Ballade géométrique au pays des groupes de type fini

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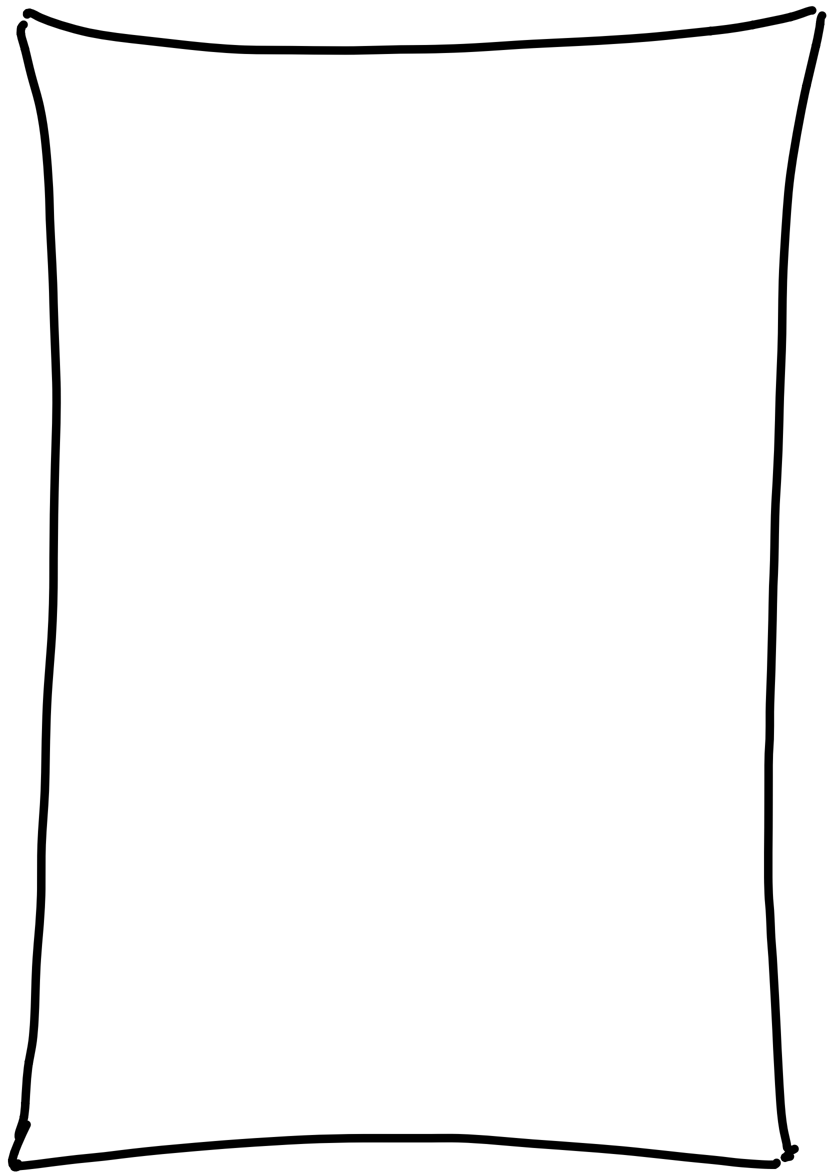
Indira Chatterji

