



Primordial Black Holes from Quantum Diffusion during Cosmological Inflation

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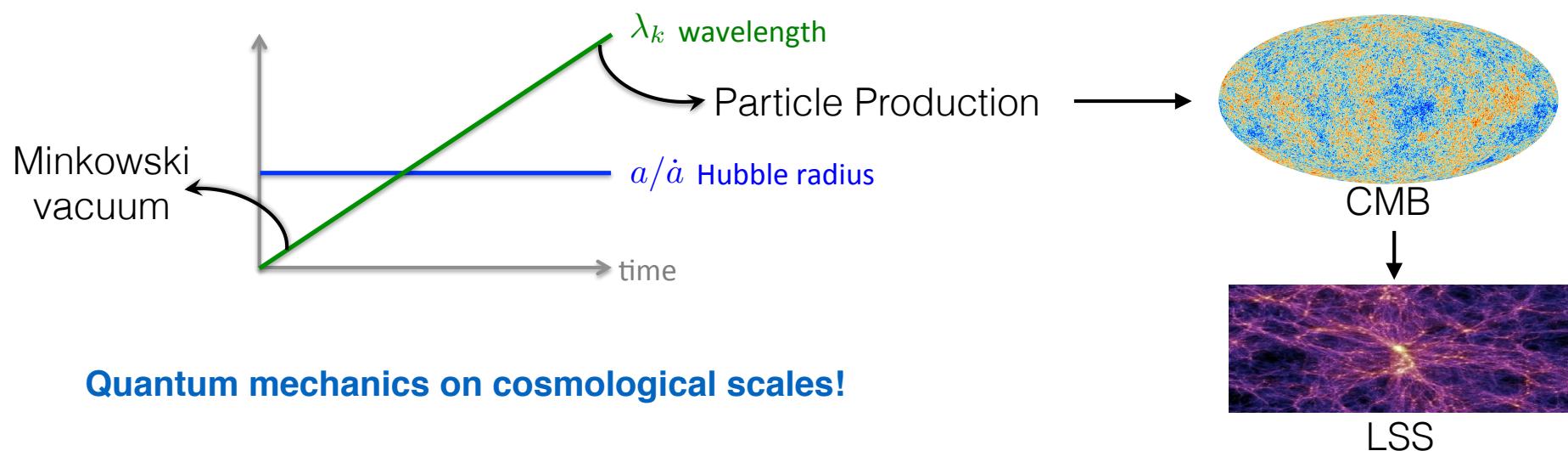
25 October 2019

Outline

- Primordial black holes as a probe of the end of inflation
- The quantum state of cosmological perturbations
- The stochastic δN formalism
- Primordial black holes & quantum diffusion

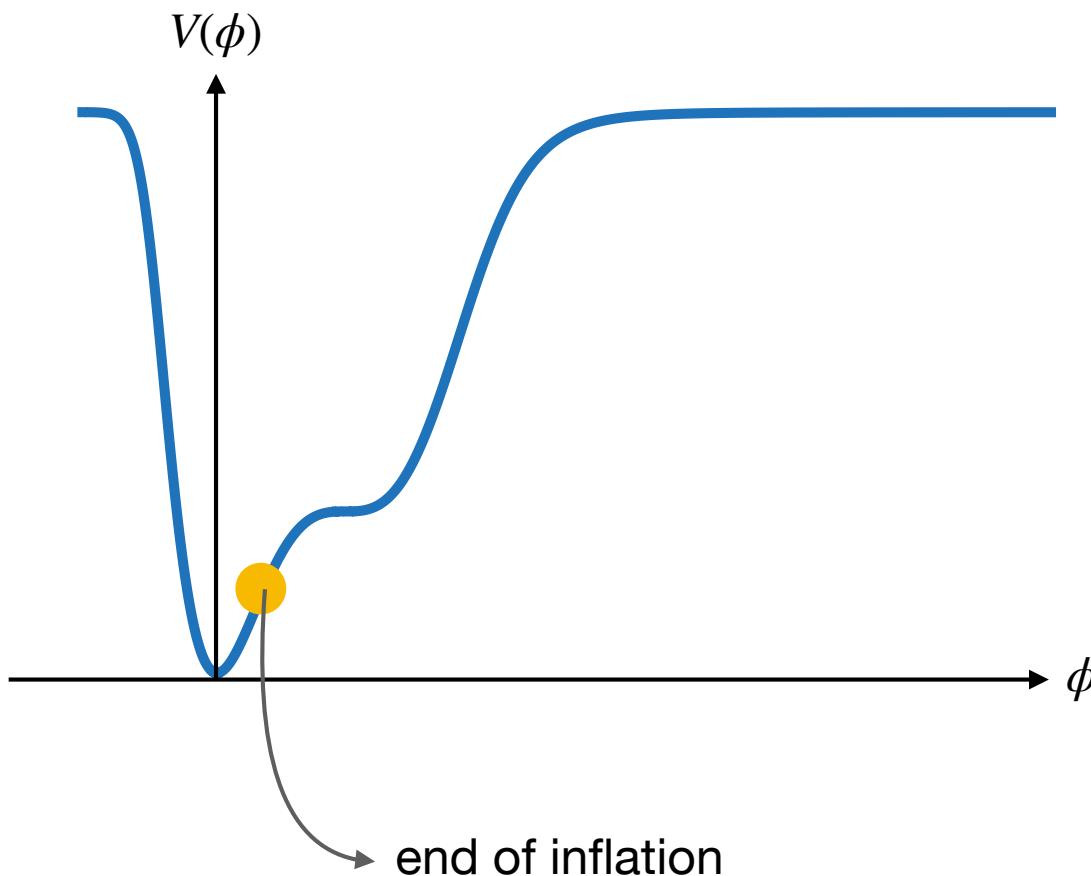
Cosmic Inflation: turning the primordial universe as a high-energy laboratory

- Inflation is a **high-energy** phase of accelerated expansion in the early Universe
$$ds^2 = -dt^2 + a^2(t) d\vec{x}^2 \quad \text{with} \quad \ddot{a} > 0 \quad \text{and} \quad \rho \leq (10^{16} \text{ GeV})^4$$
- Quantum vacuum fluctuations are stretched to cosmological distances and seed the large-scale structure of our Universe



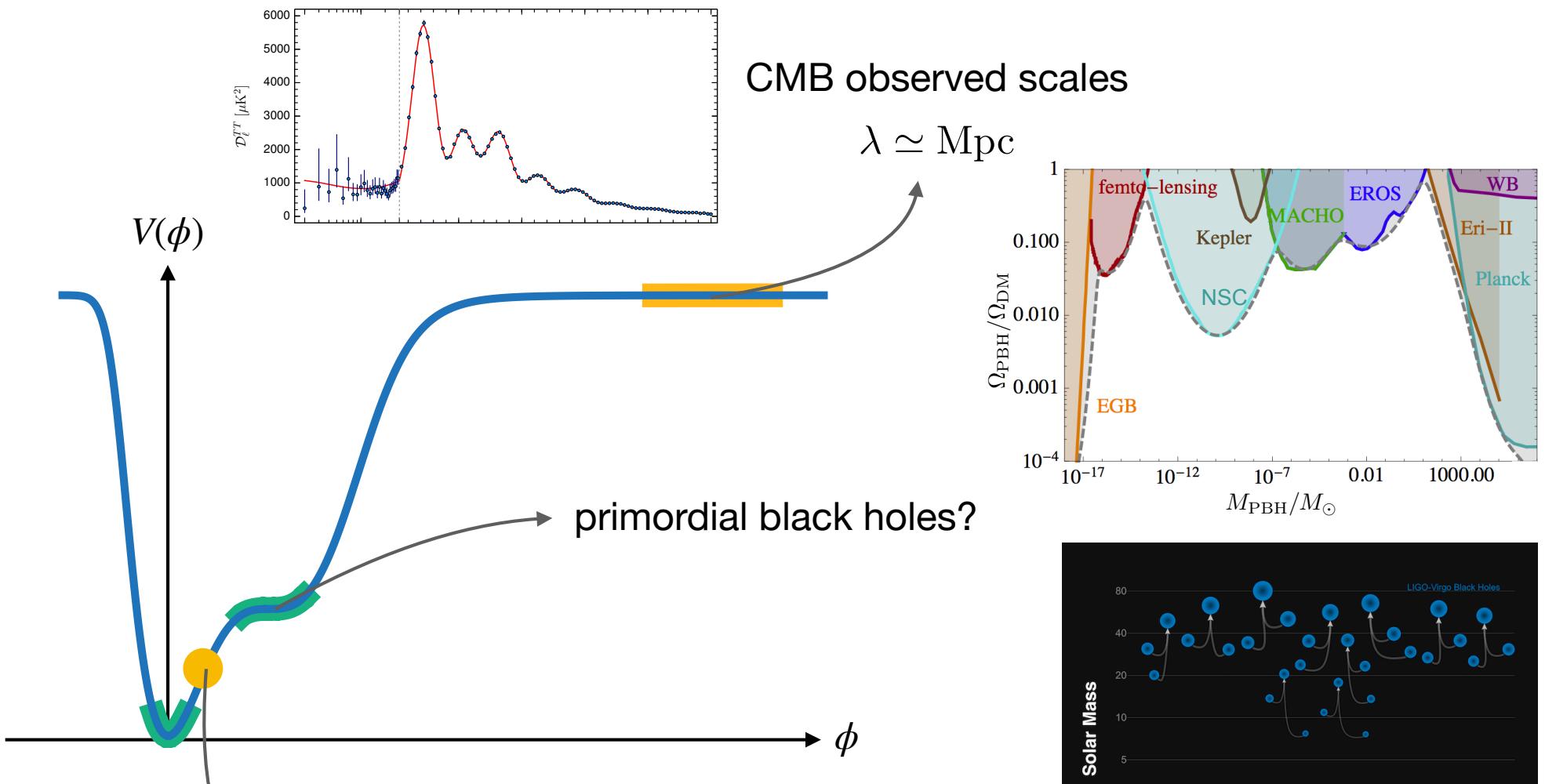
Primordial Black Holes as a probe of the end of inflation

