajectory satisfying $m\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{p} = \boldsymbol{\nabla} S$

properties

- strictly equivalent to Copenhagen QM

  probability distribution $\exists t; \rho(\boldsymbol{x}, t) = |\psi(\boldsymbol{x}, t)|^2$

  (attractor)

- classical limit well defined $Q \to 0$
- state dependent
- $\exists$ intrinsic reality (ontology)

  non local...

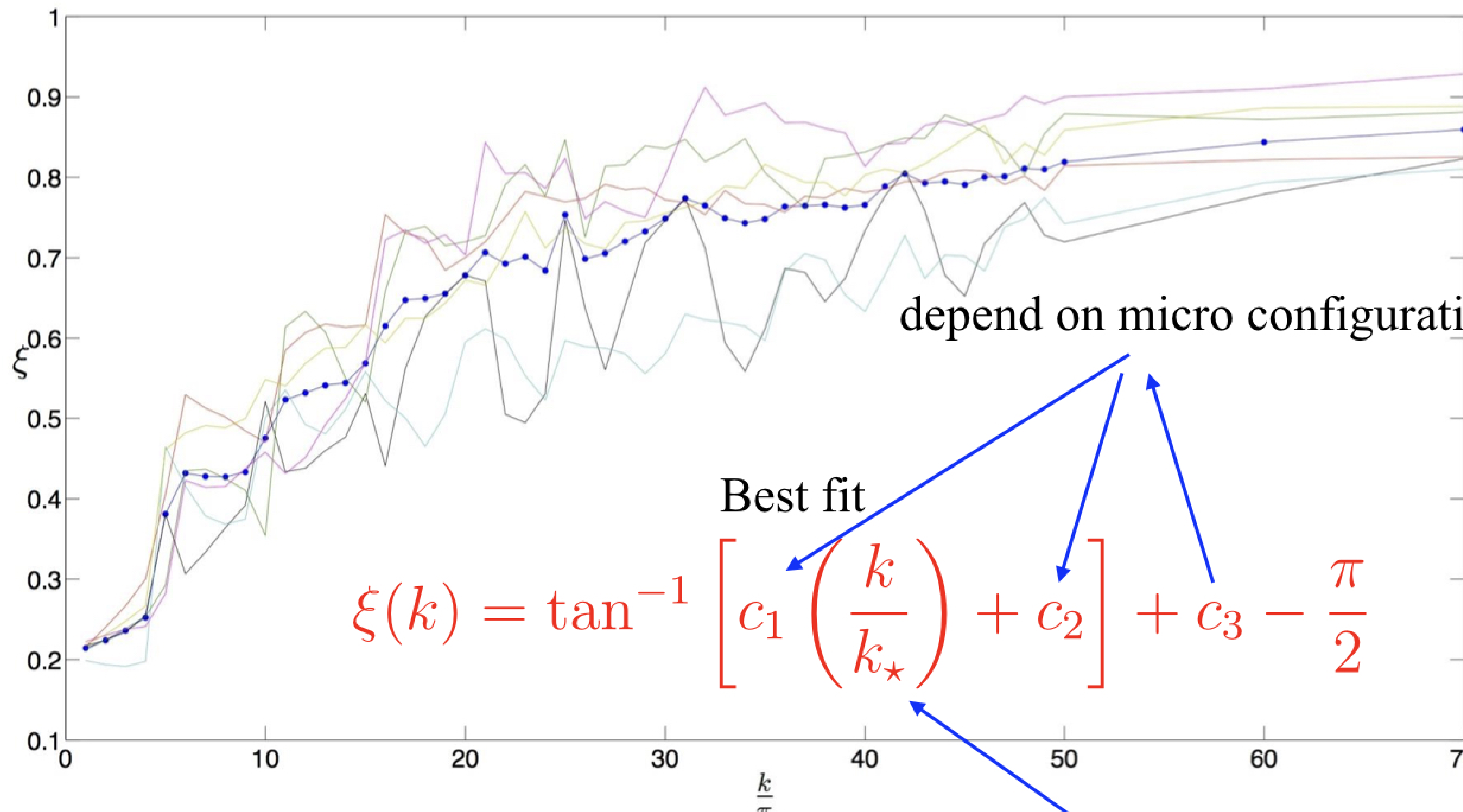
- no need for external classical domain/observer!