Un groupe G

Un groupe G

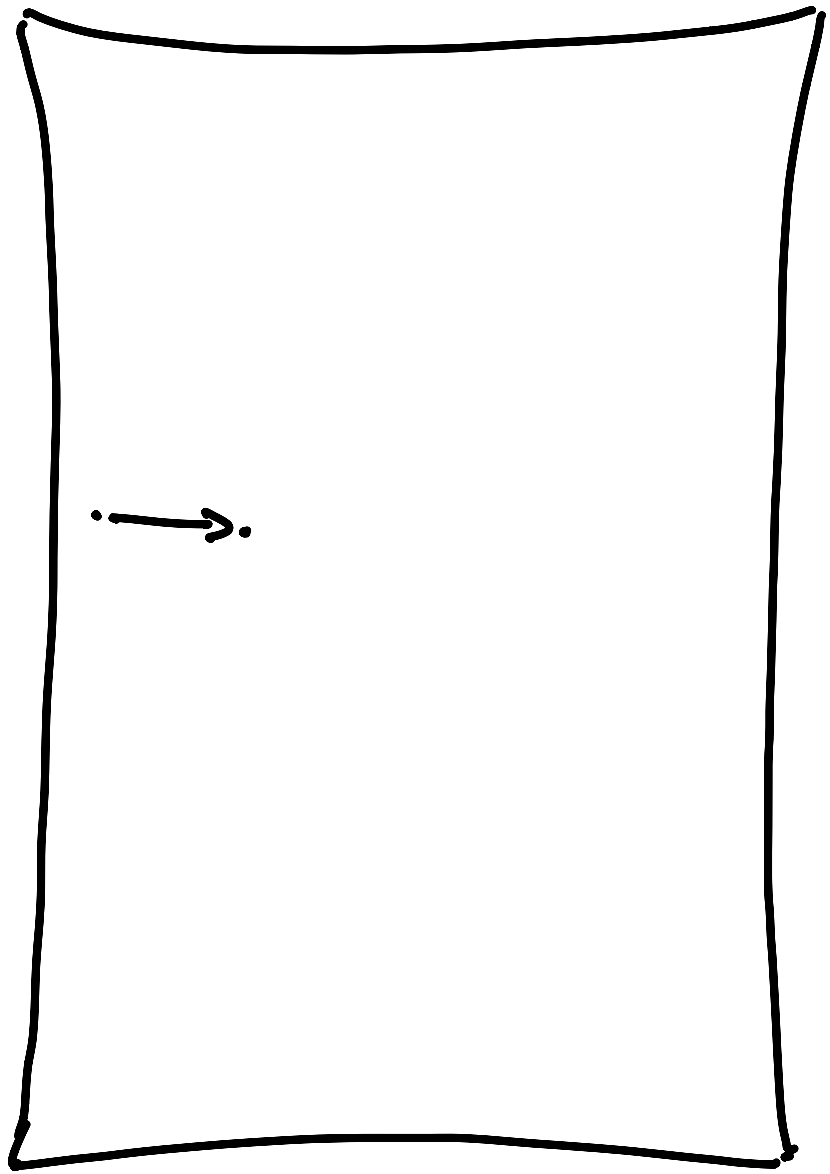
ensemble



Un groupe G

ensemble

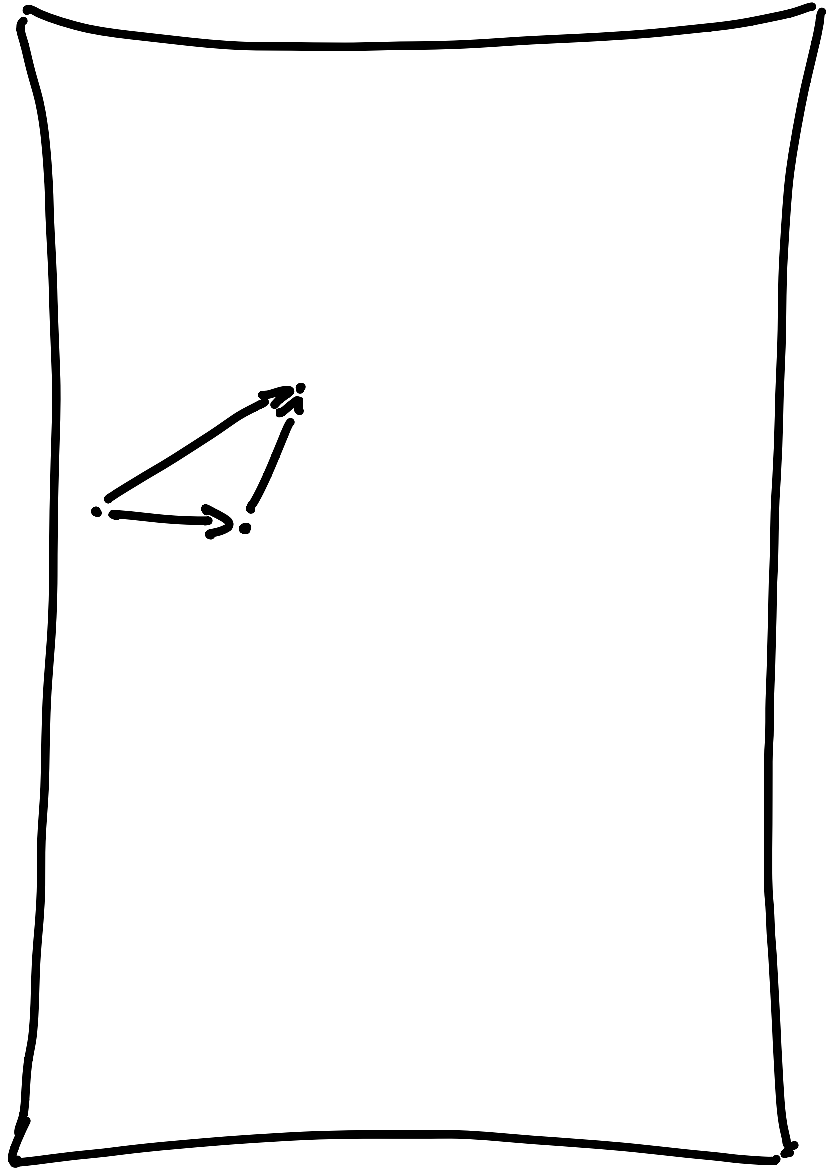
loi



Un groupe G

ensemble

loi associative

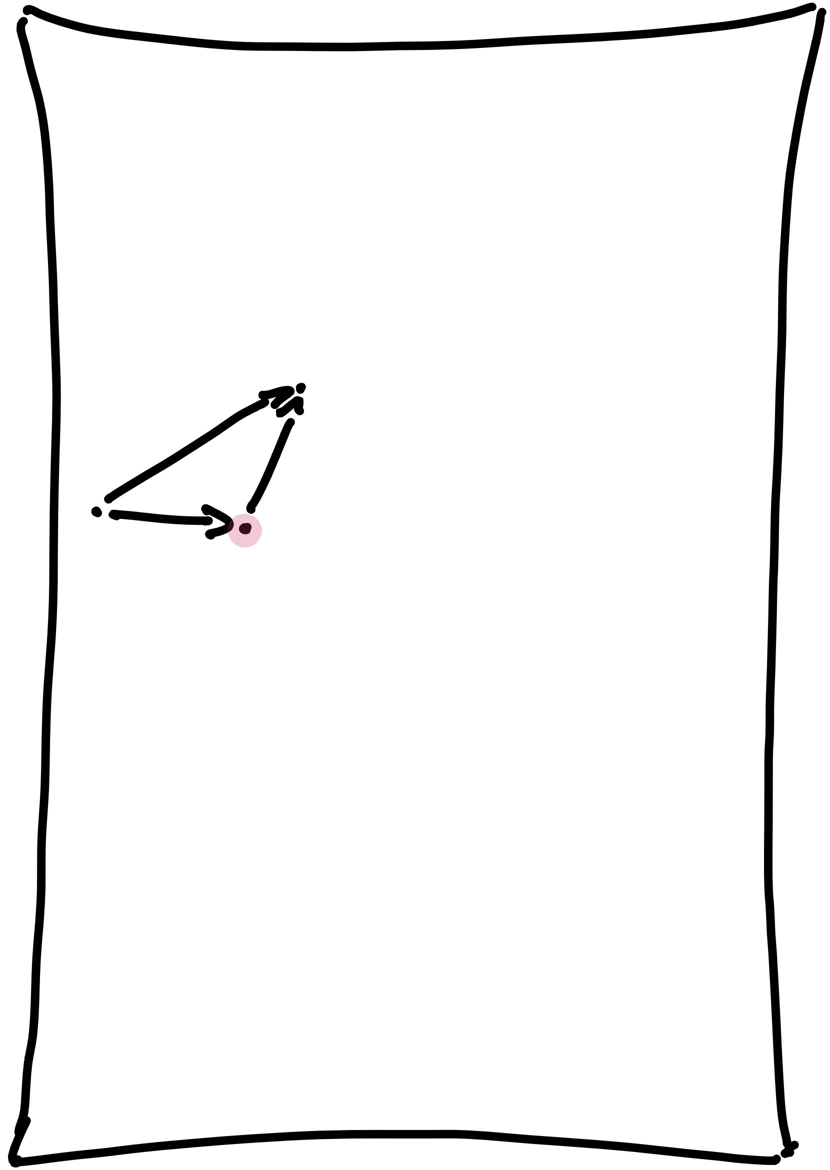


Un groupe G

ensemble

loi associative

élément neutre



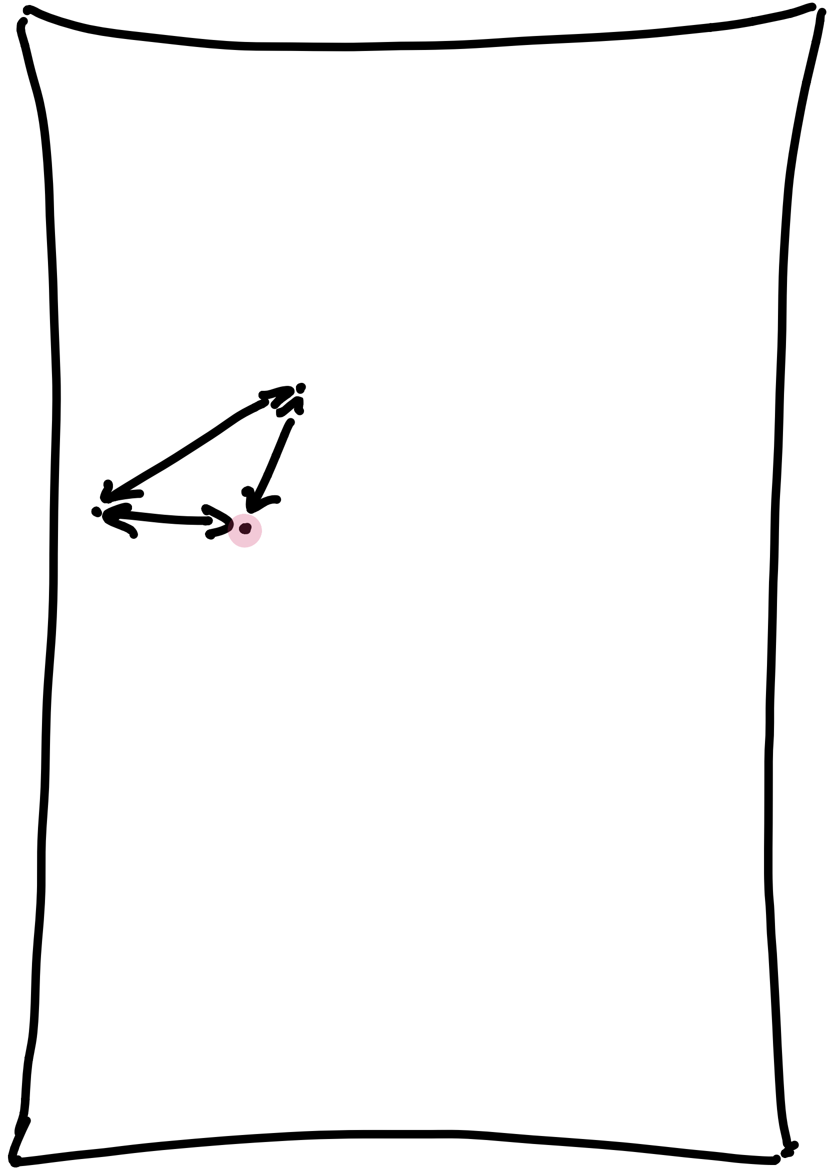
Un groupe G

ensemble

loi associative

élément neutre

inverses



Un groupe G

ensemble

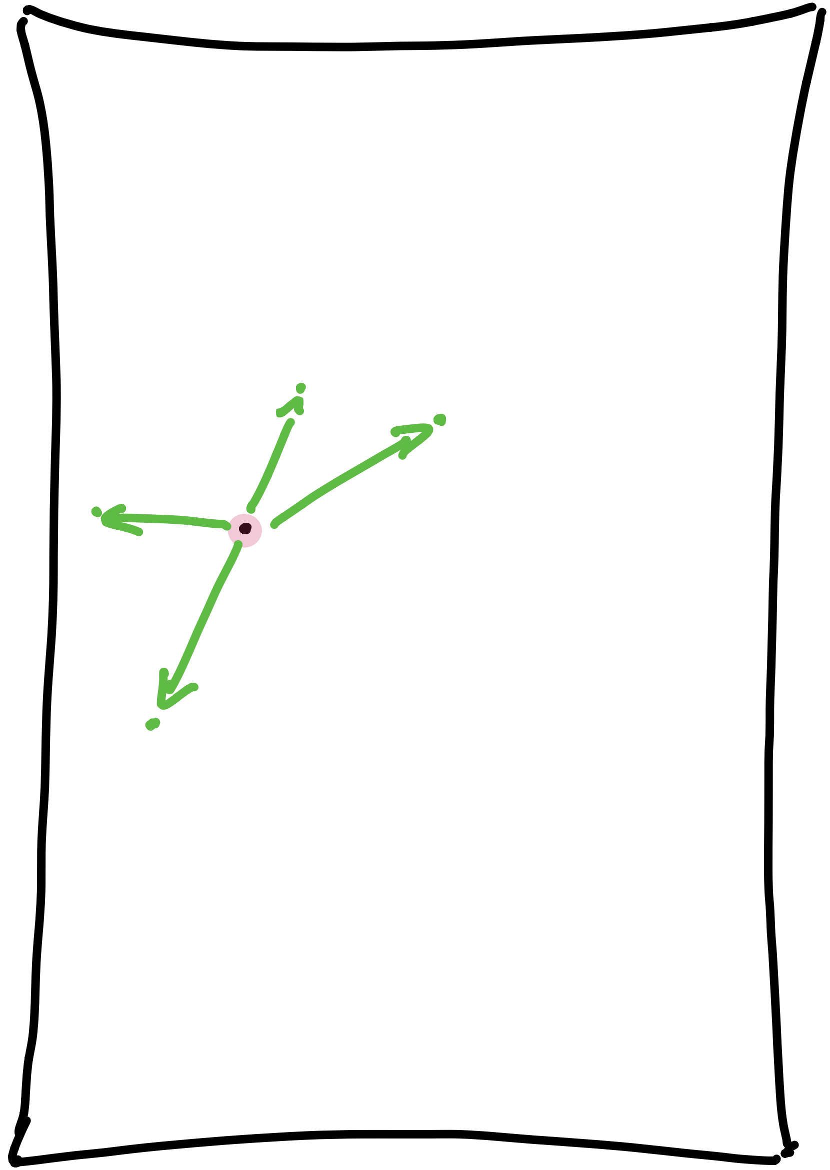
loi associative

élément neutre

inverses

type fini

$\exists S \subseteq G$ fini



Un groupe G

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loi associative

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type fini

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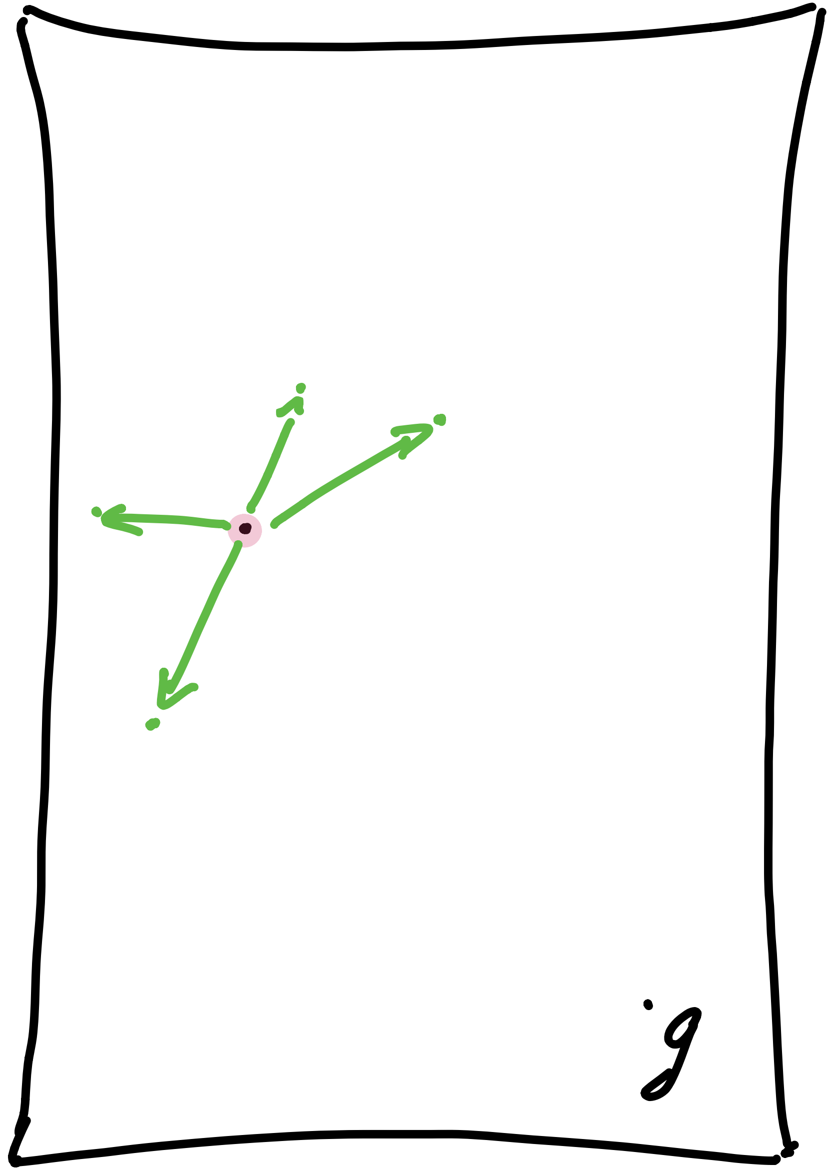
g



Un groupe G

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loi associative

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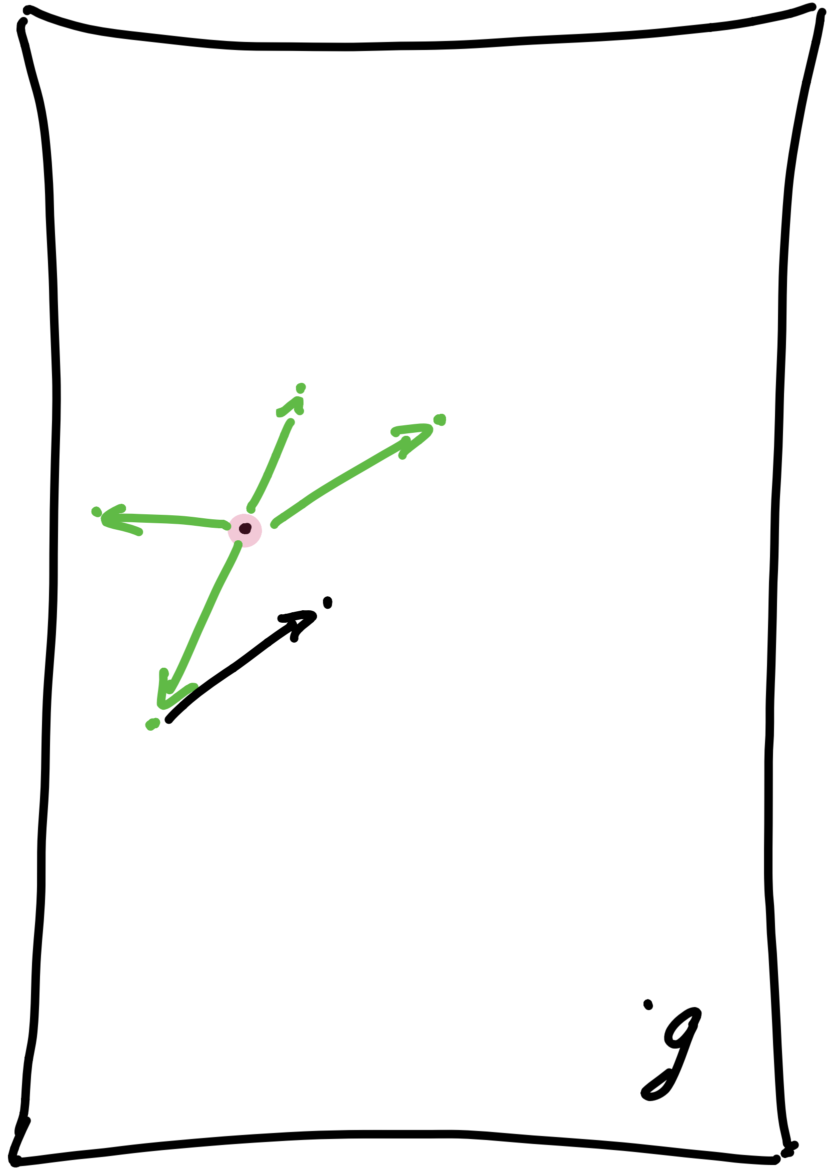
$g = s_1$



Un groupe G

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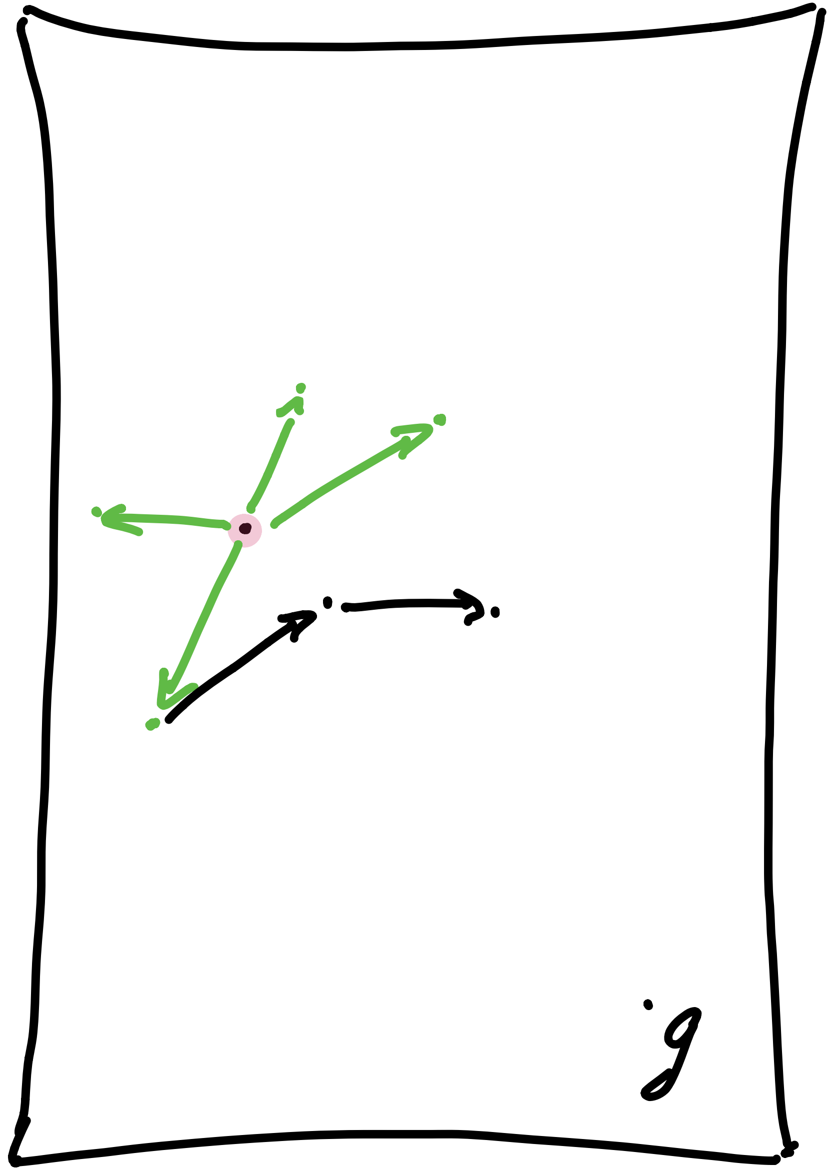
$g = s_1 \cdots s_n$