Carr, Hawking 1974



Primordial Black Holes as a probe of the end of inflation

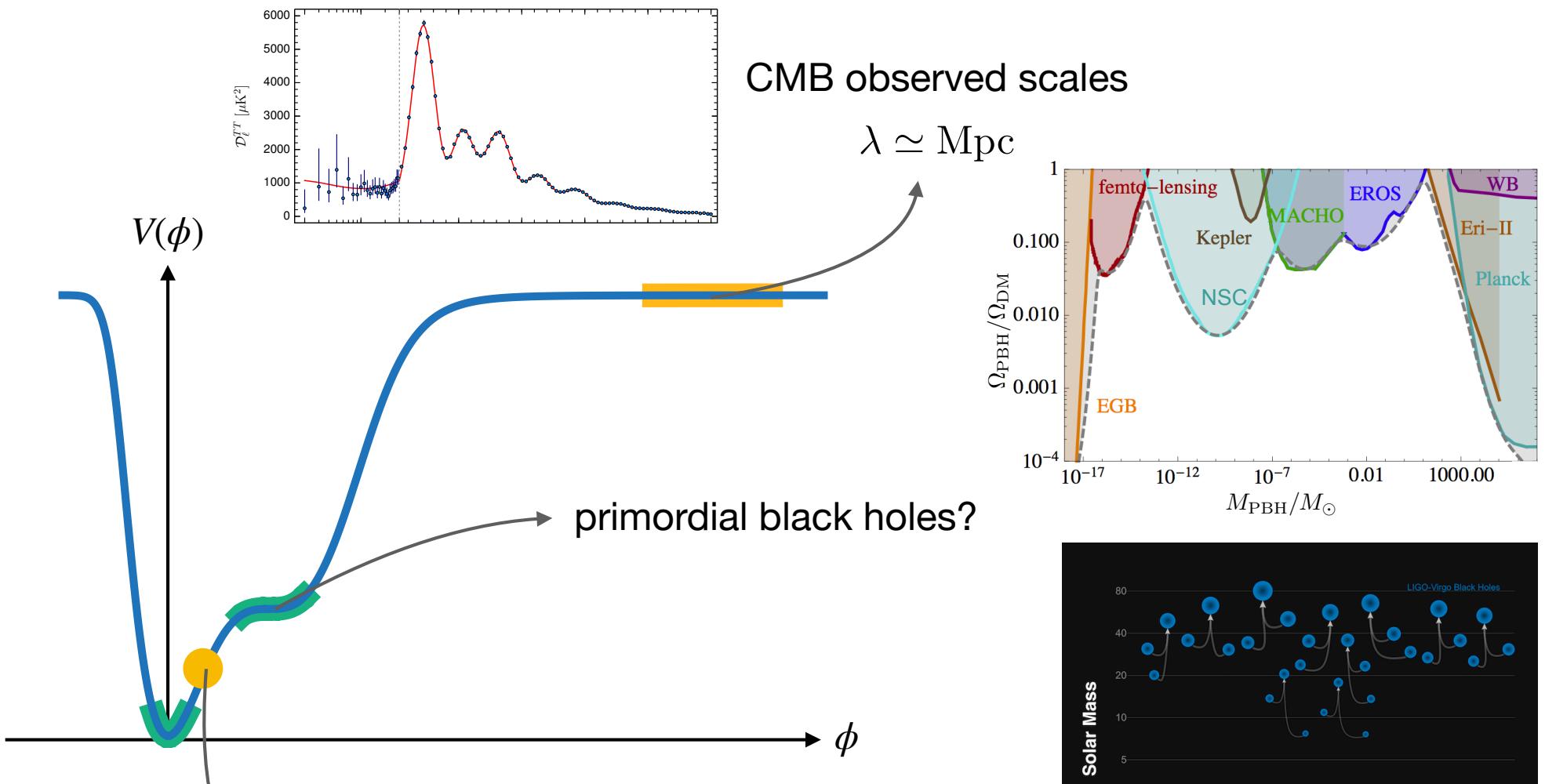
Carr, Hawking 1974



$$\frac{\lambda_{\text{end}}}{m} \simeq \frac{10^{16} \text{ GeV}}{\rho_{\text{end}}^{1/4}} R_{\text{rad}}^{-1}$$

Primordial Black Holes as a probe of the end of inflation

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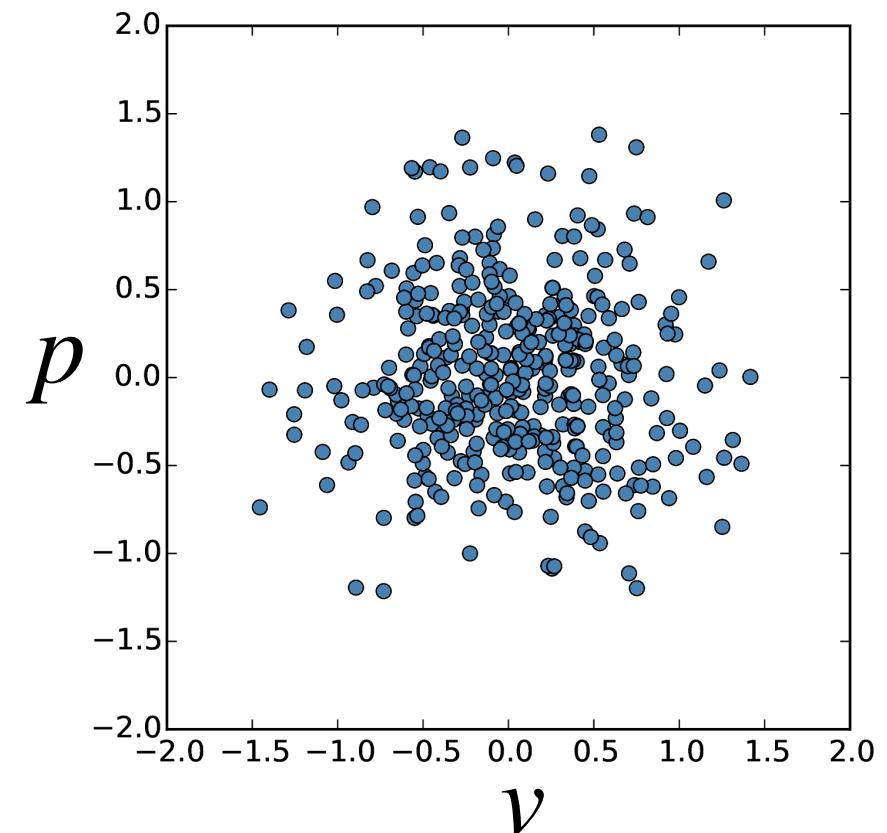
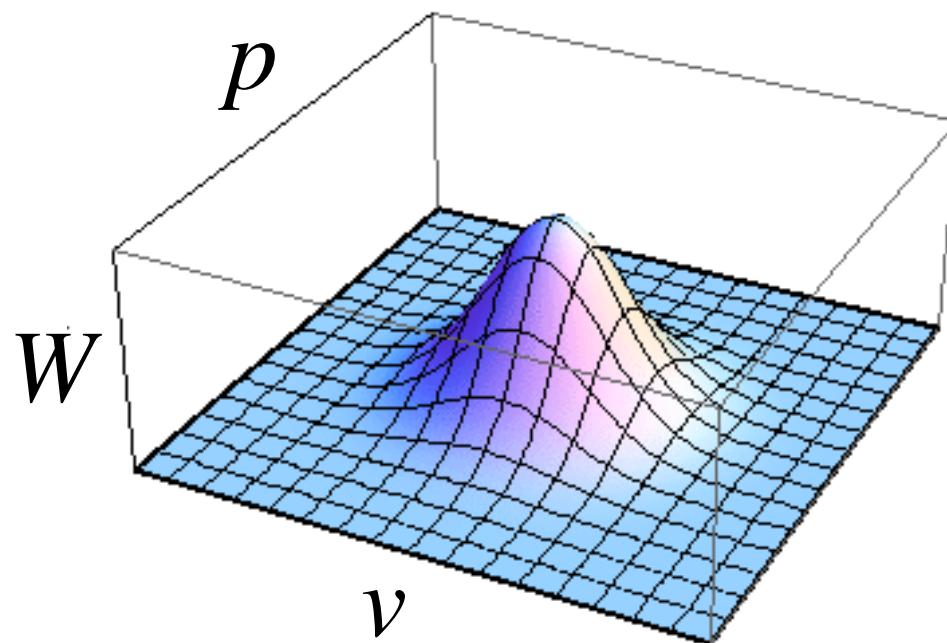
For arguments in favour of PBH DM, see e.g. García-Bellido and Clesse (1711.10458)