MOVIES…

Initial out-of-equilibrium conditions

$$\mathcal{P}(k) = \mathcal{P}(k)_{\mathrm{QE}}\,\xi(k)$$

width deficit

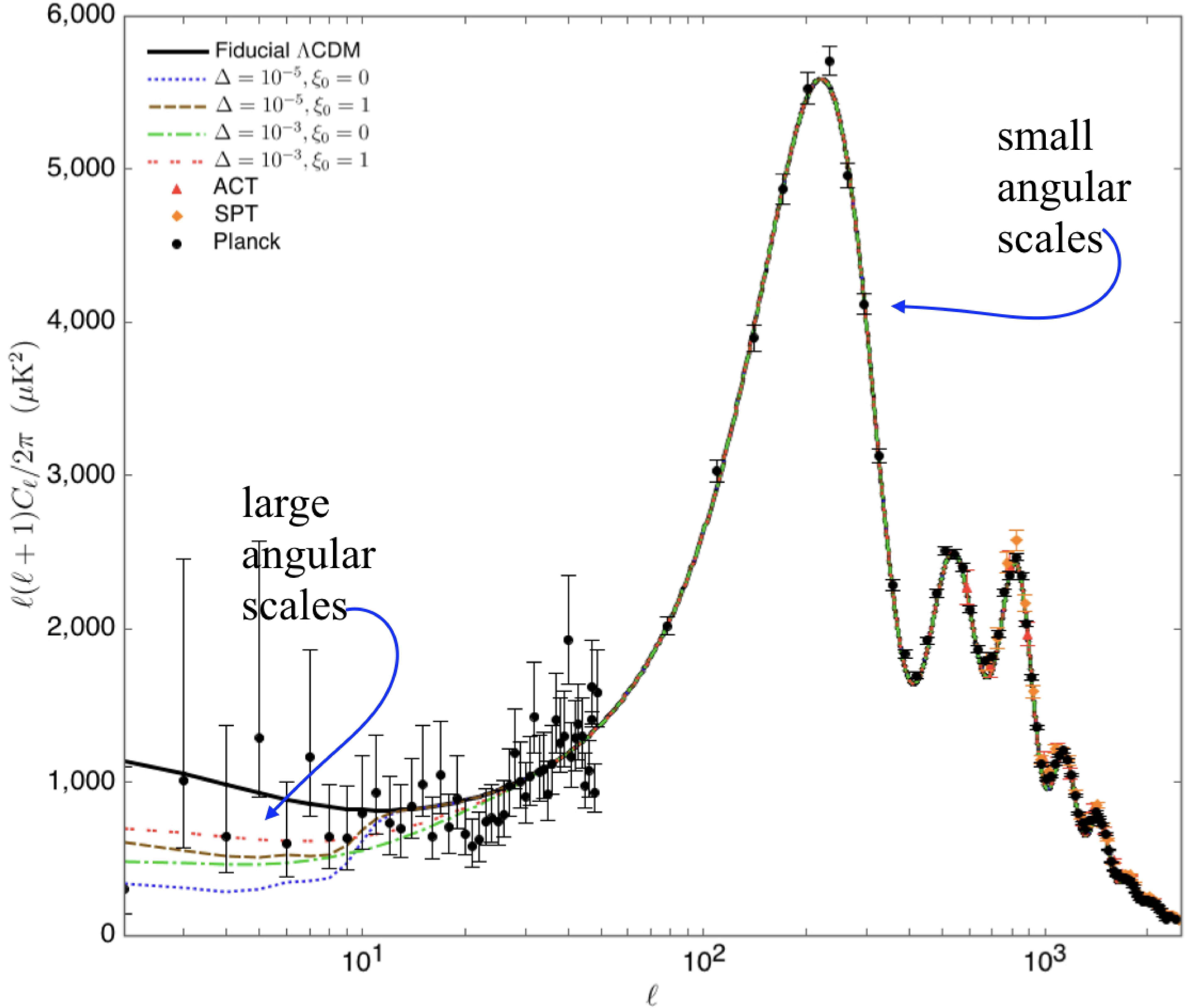