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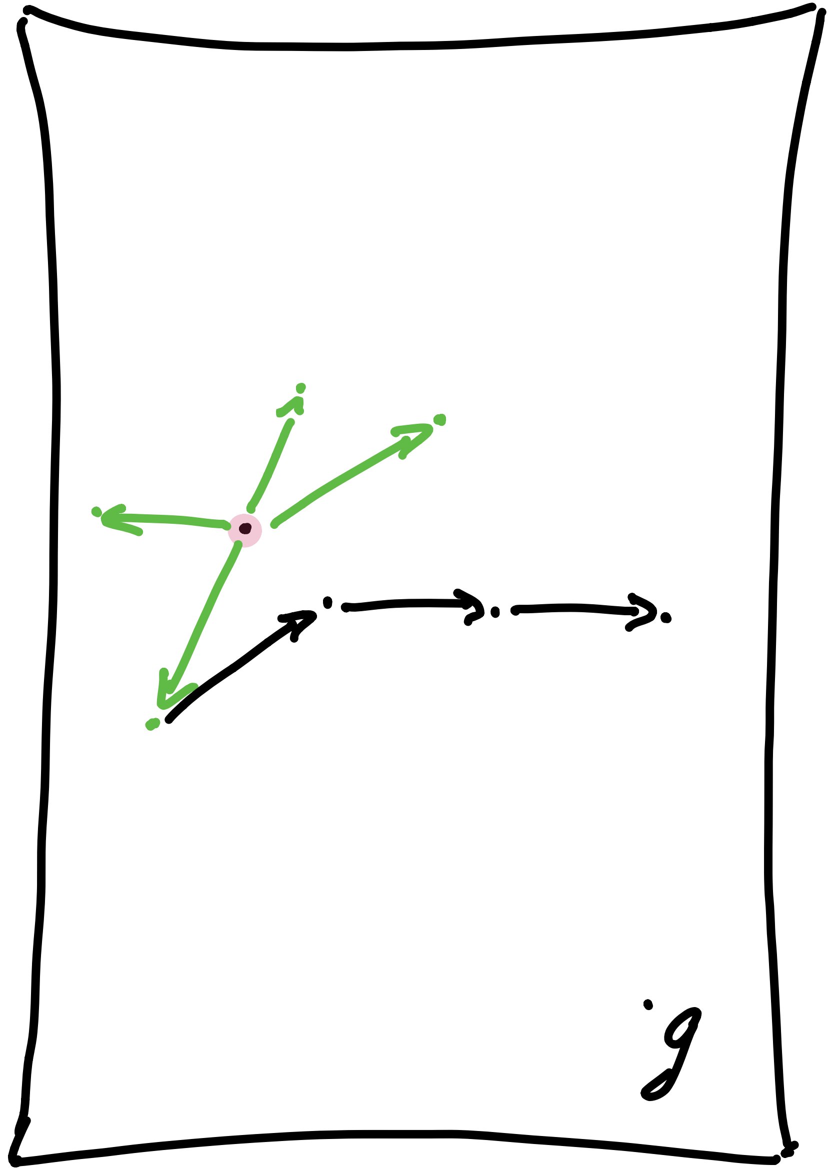

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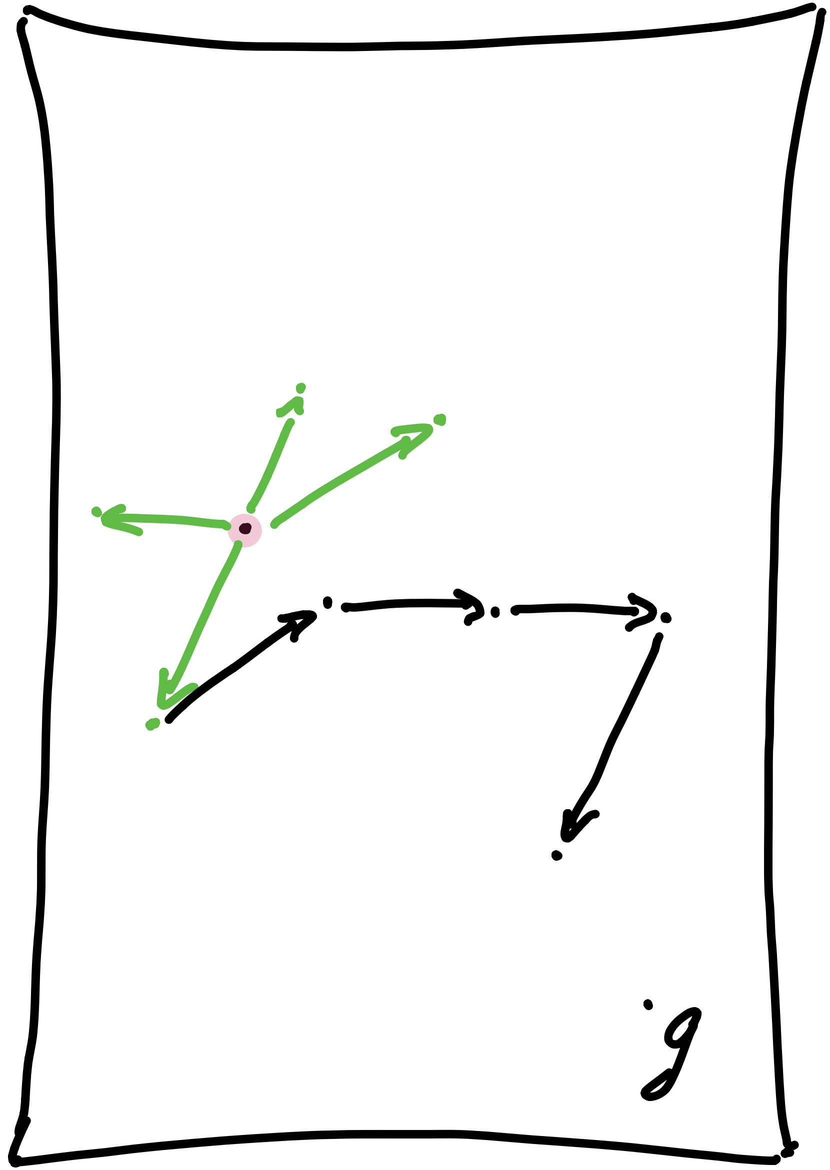

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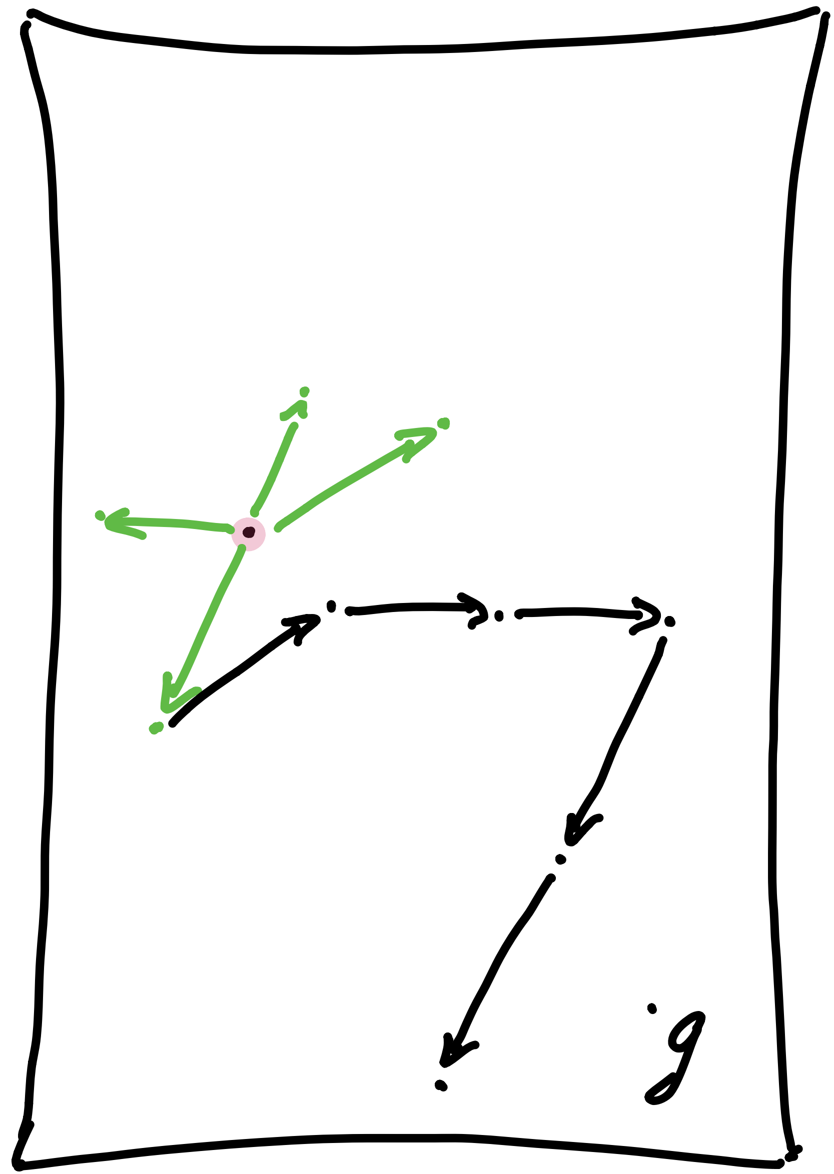

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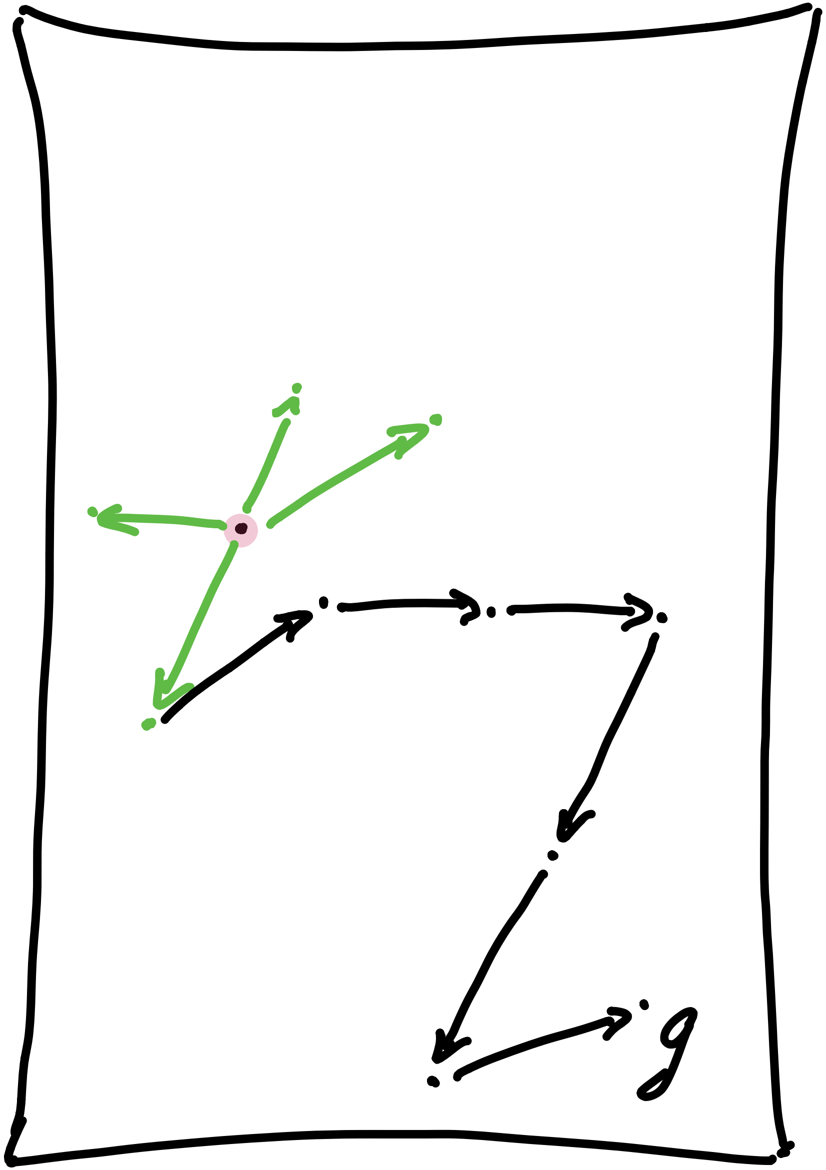

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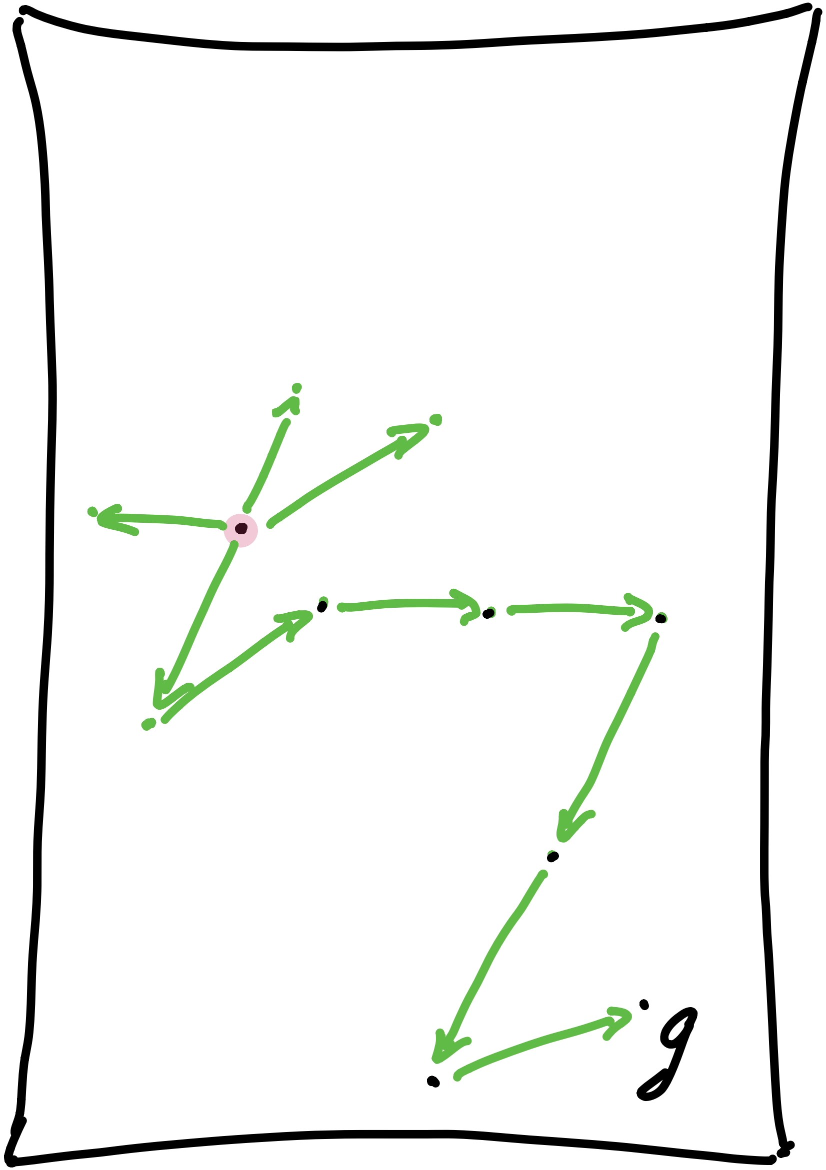