The quantum state of cosmological perturbations

- One scalar degree of freedom $v \propto \zeta$ (curvature perturbation) $\propto \delta T/T$
- $|\Psi\rangle = \bigotimes_{\mathbf{k} \in \mathbb{R}^{3+}} |\Psi_{\mathbf{k}}\rangle$ with $|\Psi_{\mathbf{k}}\rangle = \frac{1}{\cosh r_k} \sum_{n=0}^{\infty} e^{2in\varphi_k} (-1)^n \tanh^n r_k |n_{\mathbf{k}}, n_{-\mathbf{k}}\rangle$
Two-mode squeezed state (Gaussian state)
- Wigner function $W(v_{\mathbf{k}}, p_{\mathbf{k}}) = \int \frac{dx}{2\pi^2} \Psi^*(v_{\mathbf{k}} - \frac{x}{2}) e^{-ip_{\mathbf{k}}x} \Psi(v_{\mathbf{k}} + \frac{x}{2})$
- Evolution equation $\frac{\partial}{\partial t} W(v, p, t) = - \{W(v, p, t), H(v, p, t)\}_{\text{Poisson Bracket}}$
For quadratic Hamiltonians

The quantum state of cosmological perturbations

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For quadratic Hamiltonians

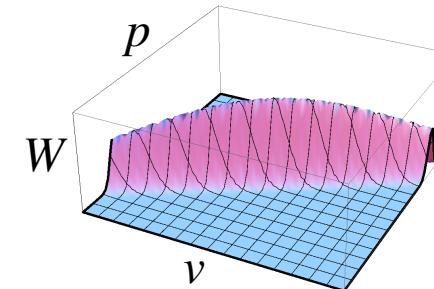


The quantum state of cosmological perturbations

- Quantum mean value and stochastic average

$$\langle \hat{\mathcal{O}}(\hat{v}, \hat{p}) \rangle_{\text{quant}} \neq \int W(v, p) \tilde{\mathcal{O}}(v, p) dv dp$$

Lesgourgues, Polarski, Starobinsky (1997)
Martin, VV (2016)



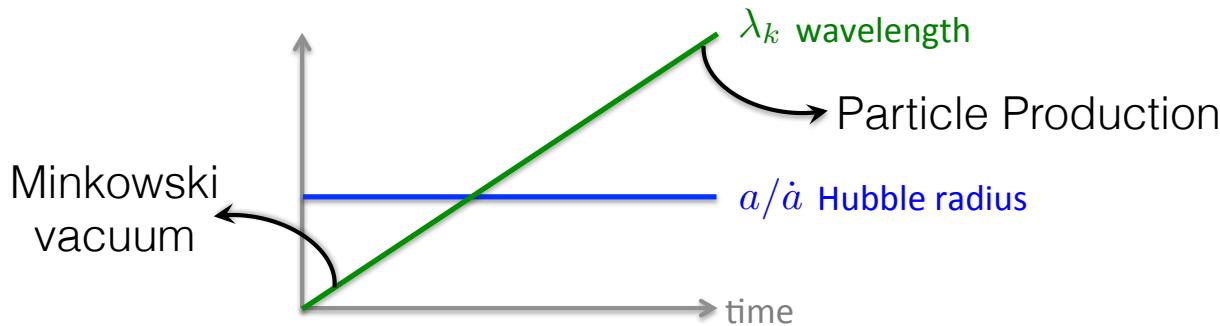
- True for $\mathcal{O}(\hat{v})$ and $\mathcal{O}(\hat{p})$
- True for Hermitian & quadratic $\mathcal{O}(\hat{v}, \hat{p})$
- True for proper $\mathcal{O}(\hat{v}, \hat{p})$ in the super-Hubble (large squeezing) limit

Example: $\hat{O} = v_k v_k^\dagger p_k p_k^\dagger + p_k p_k^\dagger v_k v_k^\dagger \longrightarrow \langle \hat{O} \rangle_{\text{quant}} = \langle \hat{O} \rangle_{\text{stoch}} + 1/4 e^{2(N - N_{\text{Hubble crossing}})}$

- Wrong for improper operators, even in the large-squeezing limit

Revzen (2006), Martin, VV (2017)

Stochastic Inflation



Quantum fluctuations
source the background

Coarse-grained field $\hat{\phi}_{\text{coarse grained}} = \int_{k < \sigma a H(N)} d^3 k \left[\phi_{\mathbf{k}}(N) e^{-i \mathbf{k} \cdot \mathbf{x}} \hat{a}_{\mathbf{k}} + \phi_{\mathbf{k}}^*(N) e^{i \mathbf{k} \cdot \mathbf{x}} \hat{a}_{\mathbf{k}}^\dagger \right]$

 $N = \ln(a)$

At leading order in slow roll: $\frac{d}{dN} \phi_{cg} = -\frac{V'(\phi_{cg})}{3H^2(\phi_{cg})} + \frac{H(\phi_{cg})}{2\pi} \xi(N)$

Starobinsky, 1986

Over one e-fold: $\frac{\Delta \phi_{\text{quant}}}{\Delta \phi_{\text{class}}} \sim \zeta$

Primordial black holes from inflation

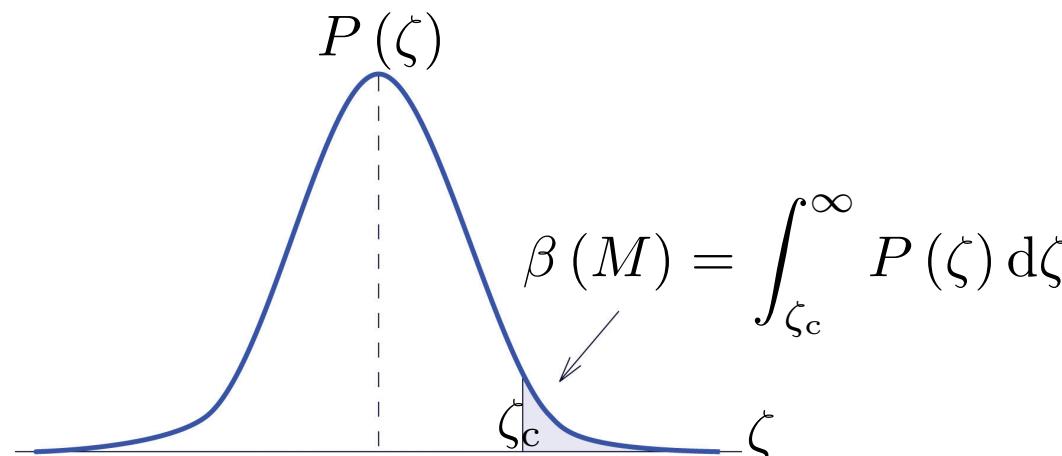
Primordial density perturbations when modes
re-enter the Hubble radius after inflation

$$\left. \frac{\delta\rho}{\rho} \right|_{k=aH} \sim \zeta \sim \frac{\Delta\phi_{\text{qu}}}{\Delta\phi_{\text{cl}}}$$