depend on micro configuration

Best fit

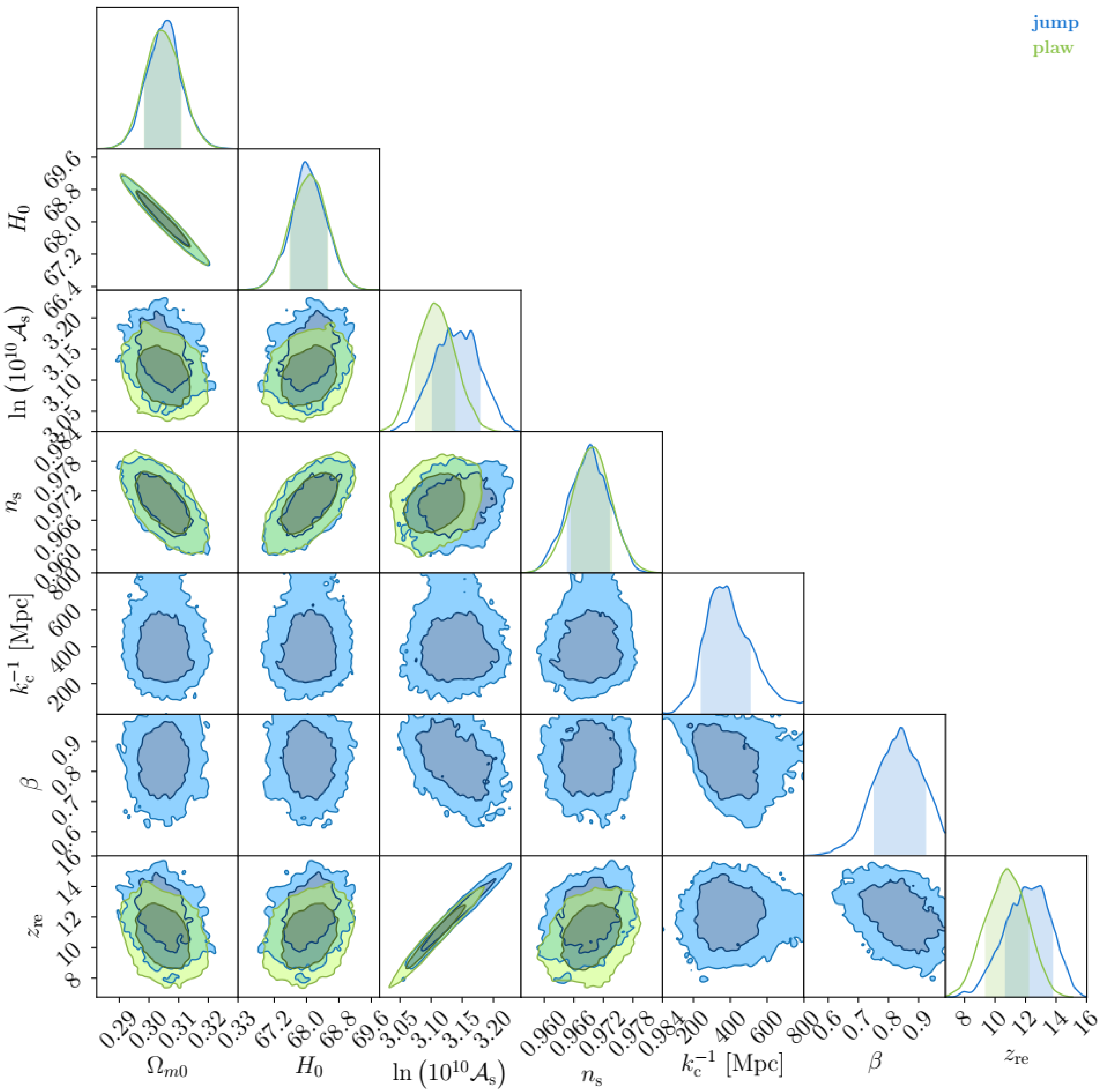
$$\xi(k) = \tan^{-1}\left[c_1\left(\frac{k}{k_\star}\right) + c_2\right] + c_3 - \frac{\pi}{2}$$