Un groupe G de type fini

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$$\Longrightarrow$$

espace métrique X localement compact

Un groupe G de type fini

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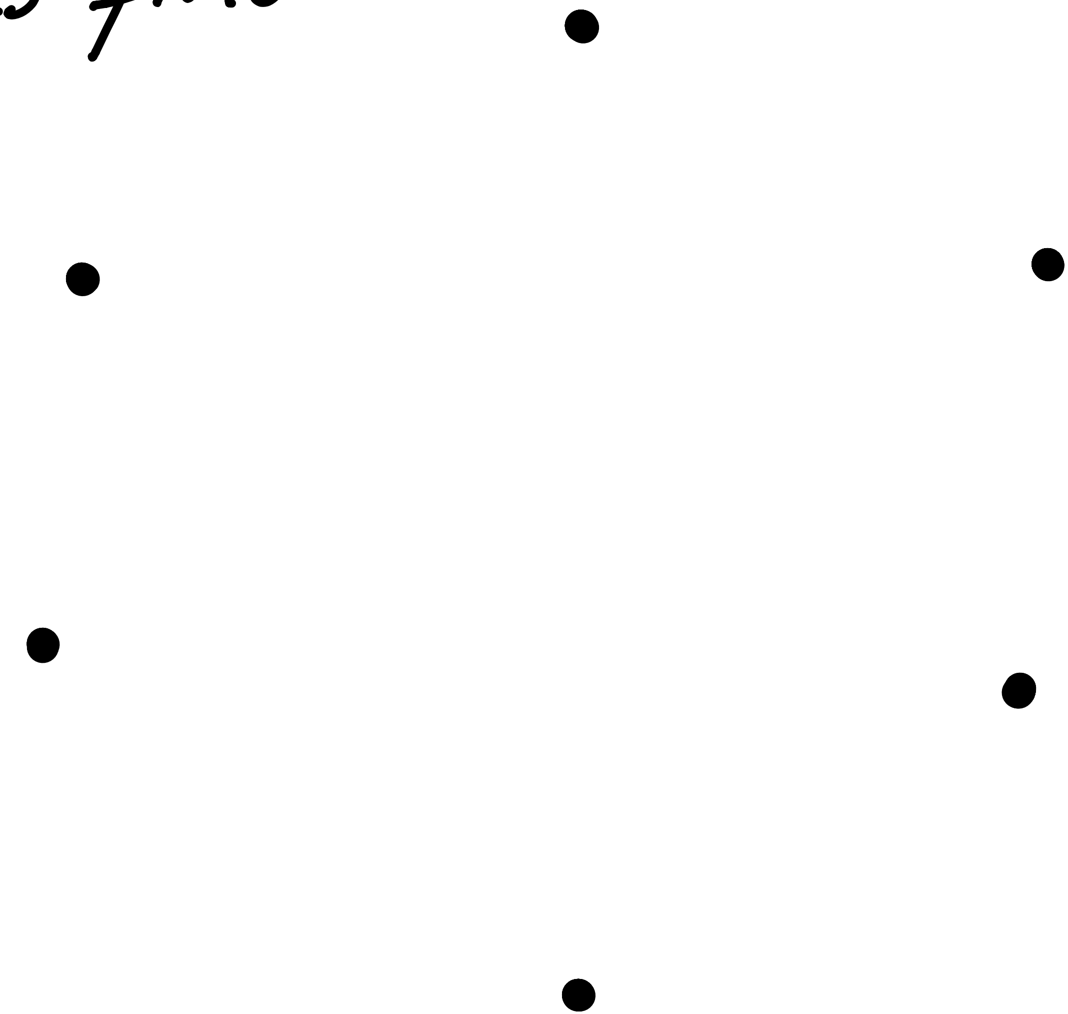
espace métrique X localement compact, $G < \mathrm{Isom}(X)$ et $X/_G$ compact.

Un groupe G de type fini

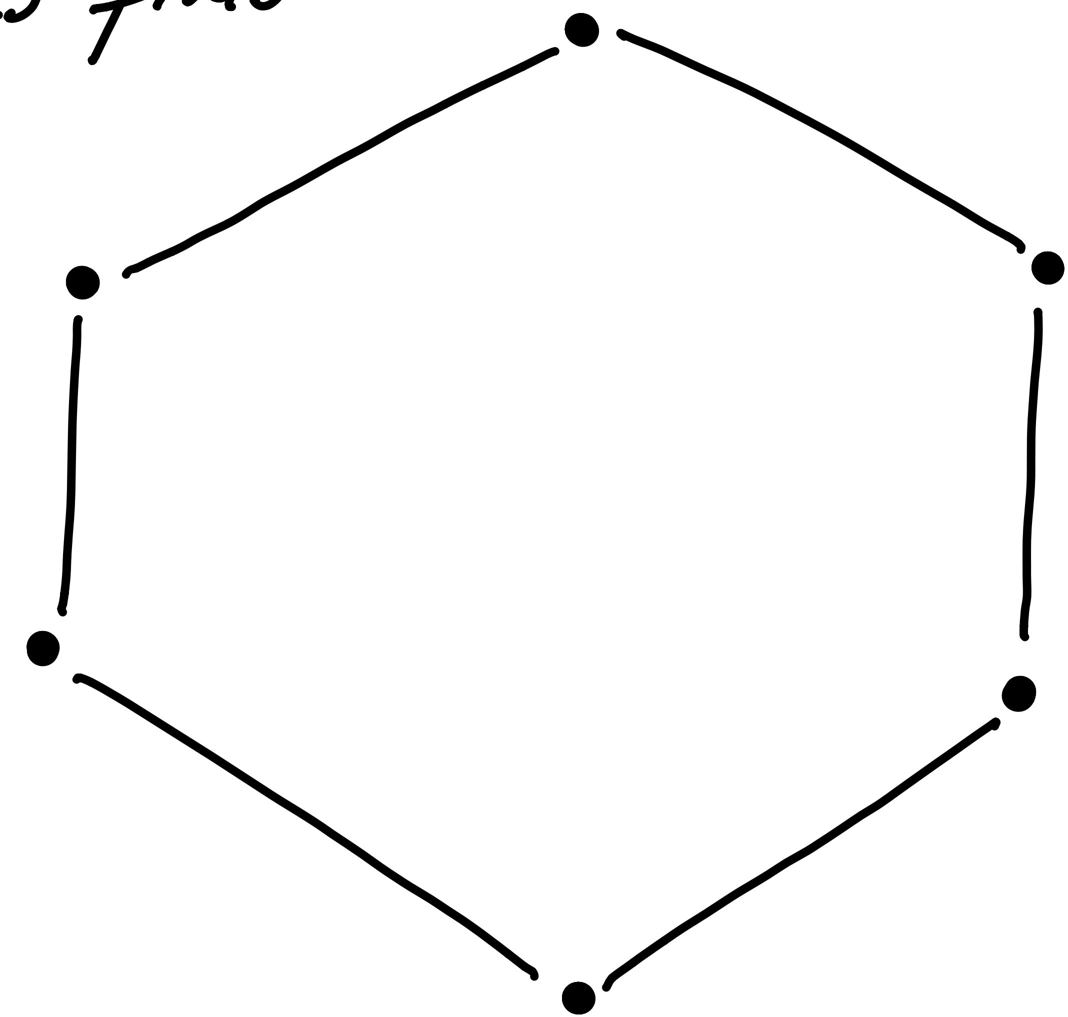
$$\Longleftrightarrow$$

$\exists$ espace métrique X localement compact, $G < \mathrm{Isom}(X)$ et X/G compact.

groupes finis



groupes finis



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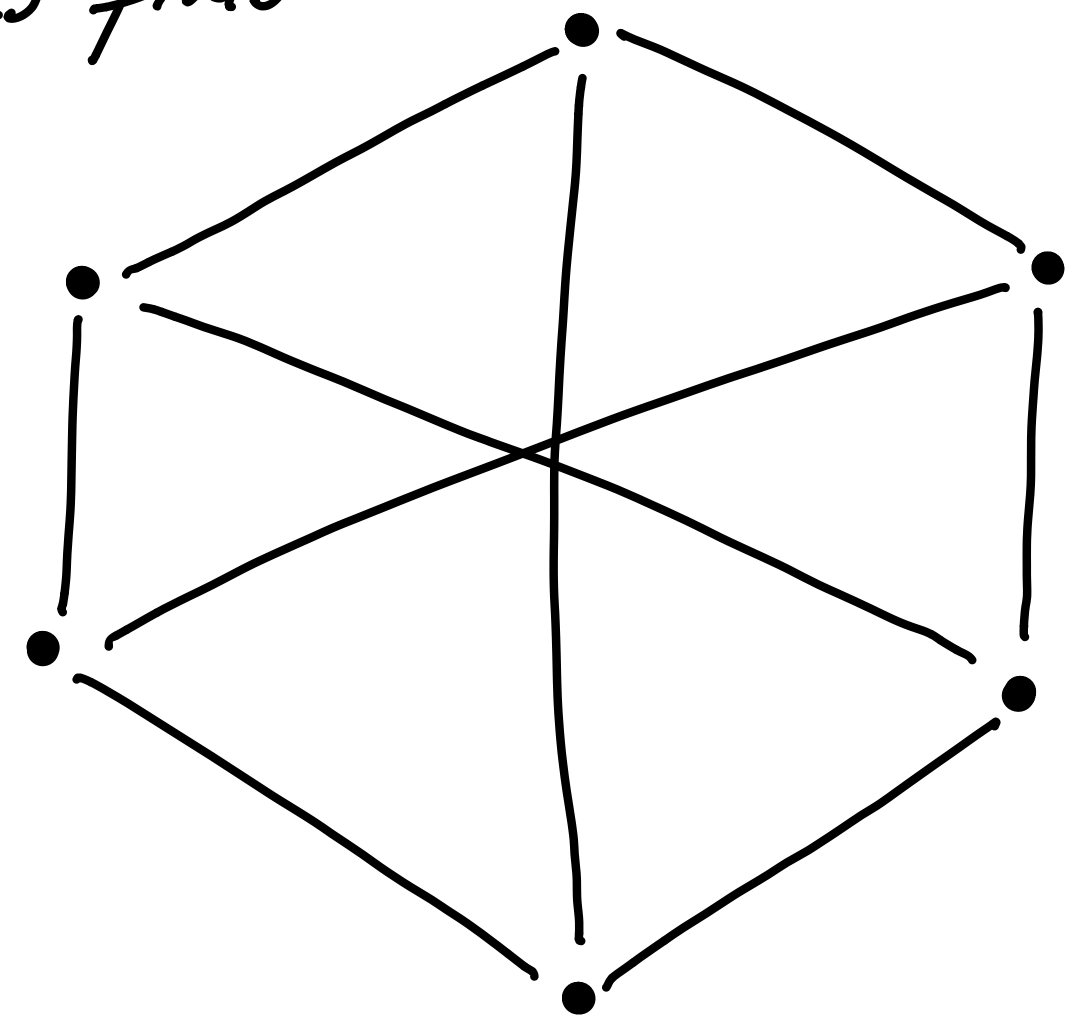

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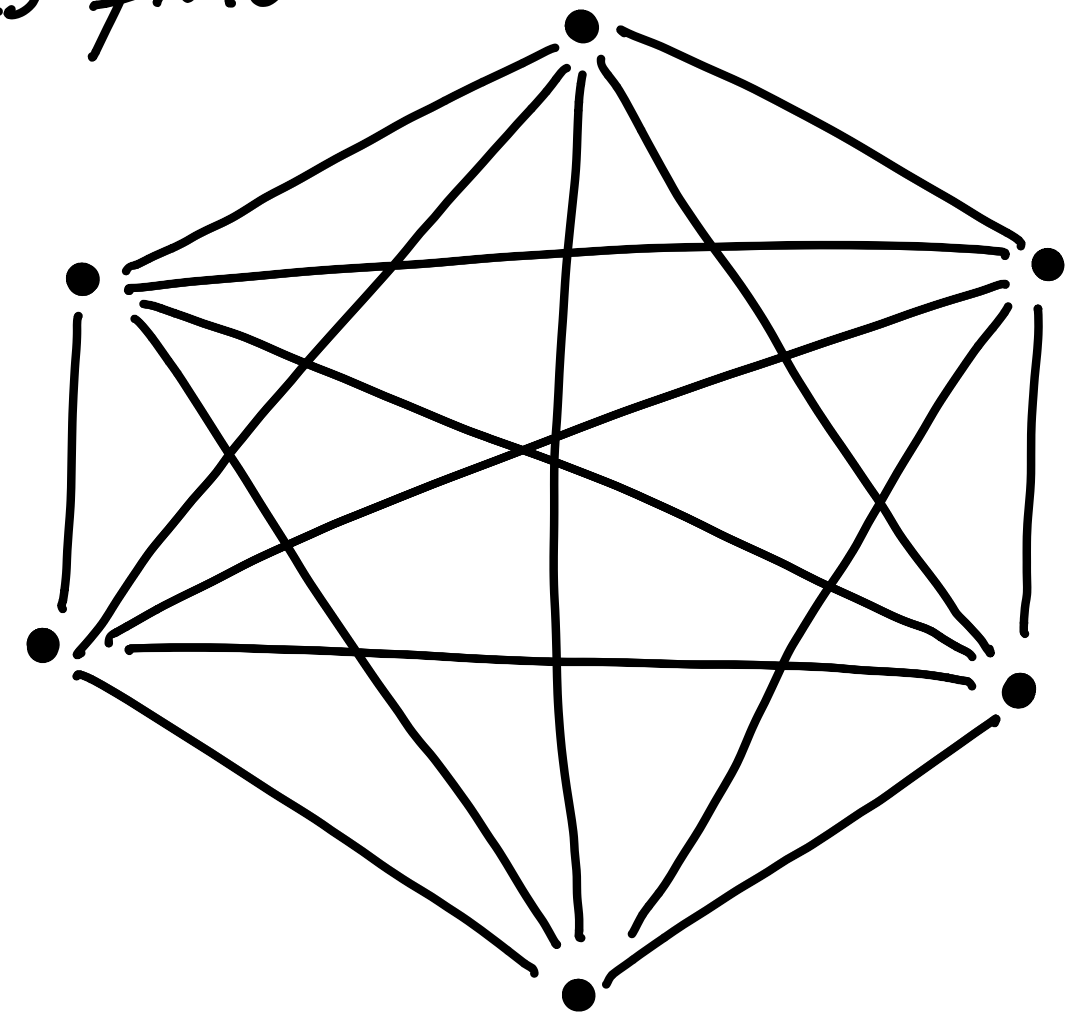

Z

$\mathbb{Z} = \langle 1 \rangle$

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$\mathbb{Z} = \langle 1, 100 \rangle$

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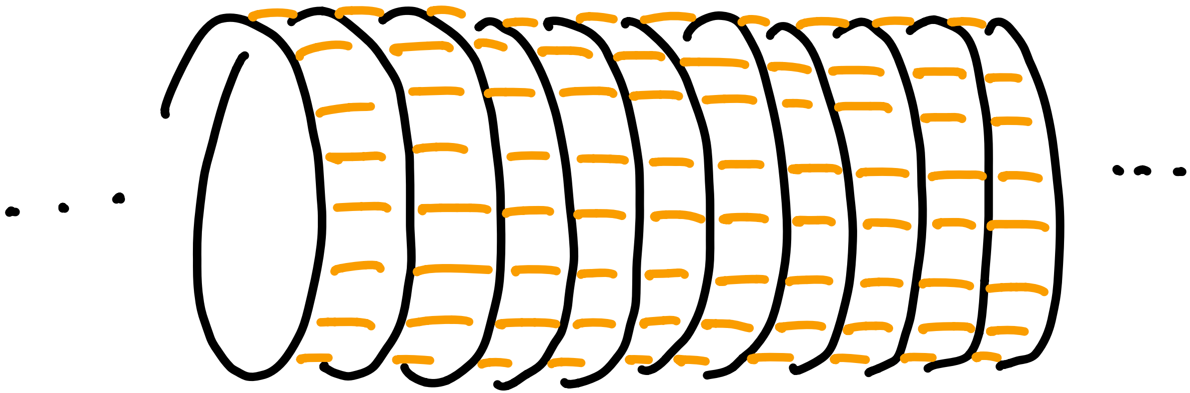