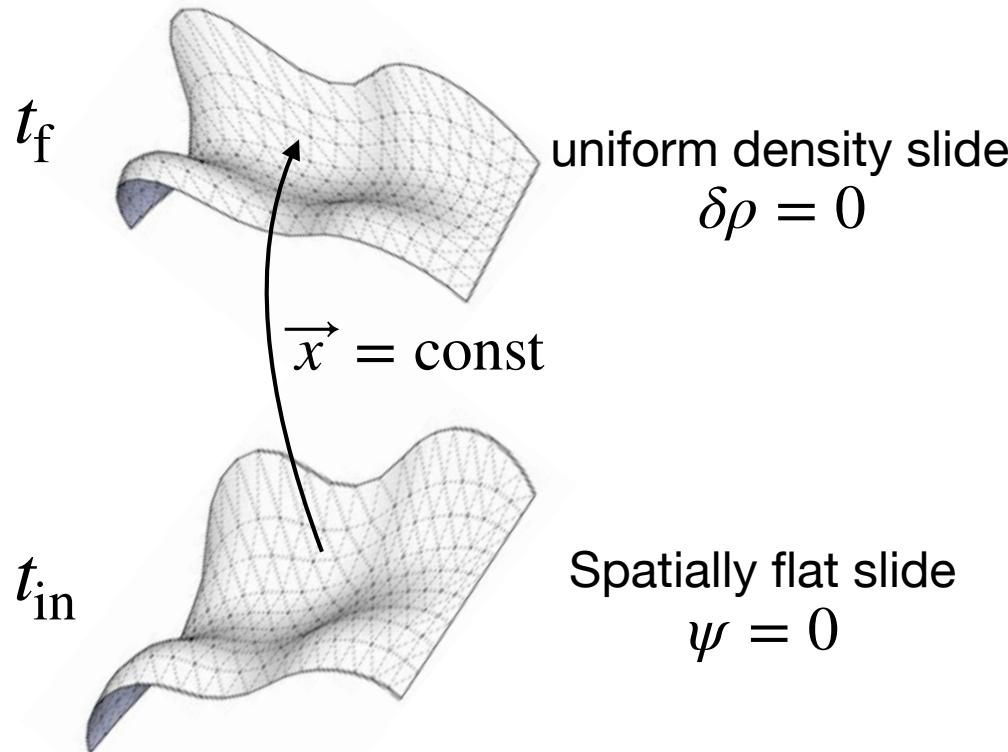
Rare fluctuations exceeding critical value
collapse to form black holes

$$\zeta > \zeta_c \sim 1$$

Mass fraction
 $\beta(M) < 10^{-8}$



Stochastic- δN formalism

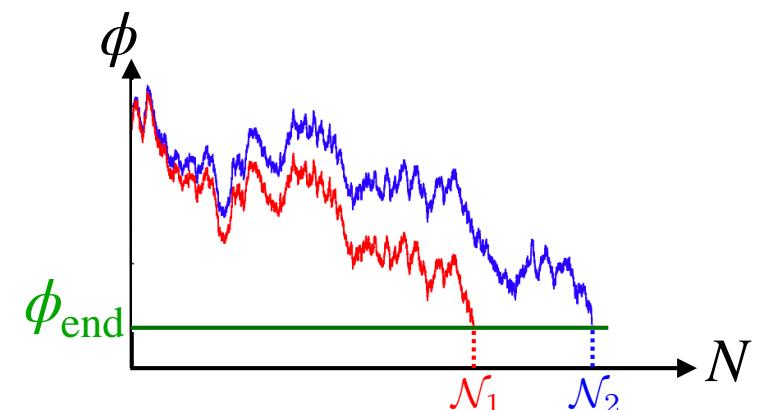


$$\zeta(t, x) = N(t, x) - N_0(t) \equiv \delta N$$

Wands, Malik, Lyth, Liddle (2000)

The realised number of e-folds
is a stochastic quantity:

$$\zeta_{\text{coarse grained}} = \mathcal{N} - \langle \mathcal{N} \rangle$$



Stochastic- δN formalism

Moments obey an interactive equation VV, Starobinsky (2015)

$$v = V/(24\pi^2 M_{\text{Pl}}^4)$$

$$\langle \mathcal{N}^n \rangle''(\phi) - \frac{v'}{v^2} \langle \mathcal{N}^n \rangle'(\phi) = - \frac{n}{v M_{\text{Pl}}^2} \langle \mathcal{N}^{n-1} \rangle(\phi)$$

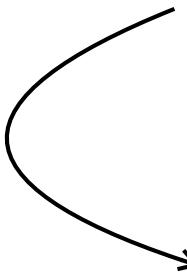
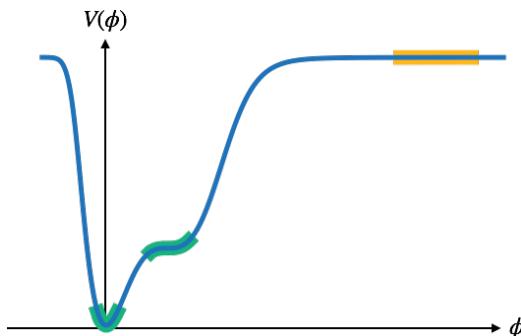
Mean number of e-folds

$$\langle \mathcal{N} \rangle(\phi) = \int_{\phi_{\text{end}}}^{\phi} \frac{dx}{M_{\text{Pl}}} \int_x^{\phi_{\text{UV}}} \frac{dy}{M_{\text{Pl}}} \frac{e^{\frac{1}{v(y)} - \frac{1}{v(x)}}}{v(y)}$$

Involves the full inflationary domain

Saddle-point expansion

$$v \ll 1, |v^2 v''/v'^2| \ll 1$$



$$\simeq \int_{\phi_{\text{end}}}^{\phi} \frac{dx}{M_{\text{Pl}}^2} \frac{v(x)}{v'(x)} \left[1 + v(x) - \frac{v''(x)v^2(x)}{v'^2(x)} + \dots \right]$$

classical result

first-order
correction

Stochastic- δN formalism

Second moment and power spectrum VV, Starobinsky (2015)

$$\mathcal{P}_\zeta(\phi) = 2 \frac{\int_\phi^{\phi_{UV}} \frac{dx}{M_{Pl}} \left\{ \int_x^{\phi_{UV}} \frac{dy}{M_{Pl}} \frac{1}{v(y)} \exp \left[\frac{1}{v(y)} - \frac{1}{v(x)} \right] \right\}^2 \exp \left[\frac{1}{v(x)} - \frac{1}{v(\phi)} \right]}{\int_\phi^{\phi_{UV}} \frac{dx}{M_{Pl}} \frac{1}{v(x)} \exp \left[\frac{1}{v(x)} - \frac{1}{v(\phi)} \right]}$$

Saddle-point
expansion
 $v \ll 1, |v^2 v''/v'^2| \ll 1$

$$\simeq \frac{2}{M_{Pl}^2} \frac{v^3(\phi)}{v'^2(\phi)} \left[1 + 5v(\phi) - 4 \frac{v^2(\phi) v''(\phi)}{v'^2(\phi)} + \dots \right]$$

Third moment and local non-Gaussianity

$$f_{NL} = \frac{5}{24} M_{Pl}^2 \left[6 \frac{v'^2}{v^2} - 4 \frac{v''}{v} + v \left(11 \frac{v'^2}{v^2} - 158 \frac{v''}{v} - 9 \frac{v'''}{v'} + 118 \frac{v''^2}{v'^2} \right) + \dots \right]$$

Need for the full PDF

$$\frac{d\phi}{dN} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\xi(N) \quad \longrightarrow \quad \frac{d}{dN}P(\phi, N) = \frac{\partial}{\partial\phi}\left(\frac{V'}{3H^2}P\right) + \frac{\partial^2}{\partial\phi^2}\left(\frac{H^2}{8\pi^2}P\right) = \mathcal{L}_\phi \cdot P$$

Langevin equation

Fokker-Planck equation

Moment equation

$$\langle \mathcal{N}^n \rangle''(\phi) - \frac{v'}{v^2} \langle \mathcal{N}^n \rangle'(\phi) = -\frac{n}{vM_{\text{Pl}}^2} \langle \mathcal{N}^{n-1} \rangle(\phi) \quad \longleftrightarrow \quad \mathcal{L}_\phi^\dagger \cdot \langle \mathcal{N}^n \rangle = -n \langle \mathcal{N}^{n-1} \rangle$$

$$\int (f \mathcal{L}_\phi \cdot g) d\phi = \int (g \mathcal{L}_\phi^\dagger \cdot f) d\phi$$