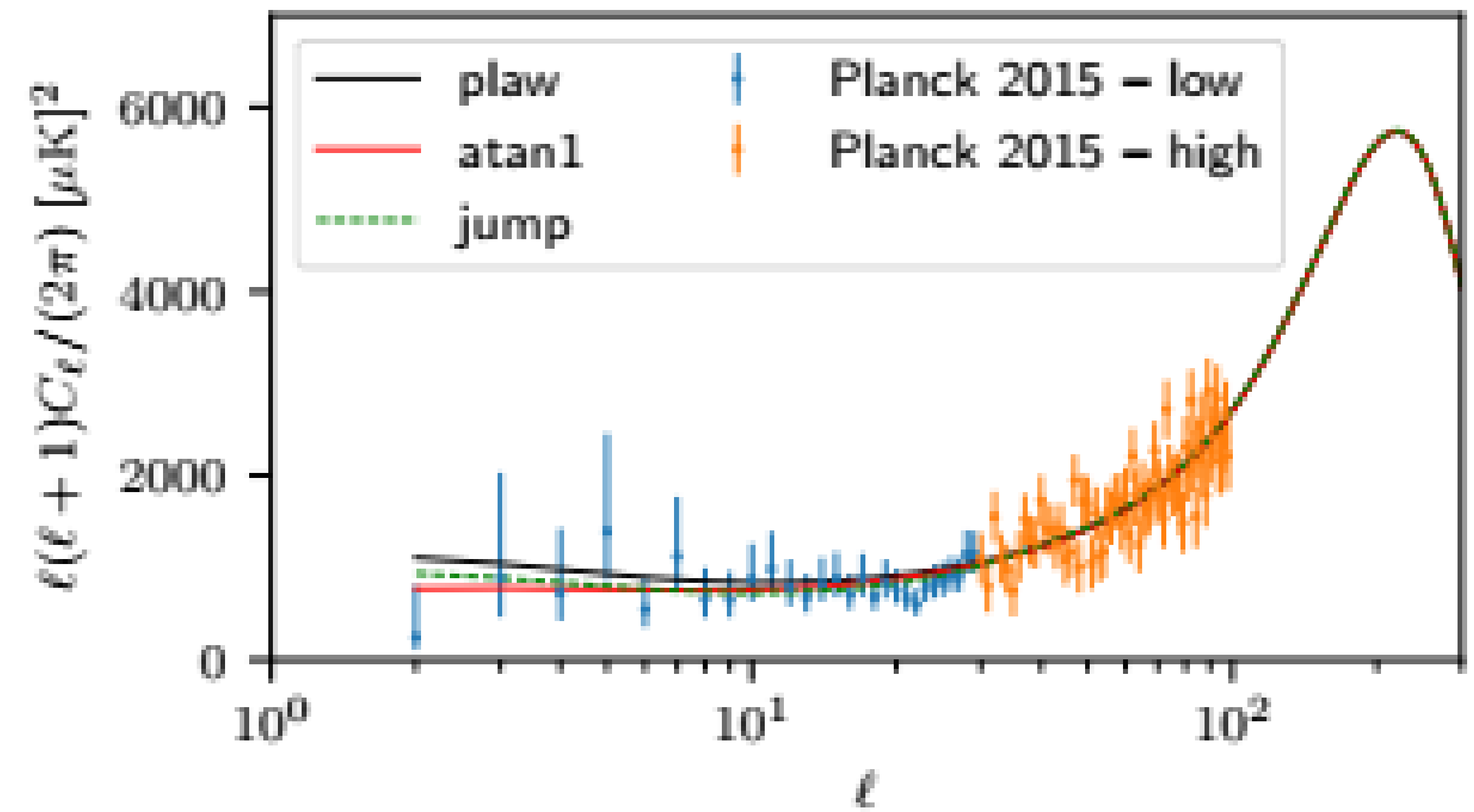
pivot scale

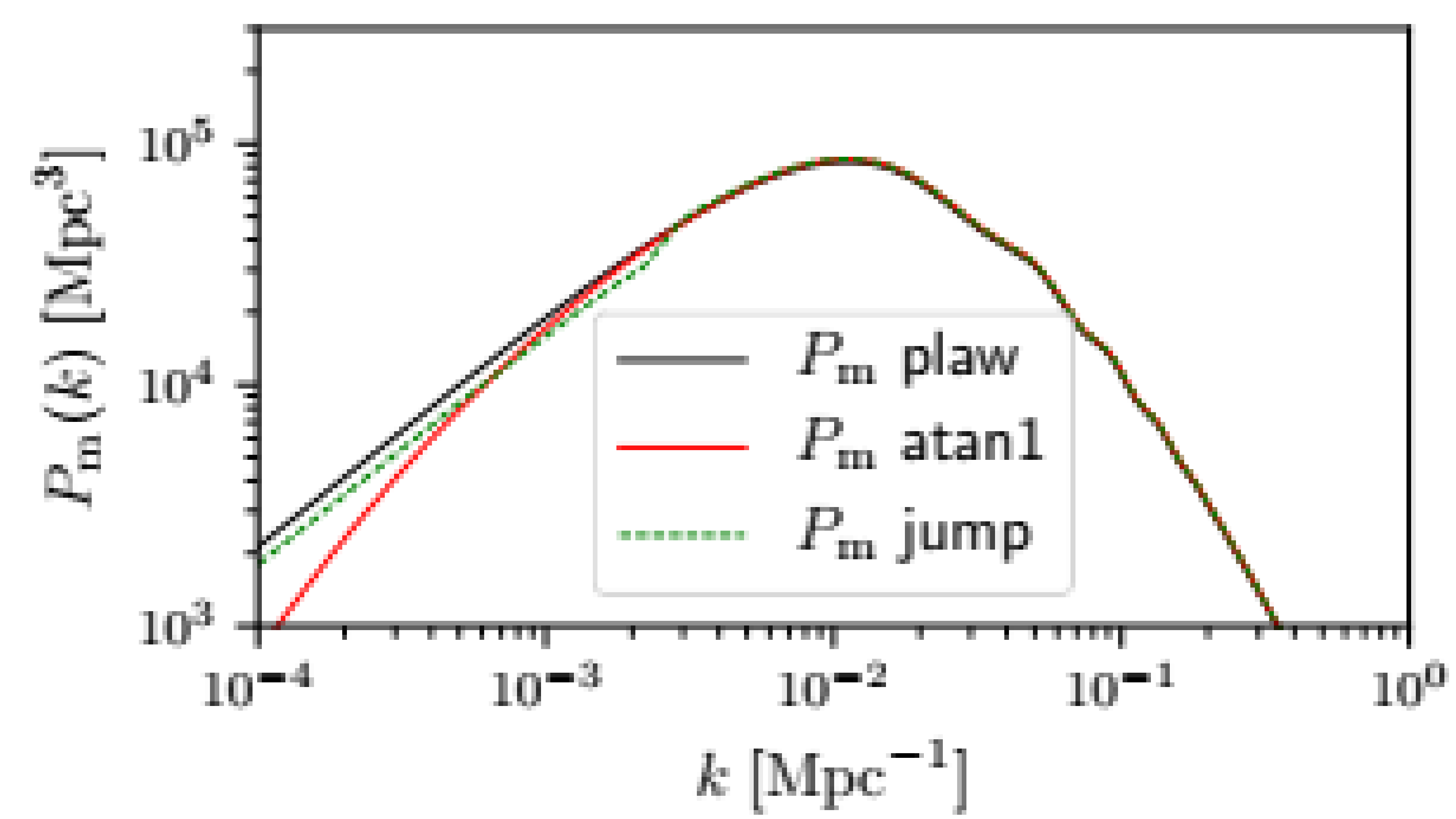


Better fit???

CosmoMC chains







Sandro D.P. Vitenti, PP and Antony Valentini, *Phys. Rev.* **D 100**, 043506 (2019).

w/ Sandro Vitenti & Przemysław Małkiewicz

Conclusions

- Internal clock formulation of QM & QC
- Clock issue in QC can be approached by WDW and set constraints on time
- Asymptotics may solve the problem… perturbations???
- Other trajectory approach = same asymptotics
- Out-of-Quantum-Equilibrium

↓

modified power spectrum

→ ***Planck best-fit…***

$\ell_{\mathrm{NEW}} \simeq 2000\mathrm{Mpc}$