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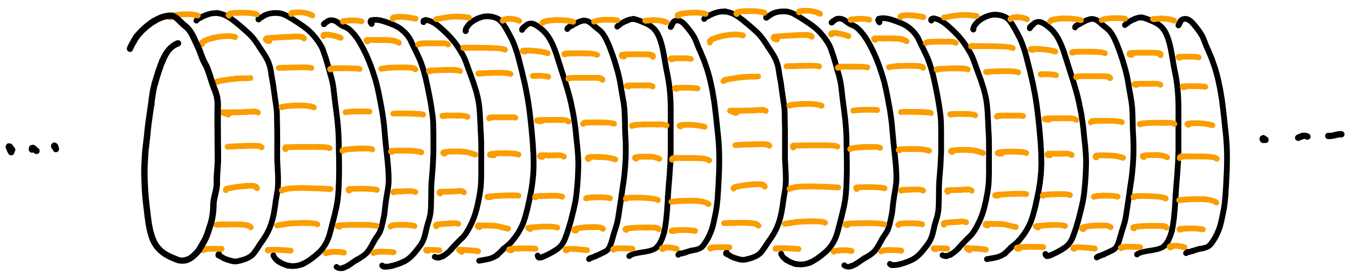

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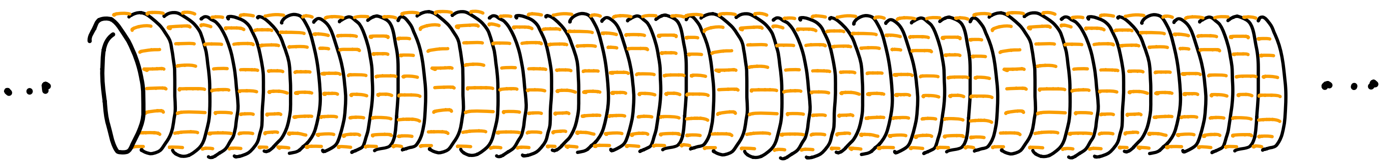

$PSL_2(\mathbb{Z})$

$$PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) / \pm I$$

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$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$$

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$$\rightsquigarrow$$

$$\mathbb{Z}_2 < PSL_2(\mathbb{Z})$$

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$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

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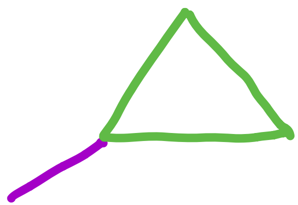

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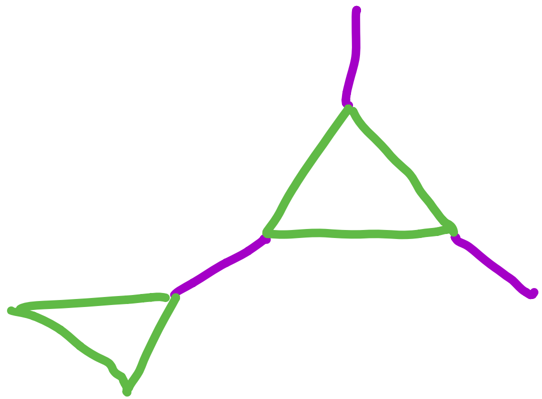

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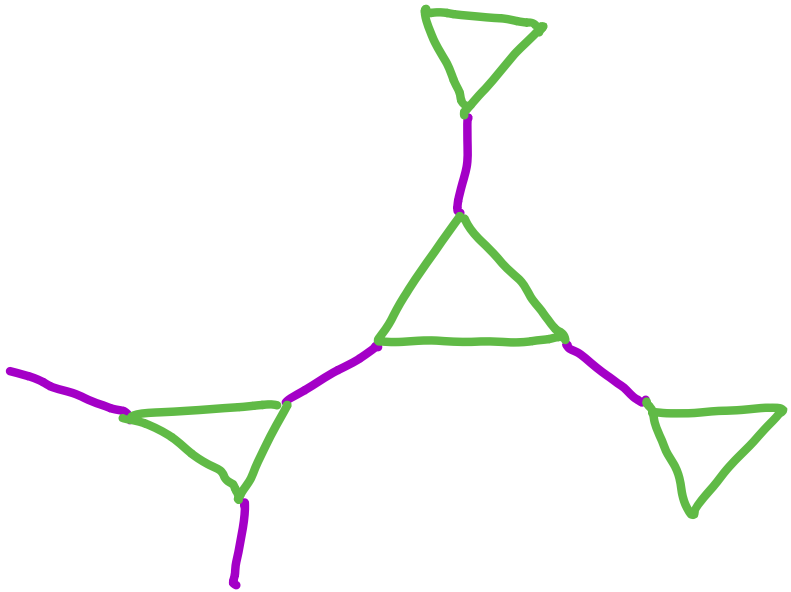

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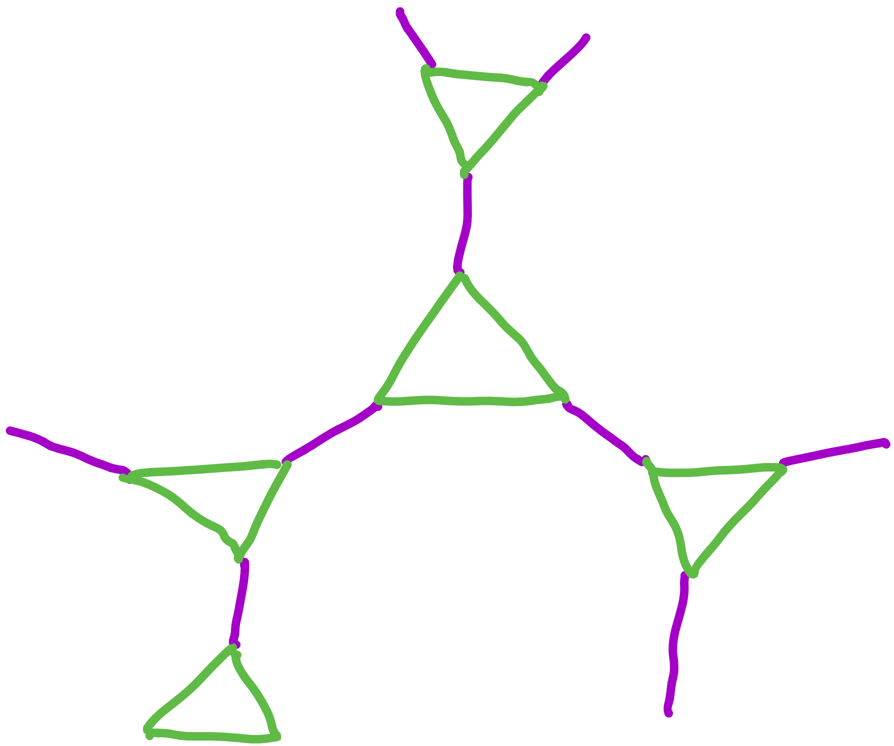

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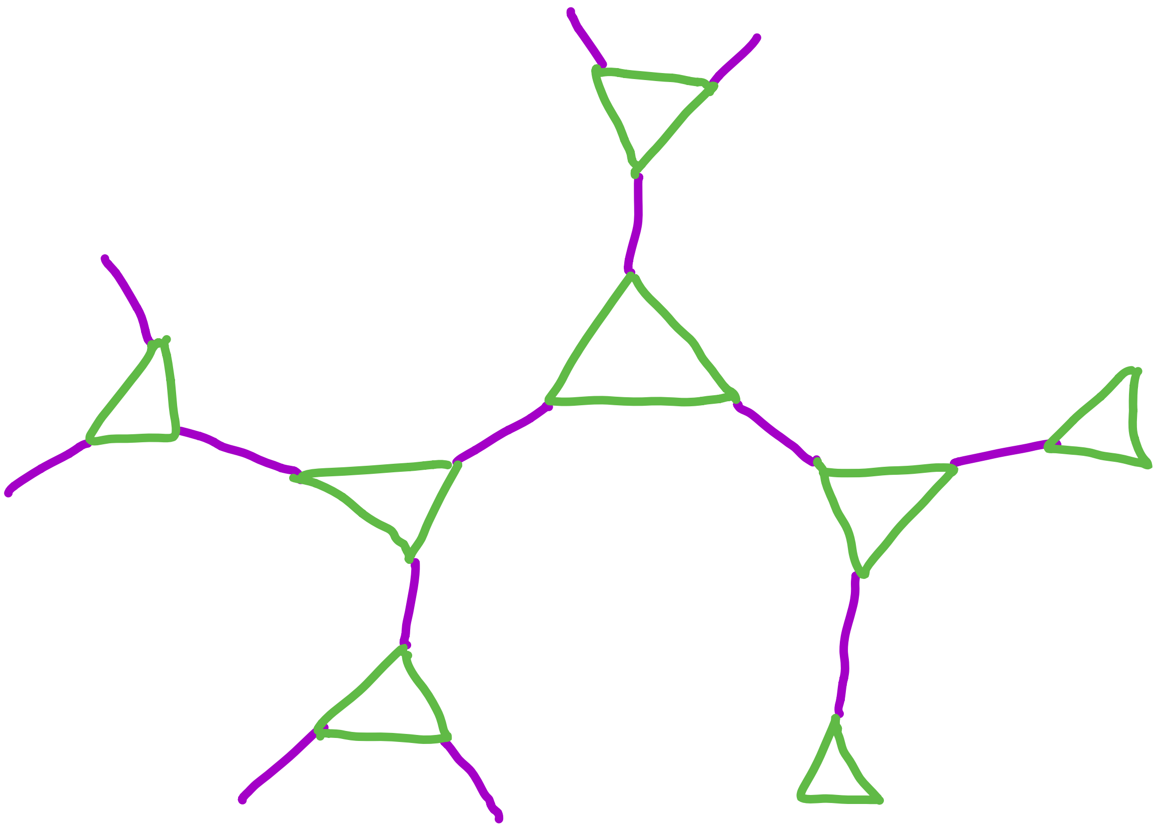

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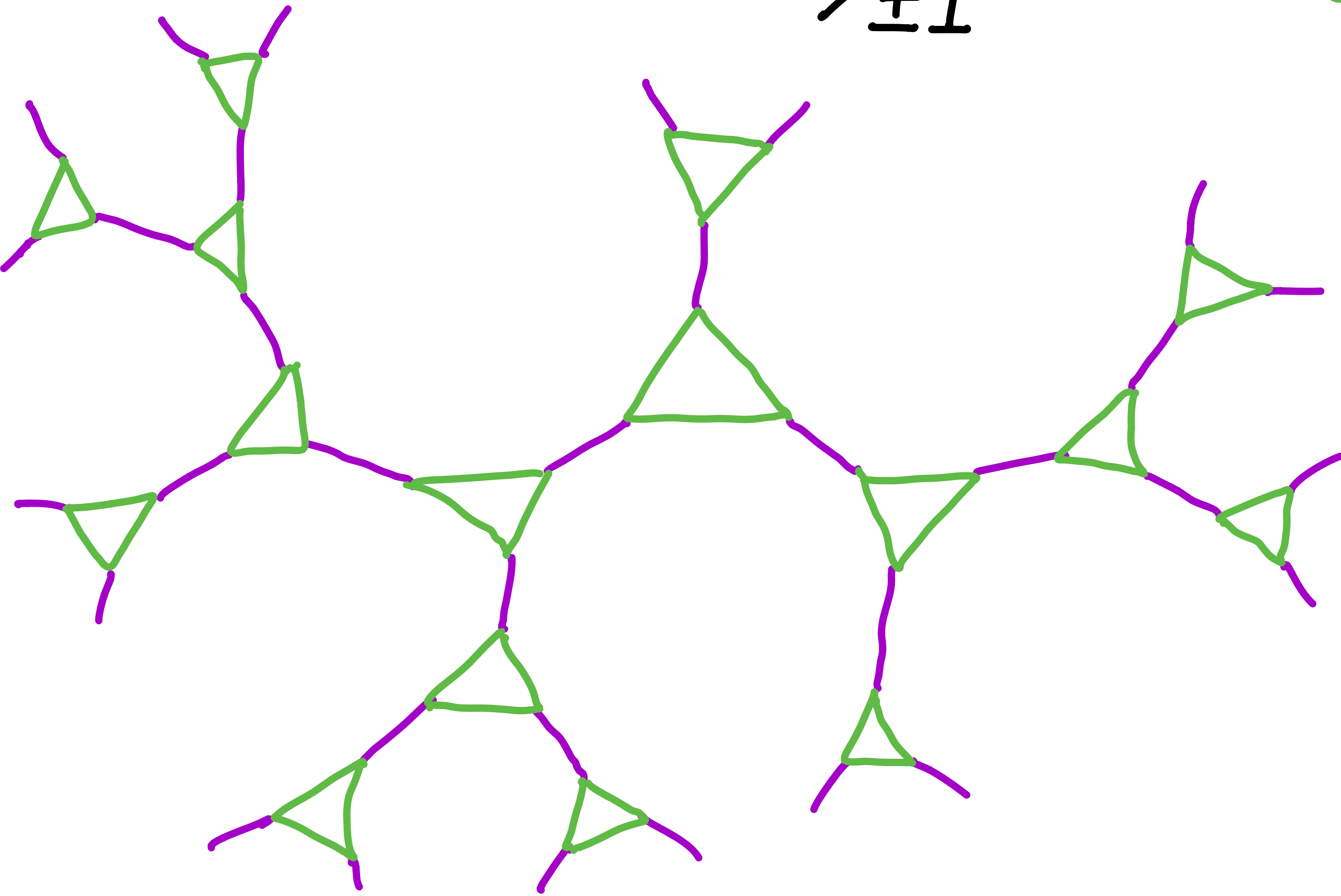

$$\mathbb{F}_2 = \langle a, b \rangle$$

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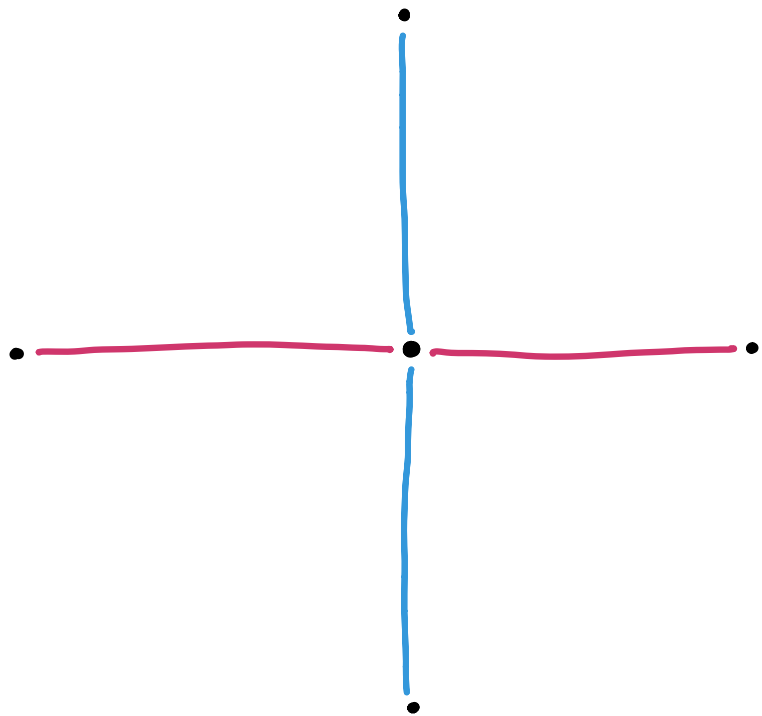