Equation for the PDF of the first passage time

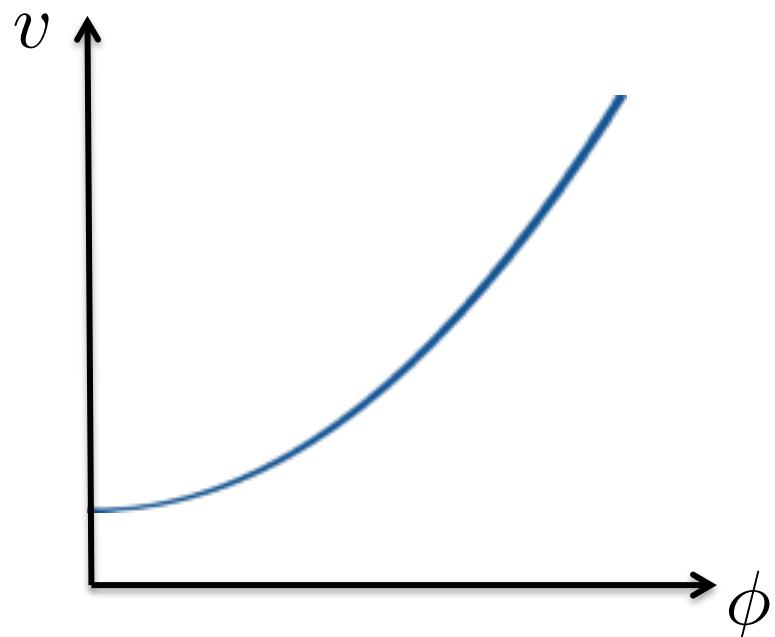
Pattison, VV, Assadullahi, Wands (2017)

$$\frac{d}{dN} \mathcal{P}(N, \phi) = \mathcal{L}_\phi^\dagger \cdot \mathcal{P}$$

Example

Pattison, VV, Assadullahi, Wands (2017)

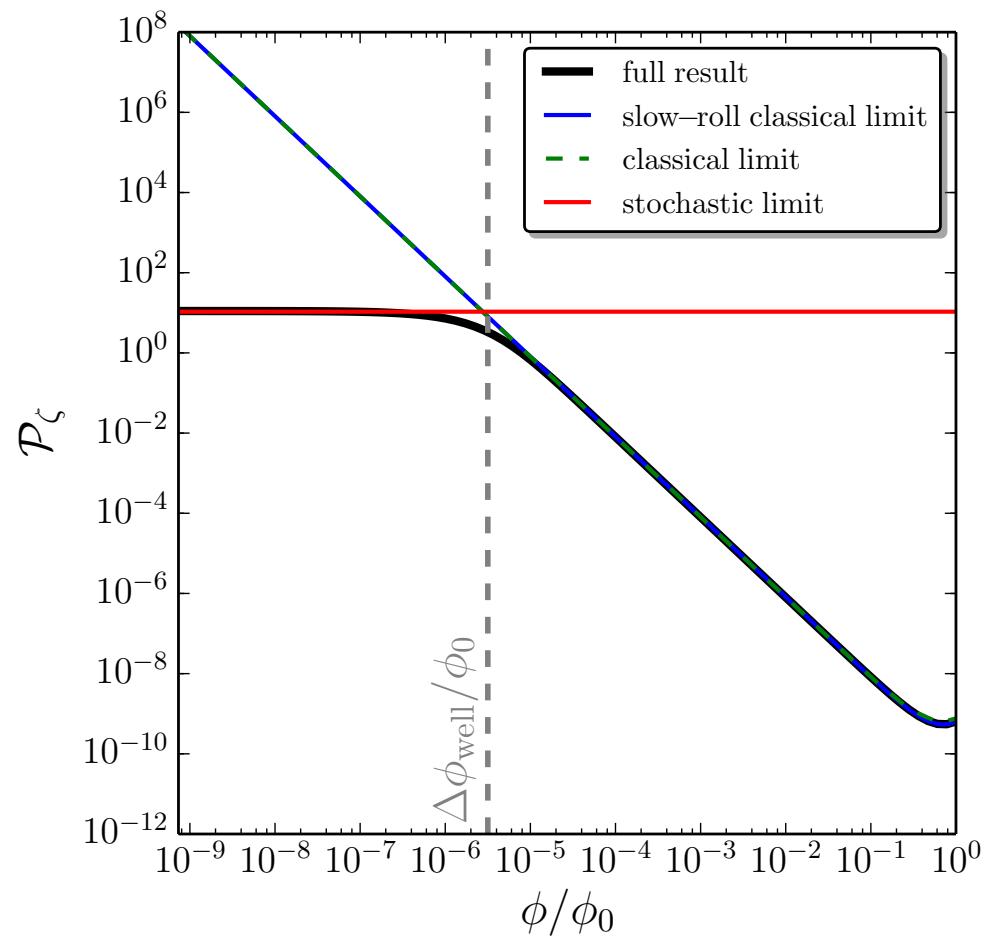
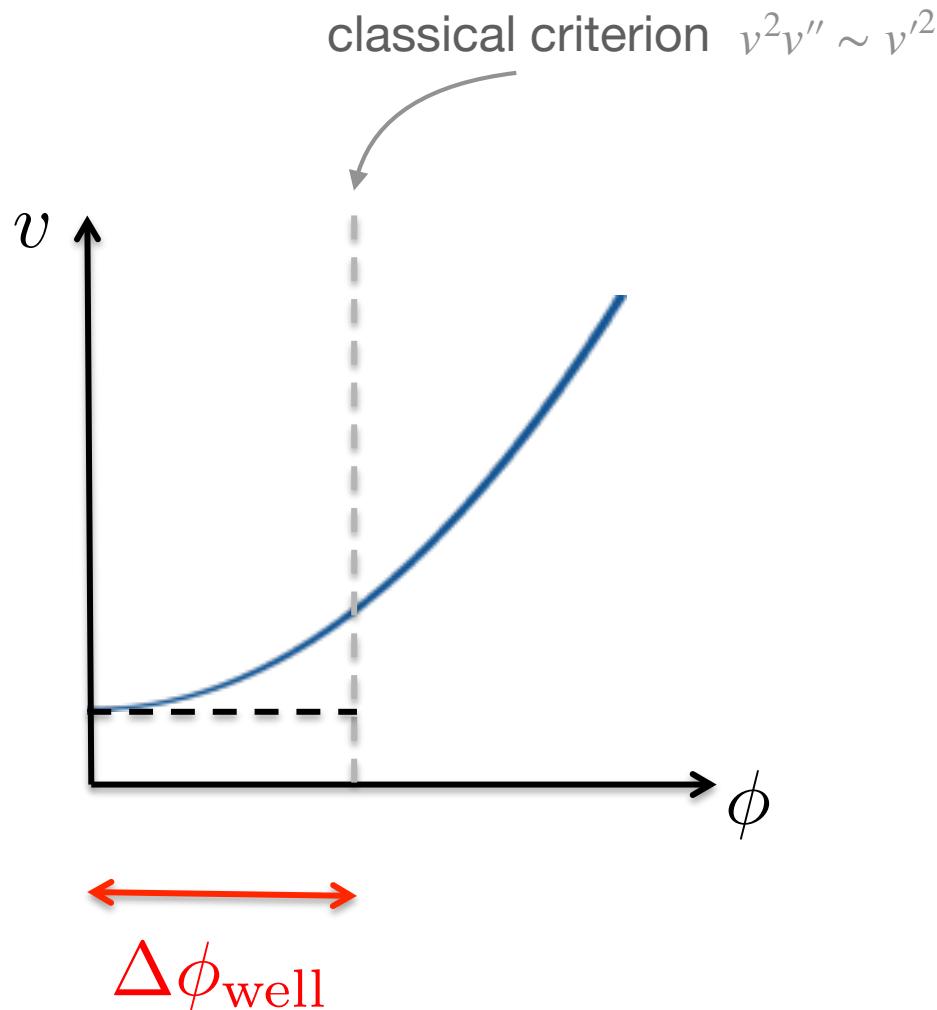
$$v(\phi) = v_0 \left[1 + \left(\frac{\phi}{\phi_0} \right)^p \right]$$



Example

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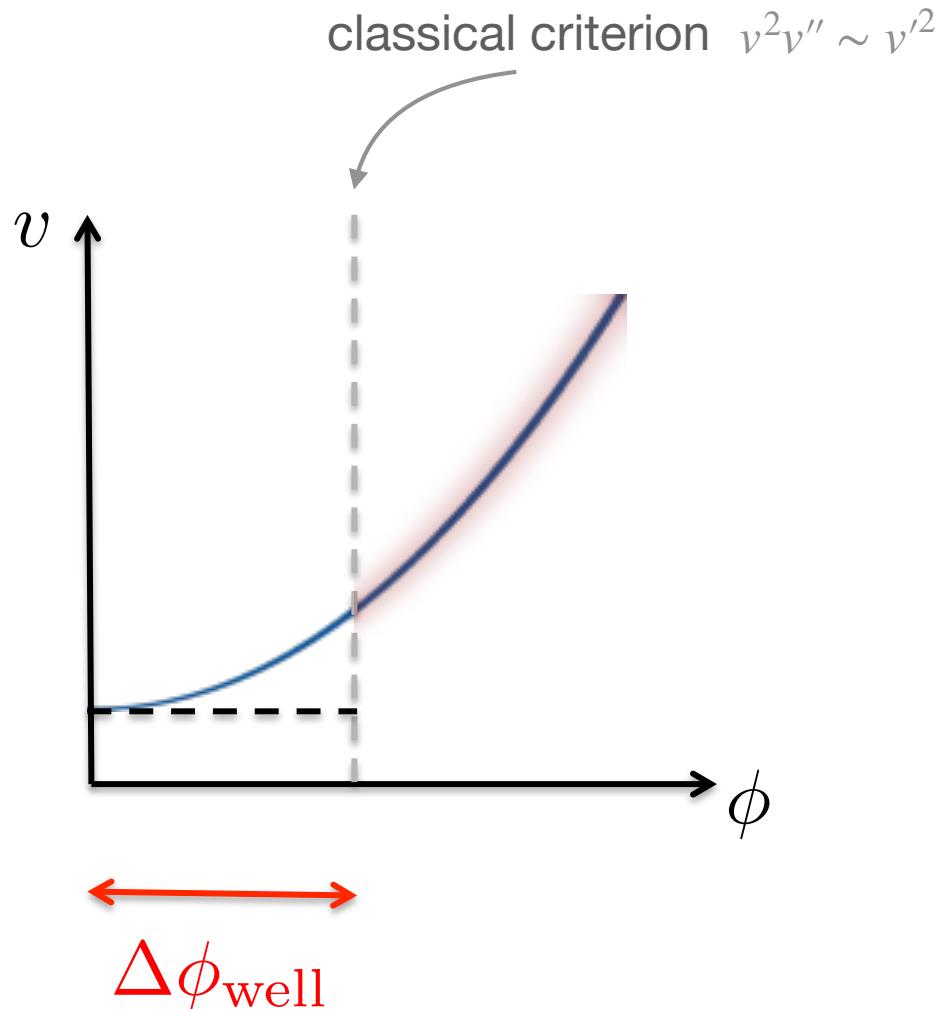
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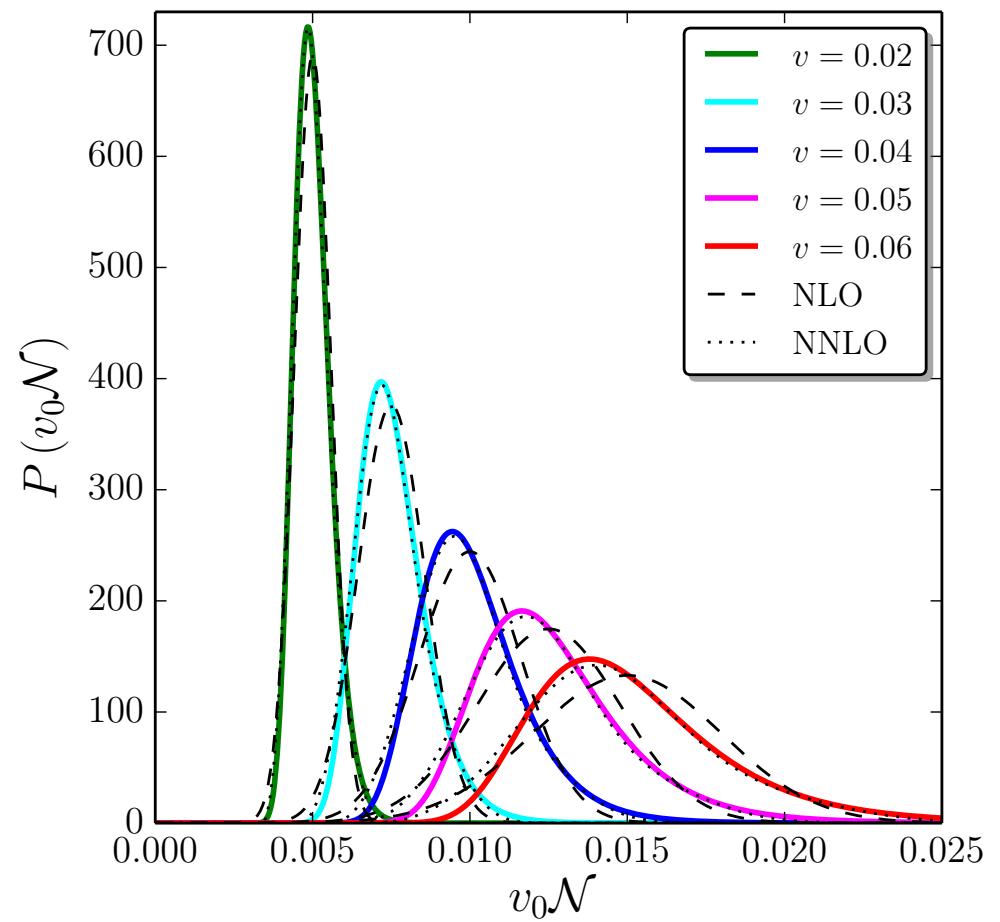
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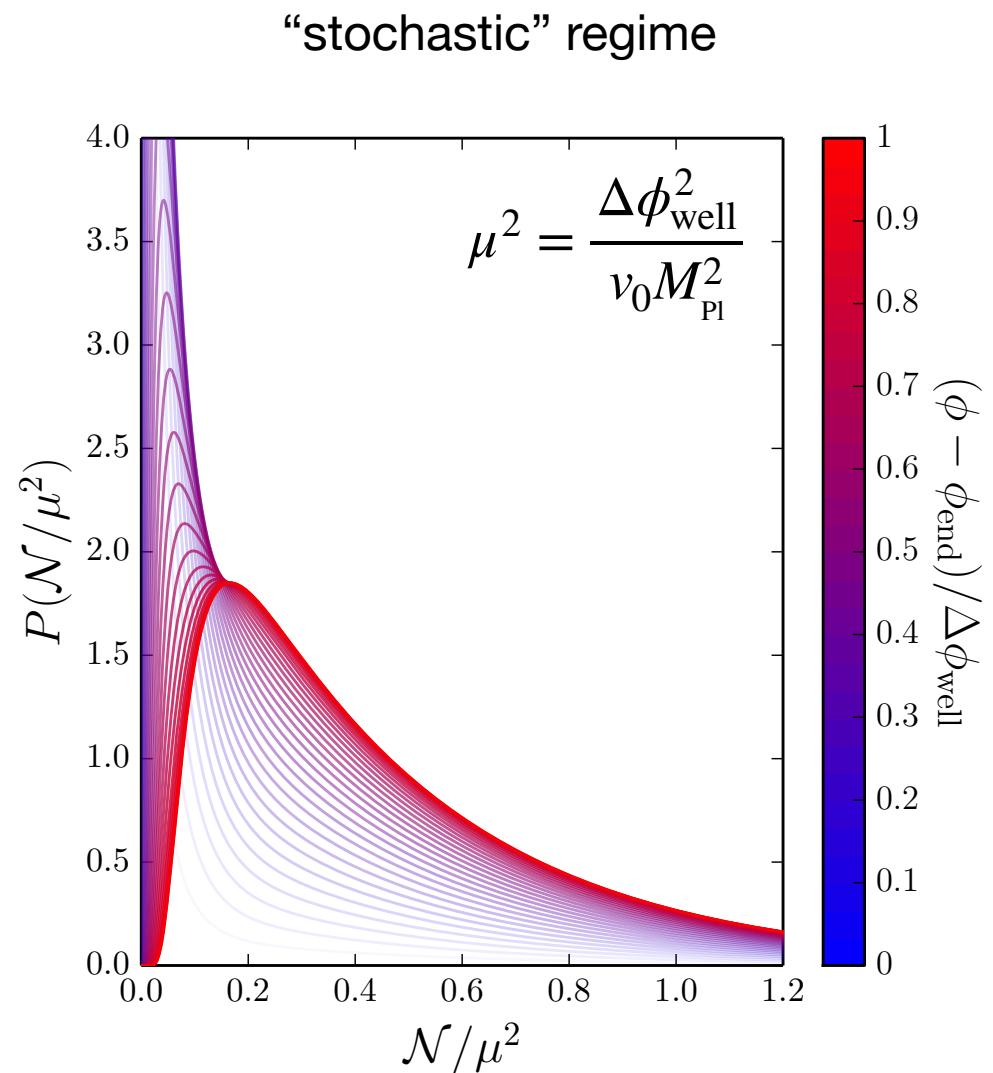
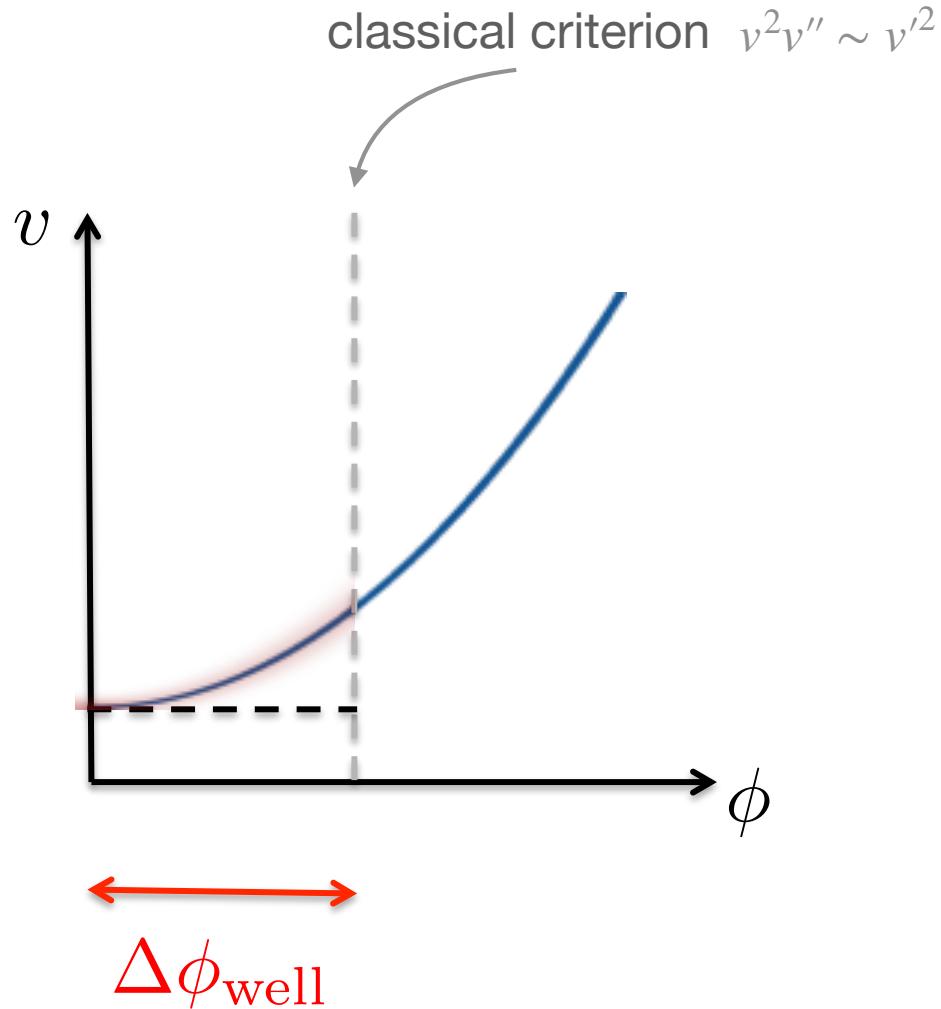
“classical” regime
Is the Gaussian approximation sufficient?