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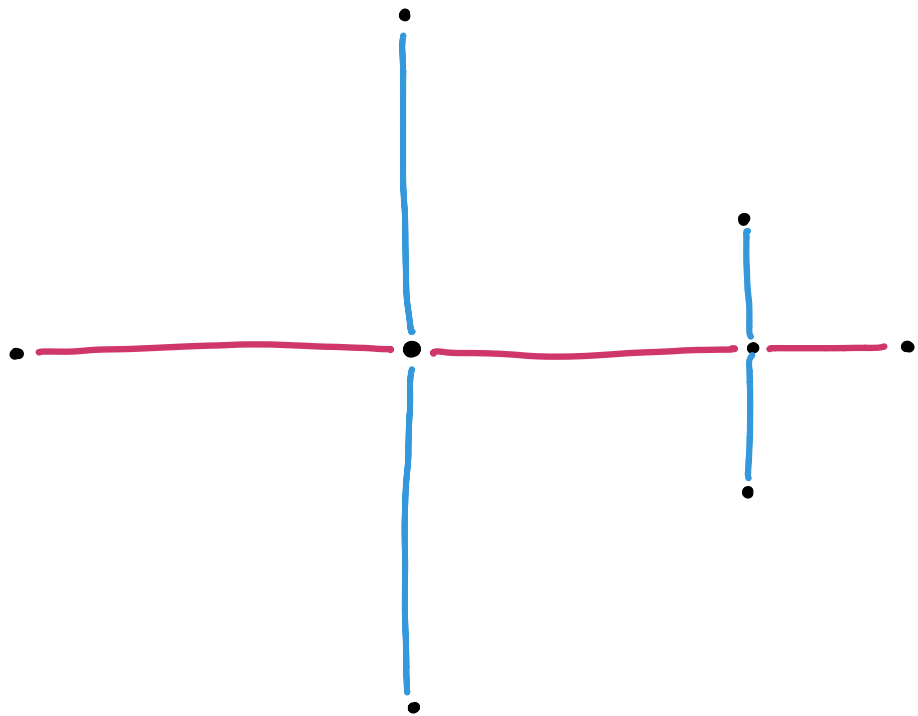

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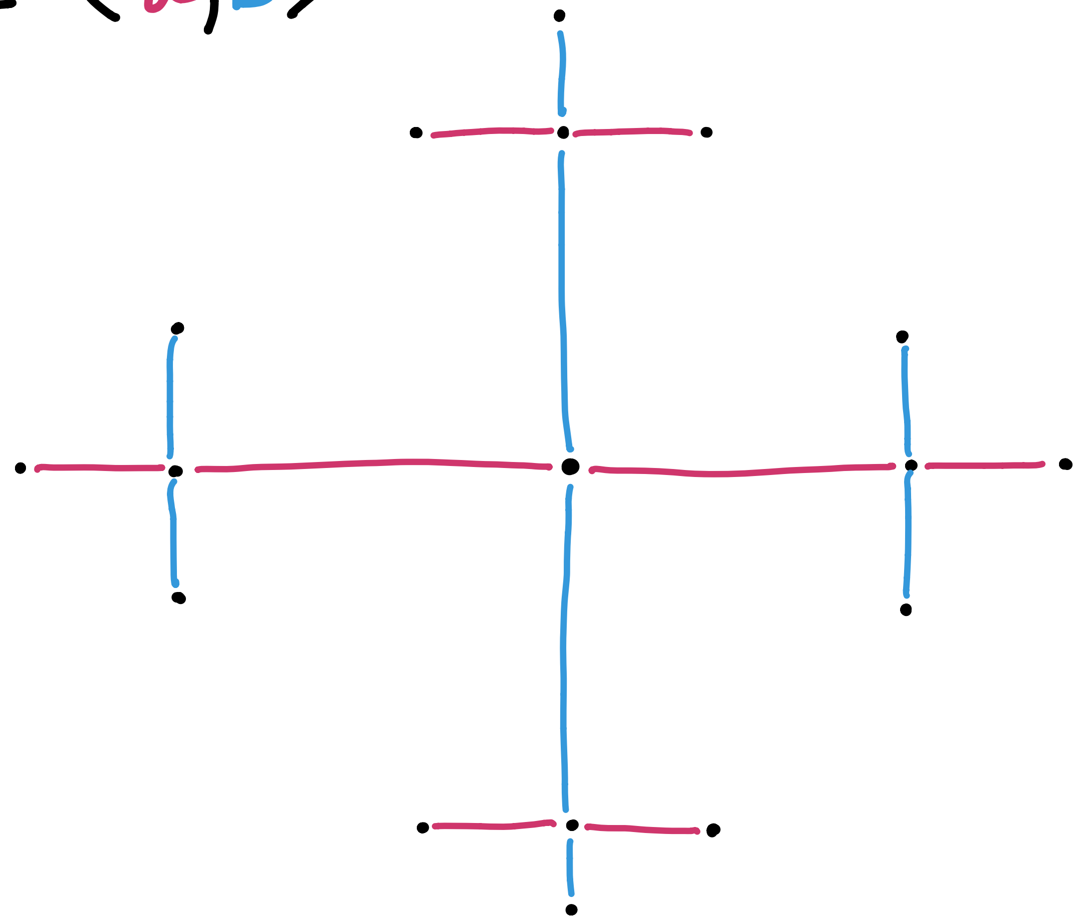

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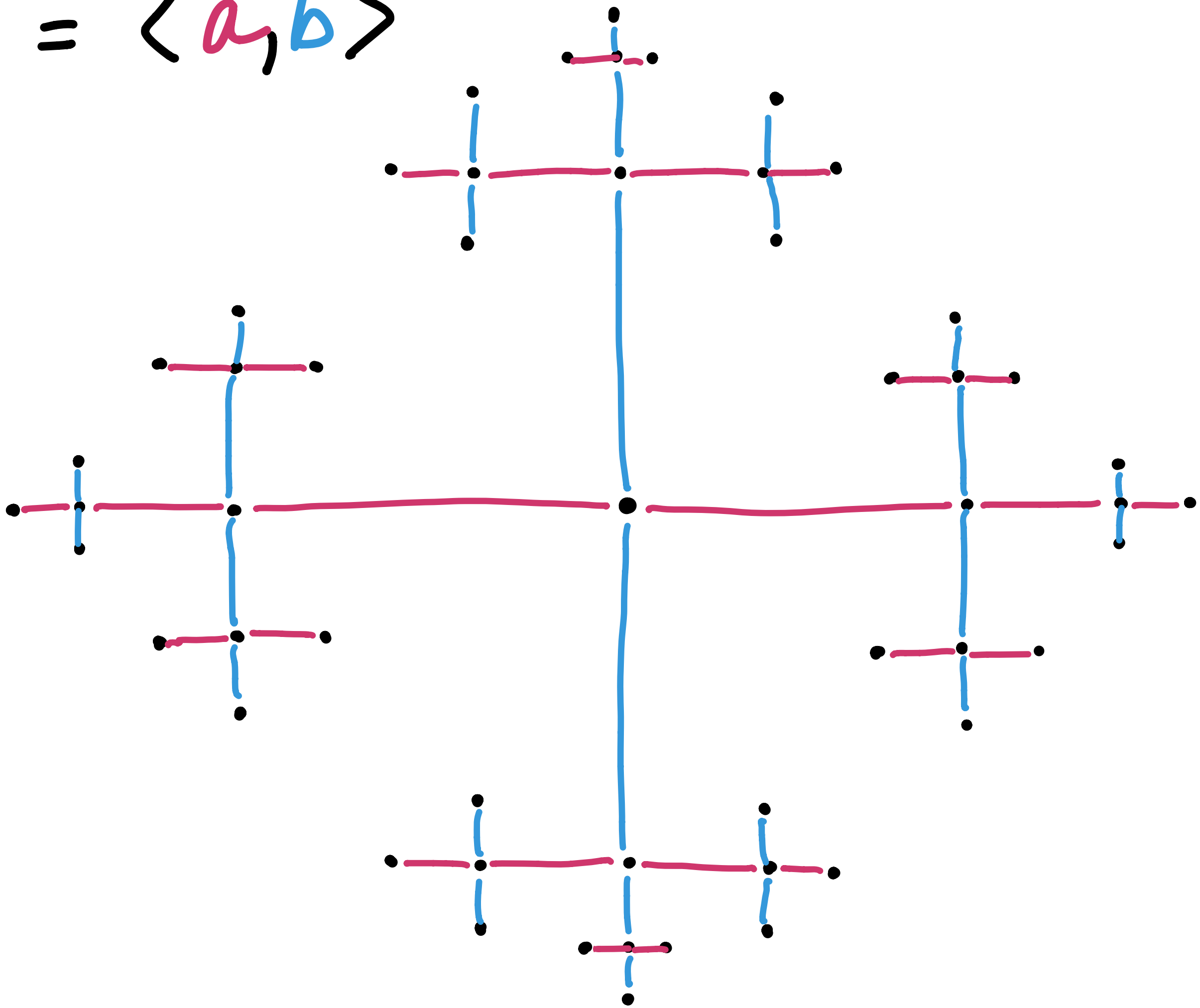

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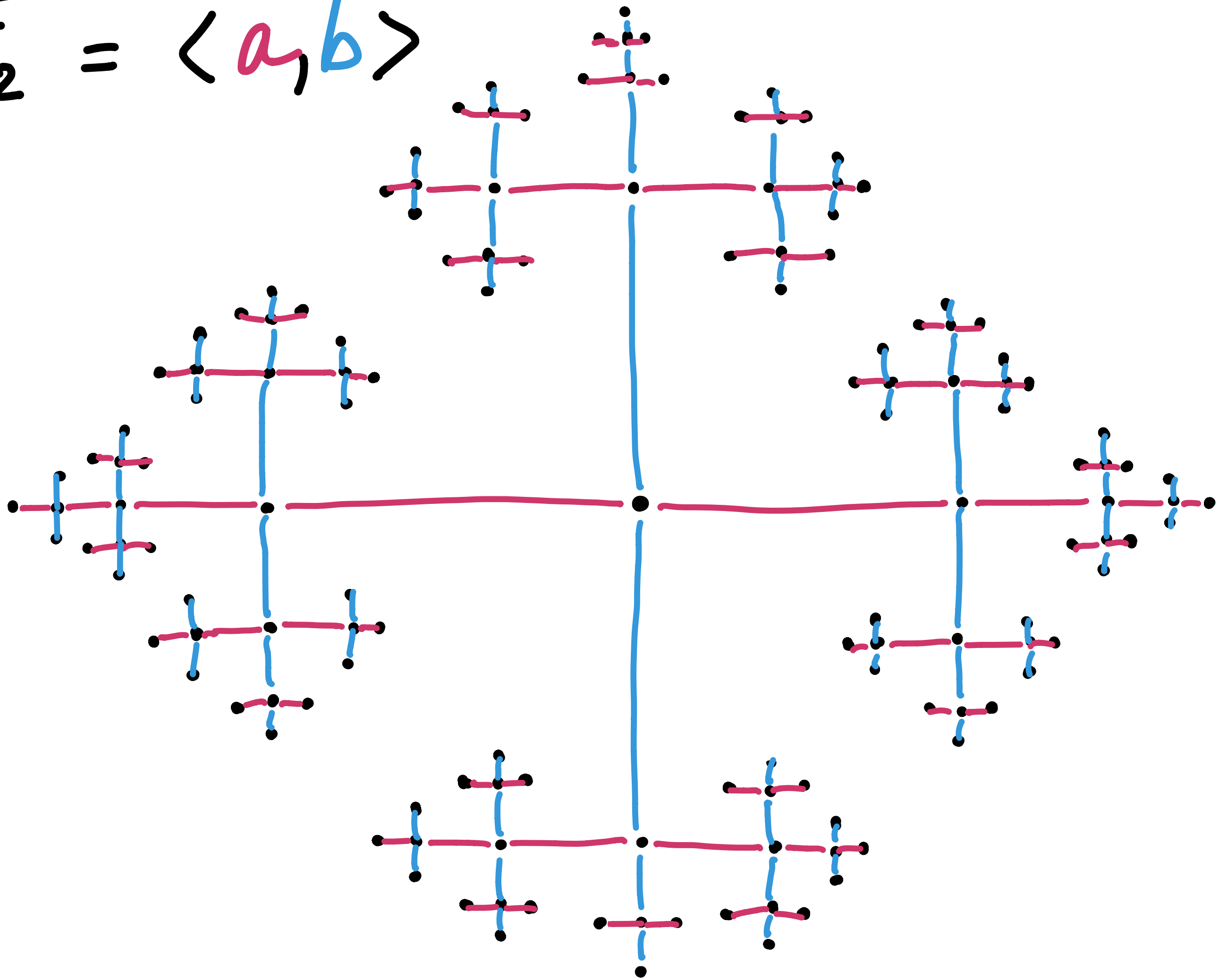
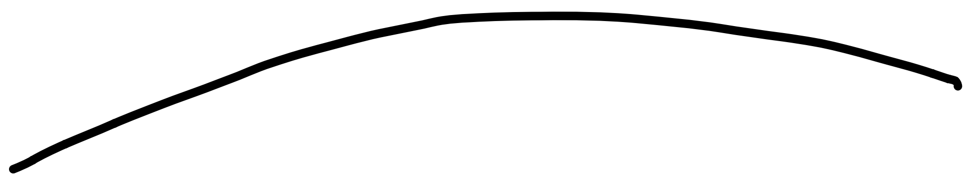
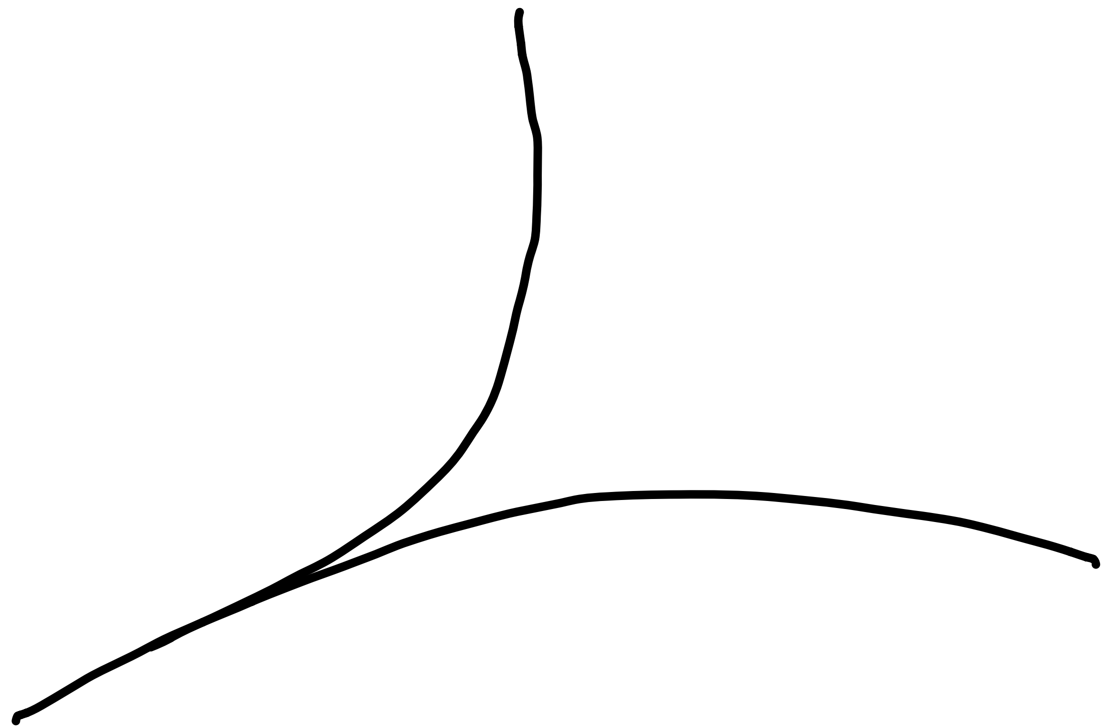