Example

Pattison, VV, Assadullahi, Wands (2017)

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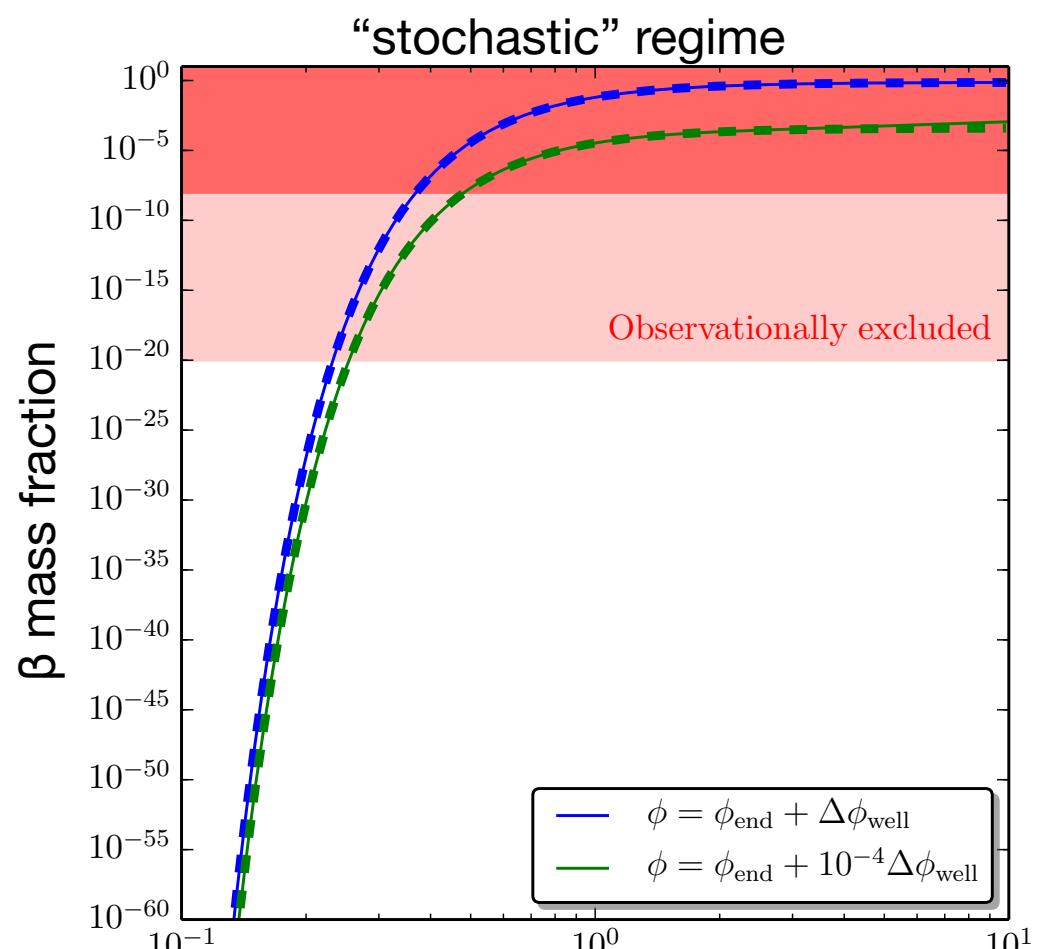
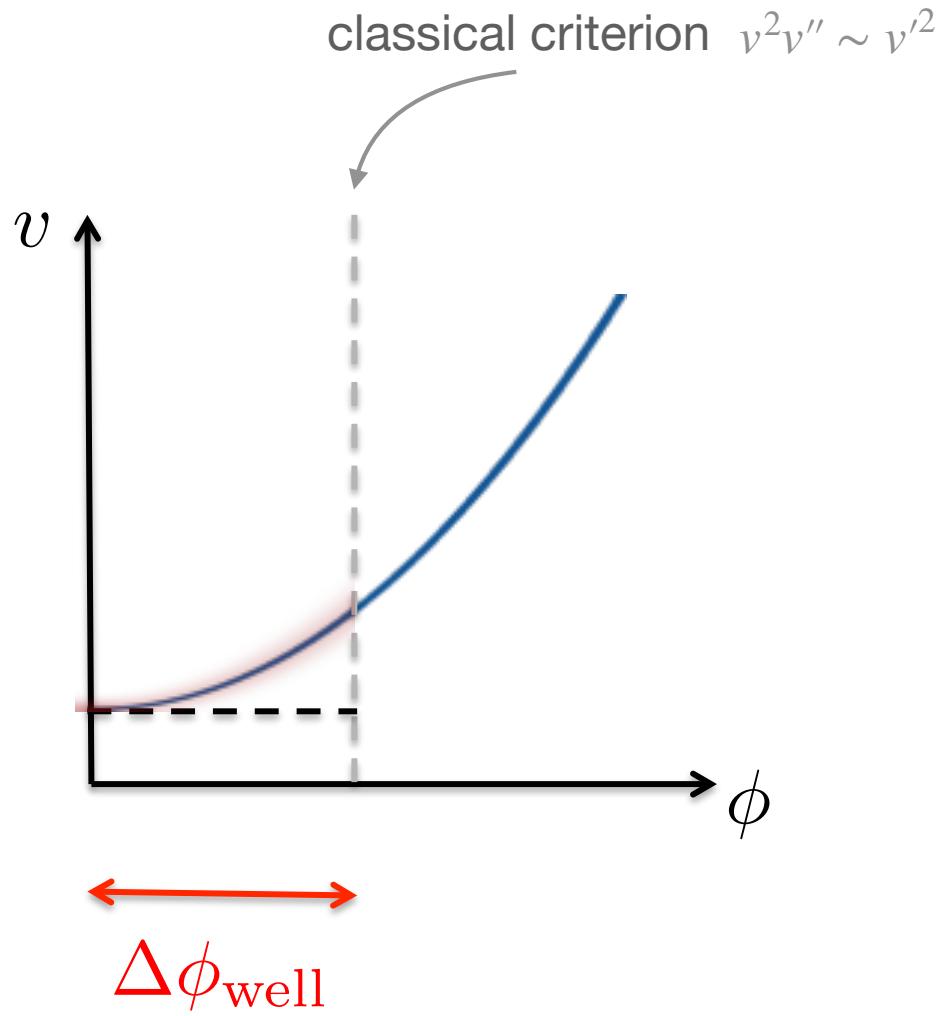