Graphe (espace métrique)
hyperbolique (à la Gromov)

Graphe (espace métrique) hyperbolique (à la Gromov)



Graphe (espace métrique) hyperbolique (à la Gromov)



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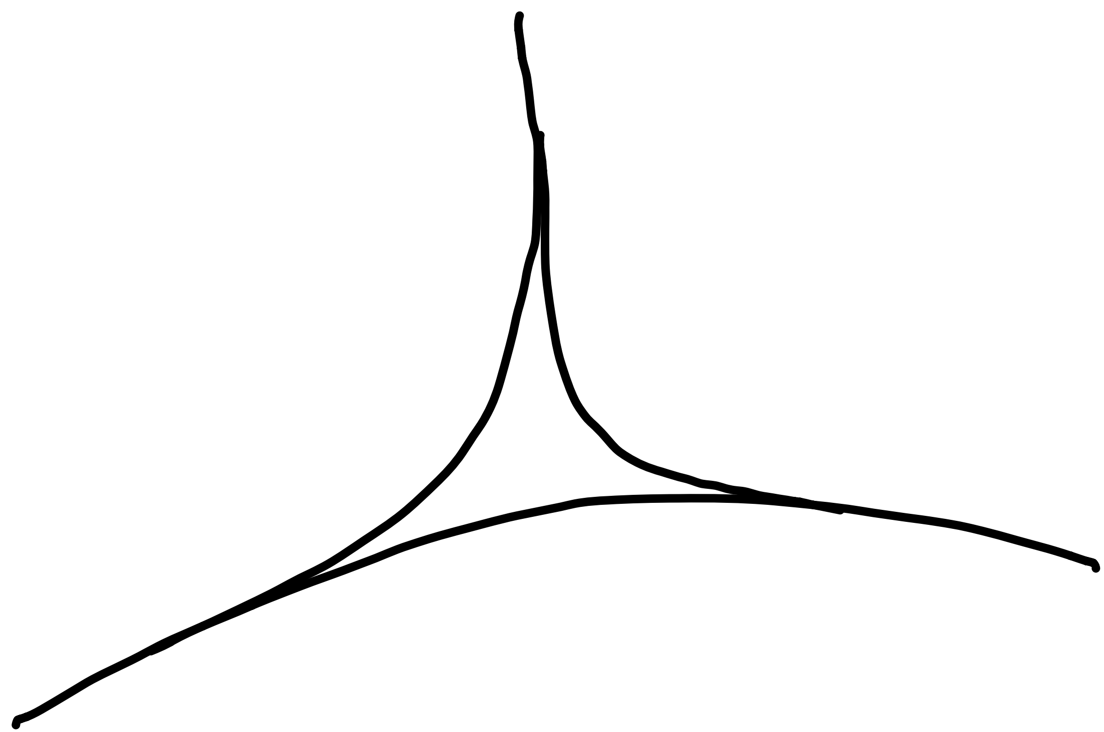

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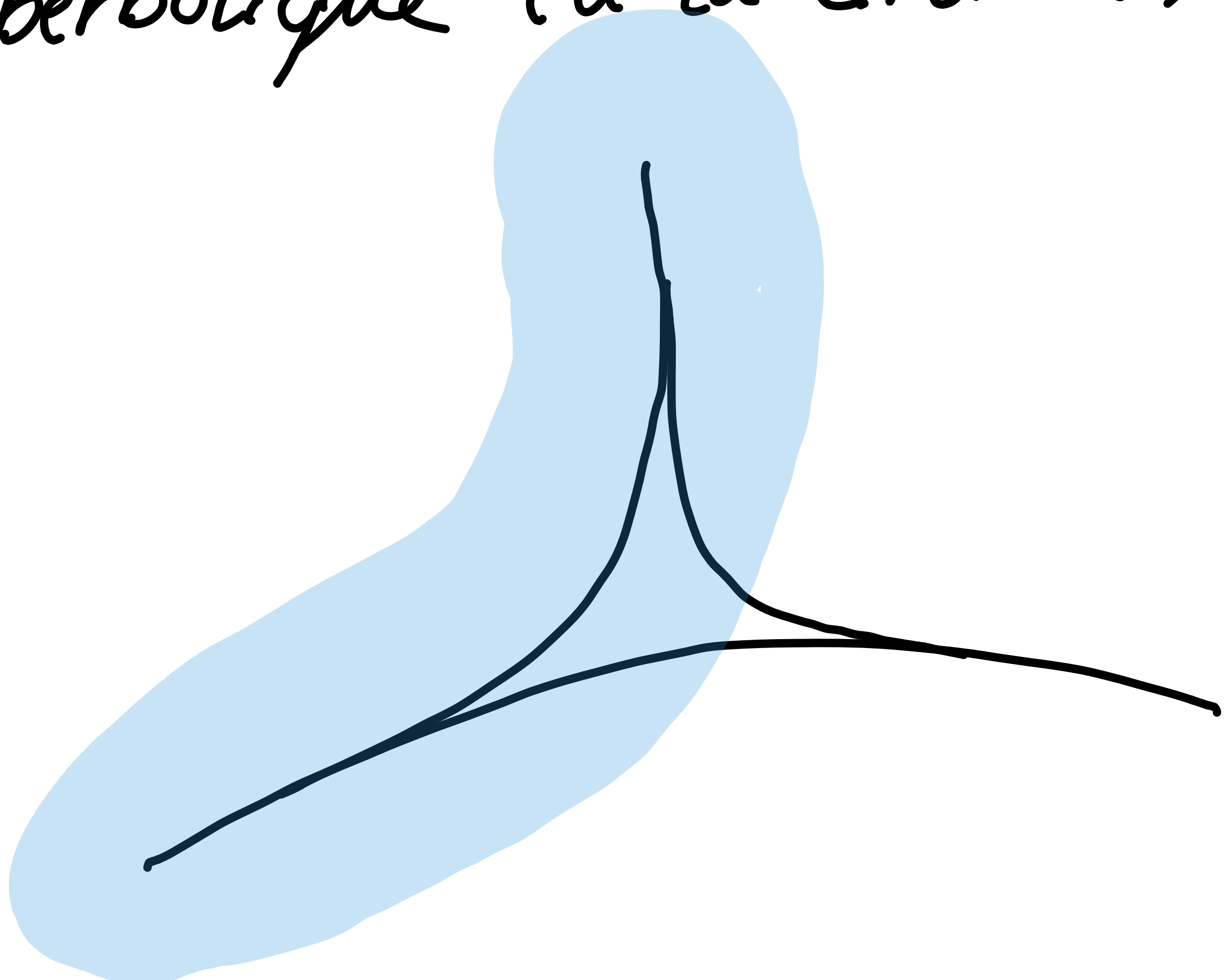

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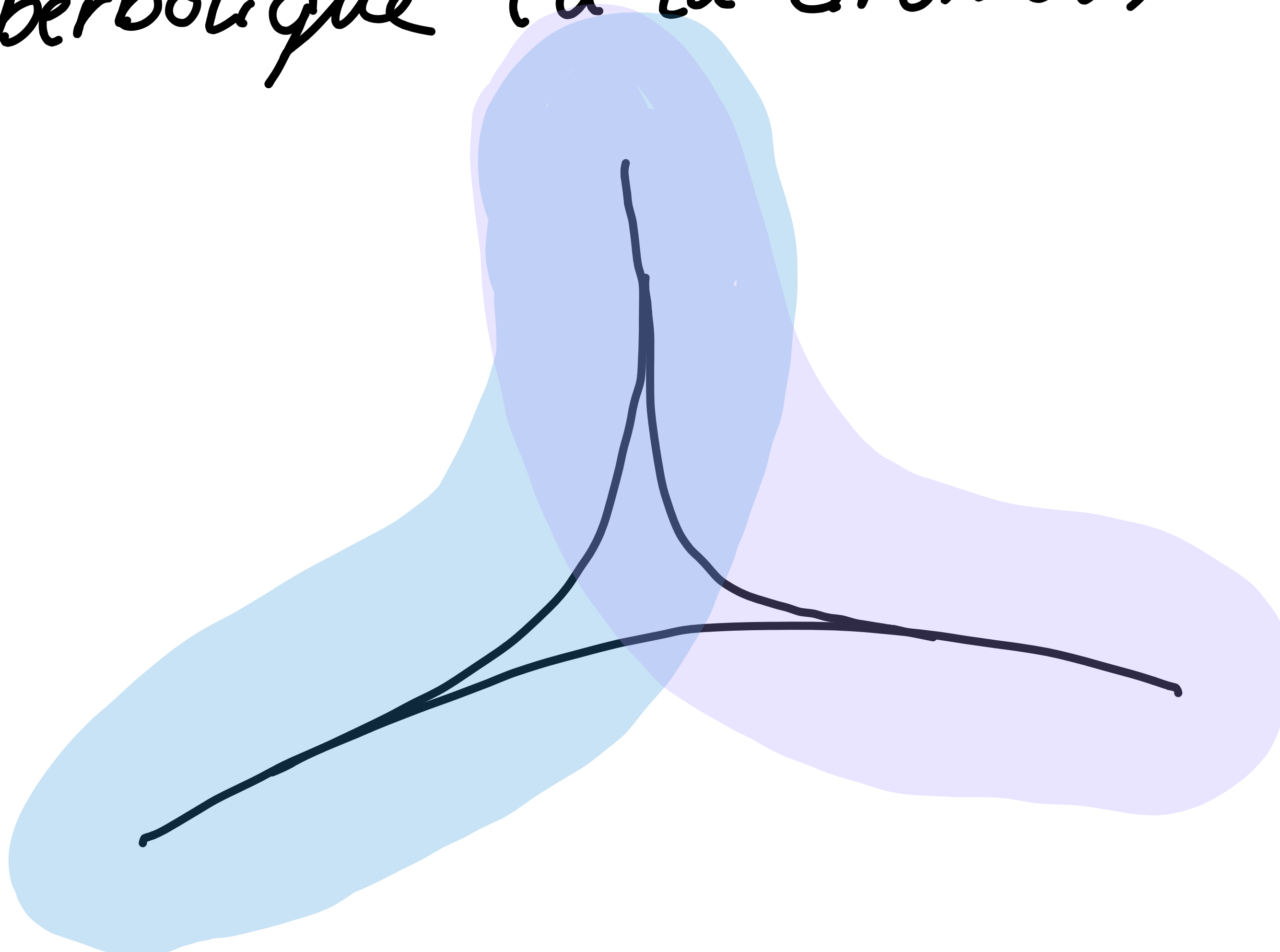
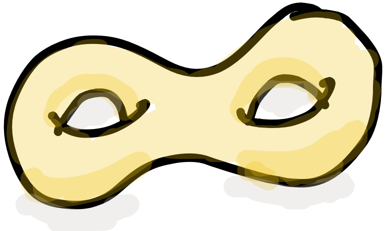
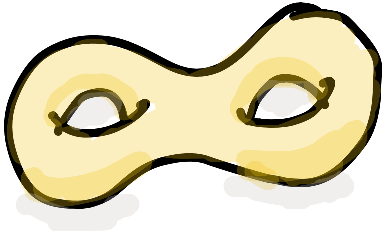
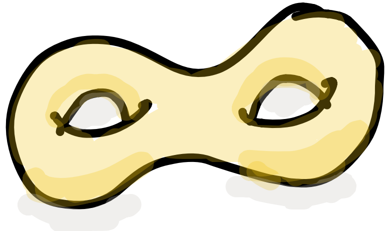
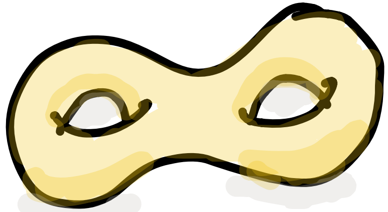
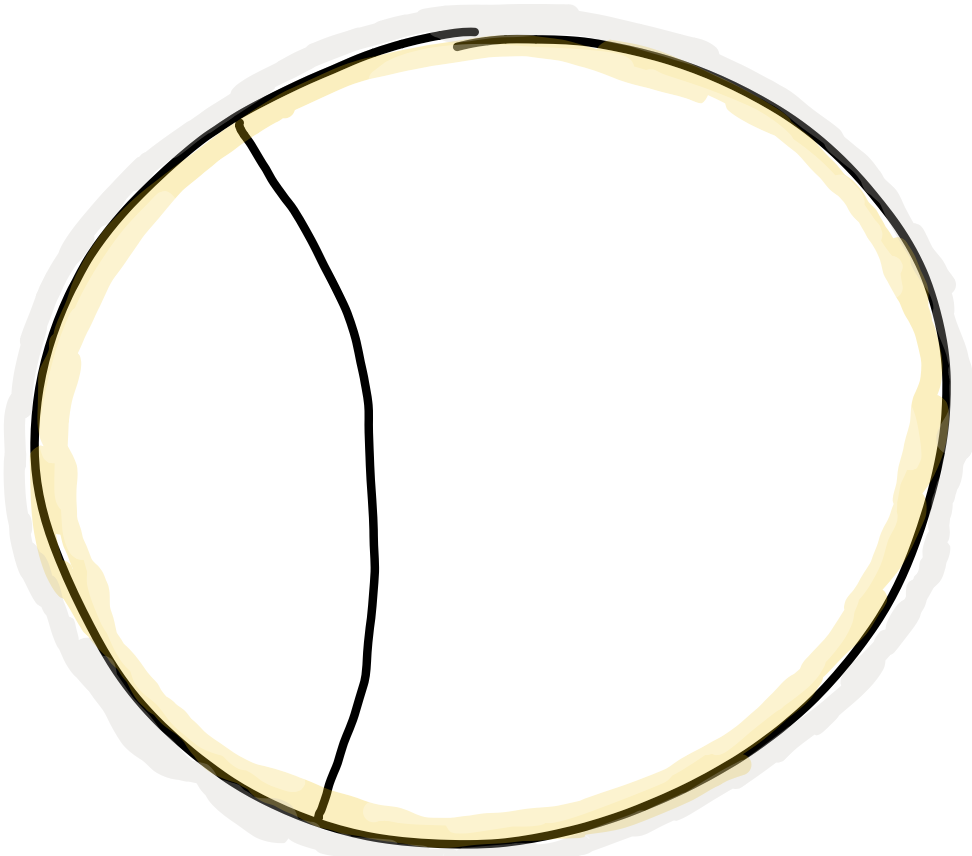