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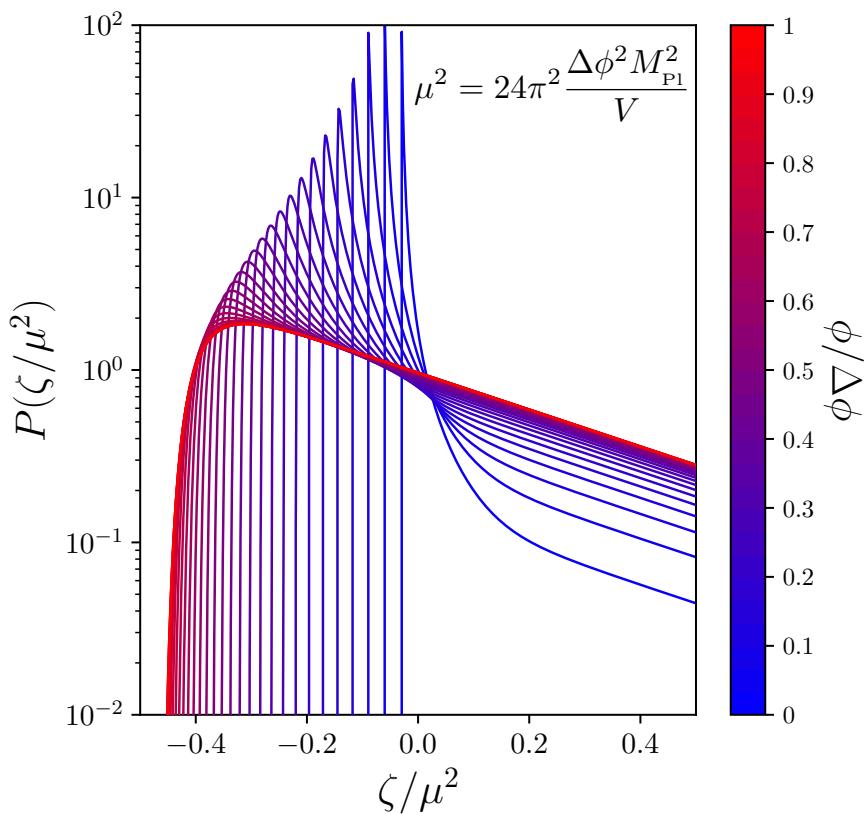
Remark: $\langle \mathcal{N} \rangle = \mu^2 \frac{\phi}{\Delta\phi} \left(1 - \frac{\phi}{2\Delta\phi} \right)$



Non-Gaussian Tails

Work in progress with Ezquiaga & Garcia Bellido

stochastic regime



exponential tails

In general: characteristic function

$$\chi(t, \phi) = \langle e^{it\mathcal{N}(\phi)} \rangle = \int e^{it\mathcal{N}} \mathcal{P}(\mathcal{N}, \phi) d\mathcal{N}$$

Ordinary differential equation

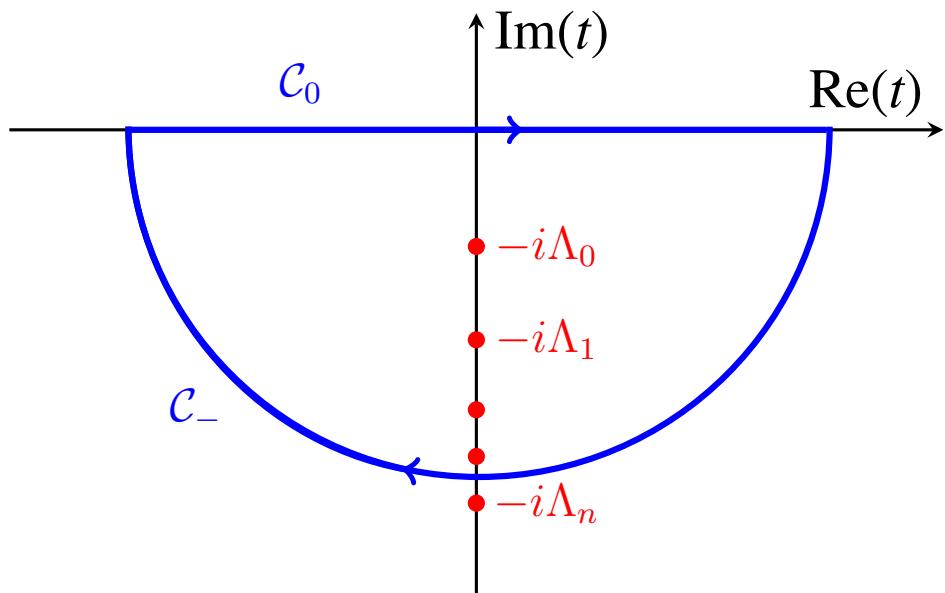
$$\mathcal{L}_\phi^\dagger \cdot \chi = -it\chi$$

Back to the PDF

$$\mathcal{P}(\mathcal{N}, \phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\mathcal{N}} \chi(t, \phi) dt$$

Non-Gaussian Tails

Work in progress with Ezquiaga & Garcia Bellido



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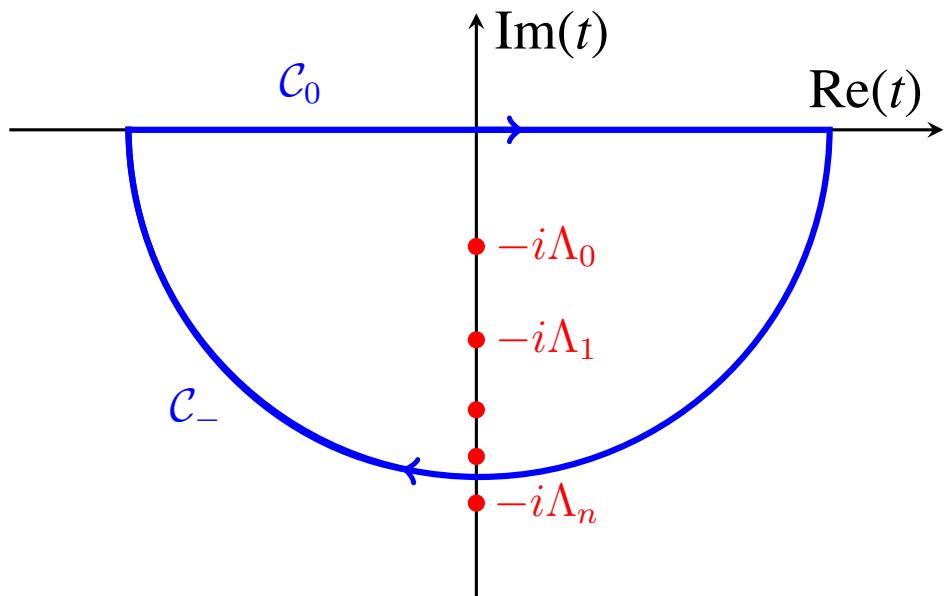
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Back to the PDF

$$\begin{aligned} \mathcal{P}(\mathcal{N}, \phi) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\mathcal{N}} \chi(t, \phi) dt \\ &= \sum_n a_n e^{-\Lambda_n \mathcal{N}} \end{aligned}$$

Non-Gaussian Tails

Work in progress with Ezquiaga & Garcia Bellido



Flat well

$$\Lambda_n = \frac{\pi^2}{\mu^2} \left(n + \frac{1}{2} \right)^2$$

Constant slope well

$$v = v_0 \left(1 + \alpha \frac{\phi}{M_{\text{Pl}}} \right)$$

$$\Lambda_n \simeq \frac{\alpha^2}{4v_0} + \frac{\pi^2}{\mu^2} \left(n + \frac{1}{2} \right)^2$$

Cubic inflection point

$$v = v_0 \left(1 + \alpha \frac{\phi^3}{M_{\text{Pl}}^3} \right)$$

$$\Lambda_n \simeq \left(\frac{3}{2} \right)^{2/3} \pi^2 (v_0 \alpha)^{1/3} \left(n + \frac{1}{2} \right)^2$$

Conclusions

- Primordial black holes can be seeded by large density perturbations that form during inflation
- When this happens, quantum diffusion, that is, the back-reaction of vacuum quantum fluctuations on the background dynamics as they get amplified and stretched to large distances, may play an important role
- Regions of the potential dominated by stochastic diffusion can be identified with a simple criterion: $v'^2 < v^2 v''$
- In stochastically-dominated regions, the system must spend less than one e-fold for PBHs not to be too abundantly produced
- Even in classically-dominated regions, the standard calculations may fail because of non-Gaussian tails, which can only be described with non-perturbative techniques such as the stochastic- δN formalism
- Extensions to non-slow roll & multiple field dynamics in progress