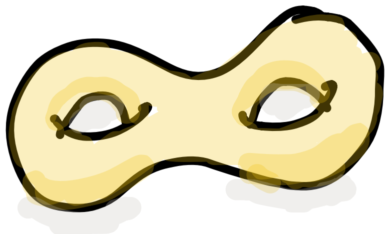
$\pi_1\Big($  $\Big)$

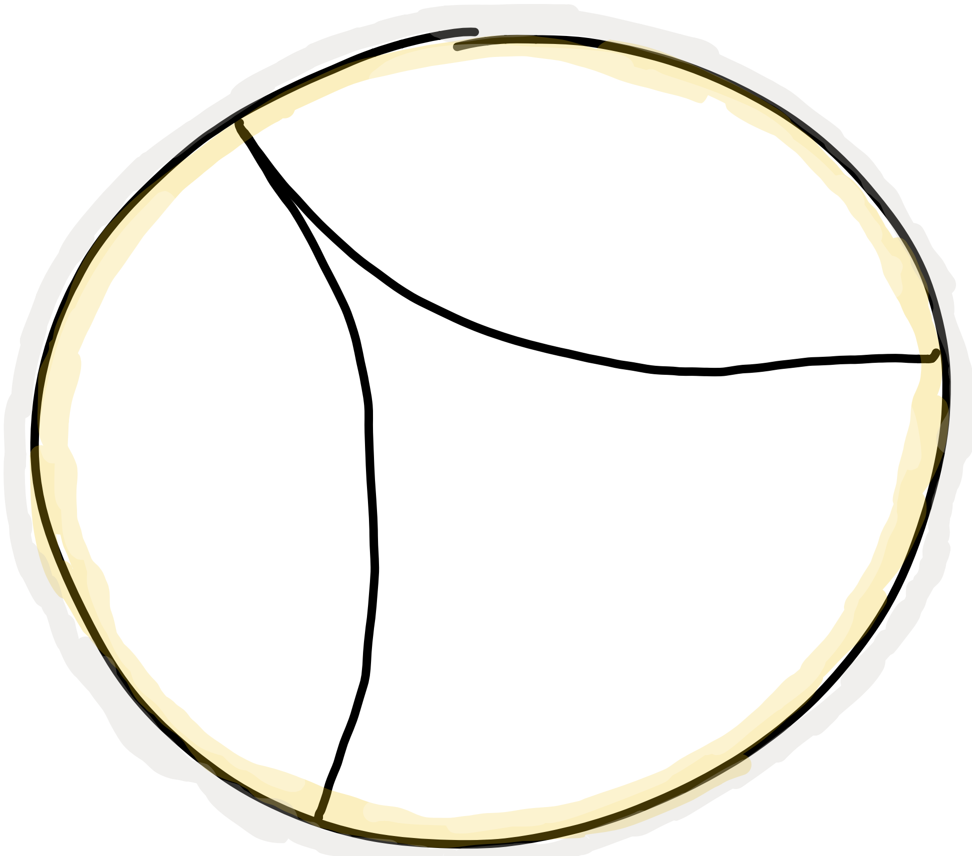
$\pi_1\Big($  $\Big) < \mathrm{Isom}(\mathbb{H}^n)$

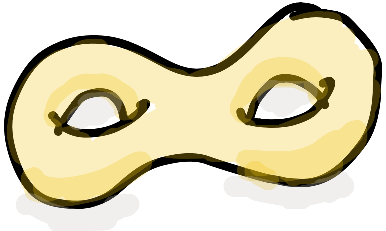
$\pi_1\Big($  $\Big) < \mathrm{Isom}(\mathbb{H}^n)$

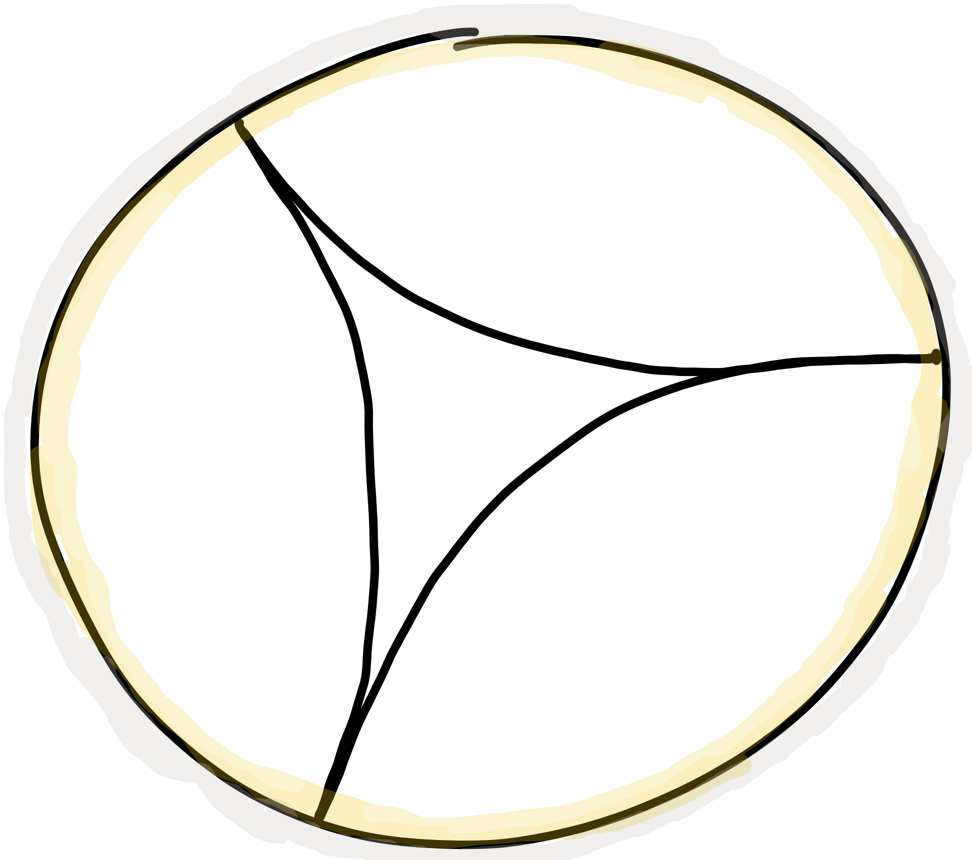
$\pi_1\Big($  $\Big) < \mathrm{Isom}(\mathbb{H}^n)$

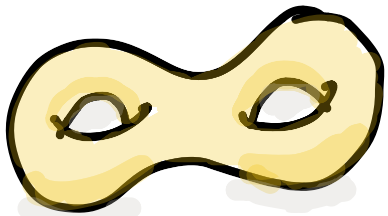


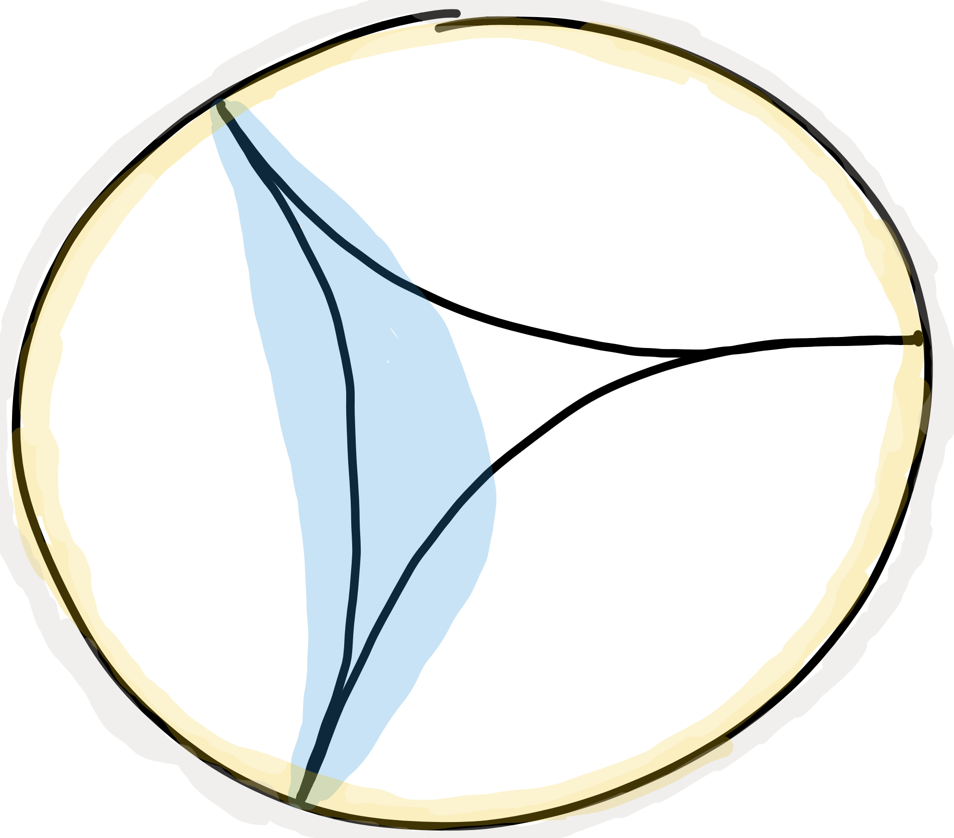
$\pi_1\Bigg($  $\Bigg) < \mathrm{Isom}(\mathbb{H}^n)$

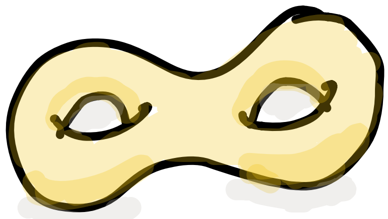


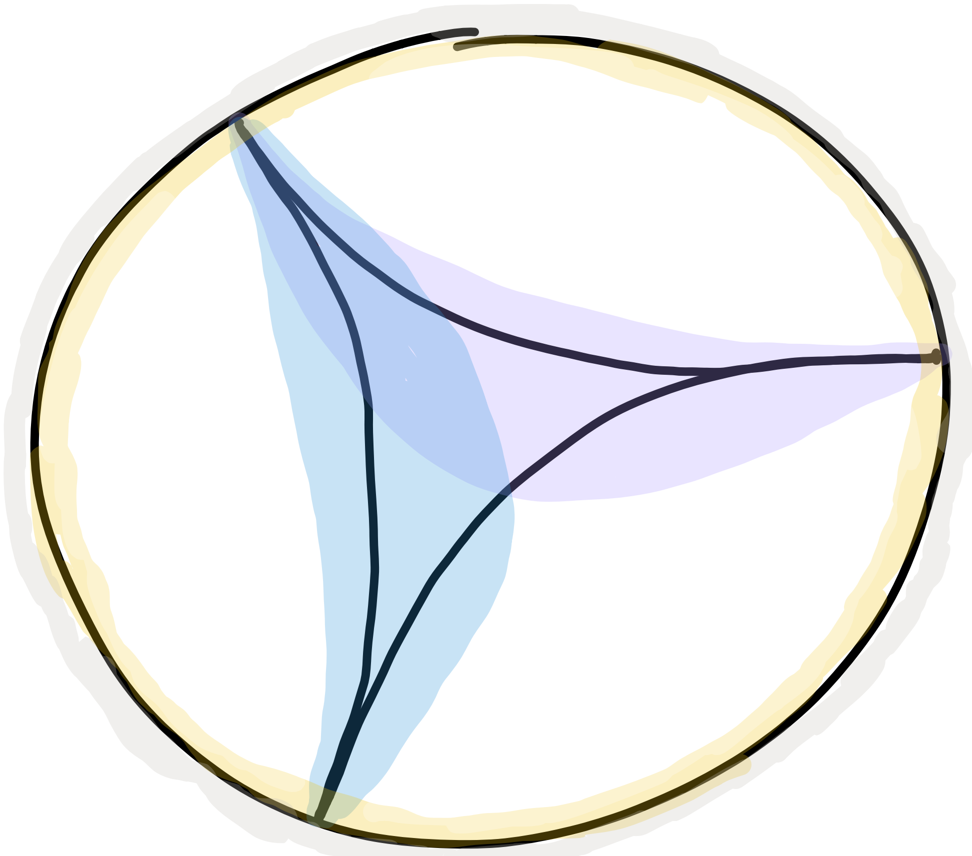
$\pi_1\big($  $\big) < \mathrm{Isom}(\mathbb{H}^n)$

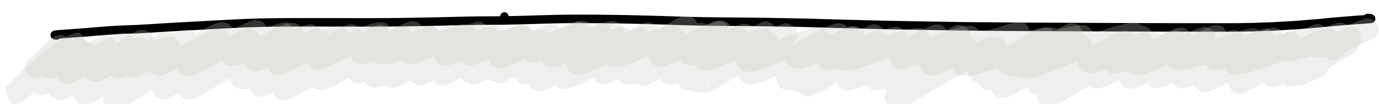


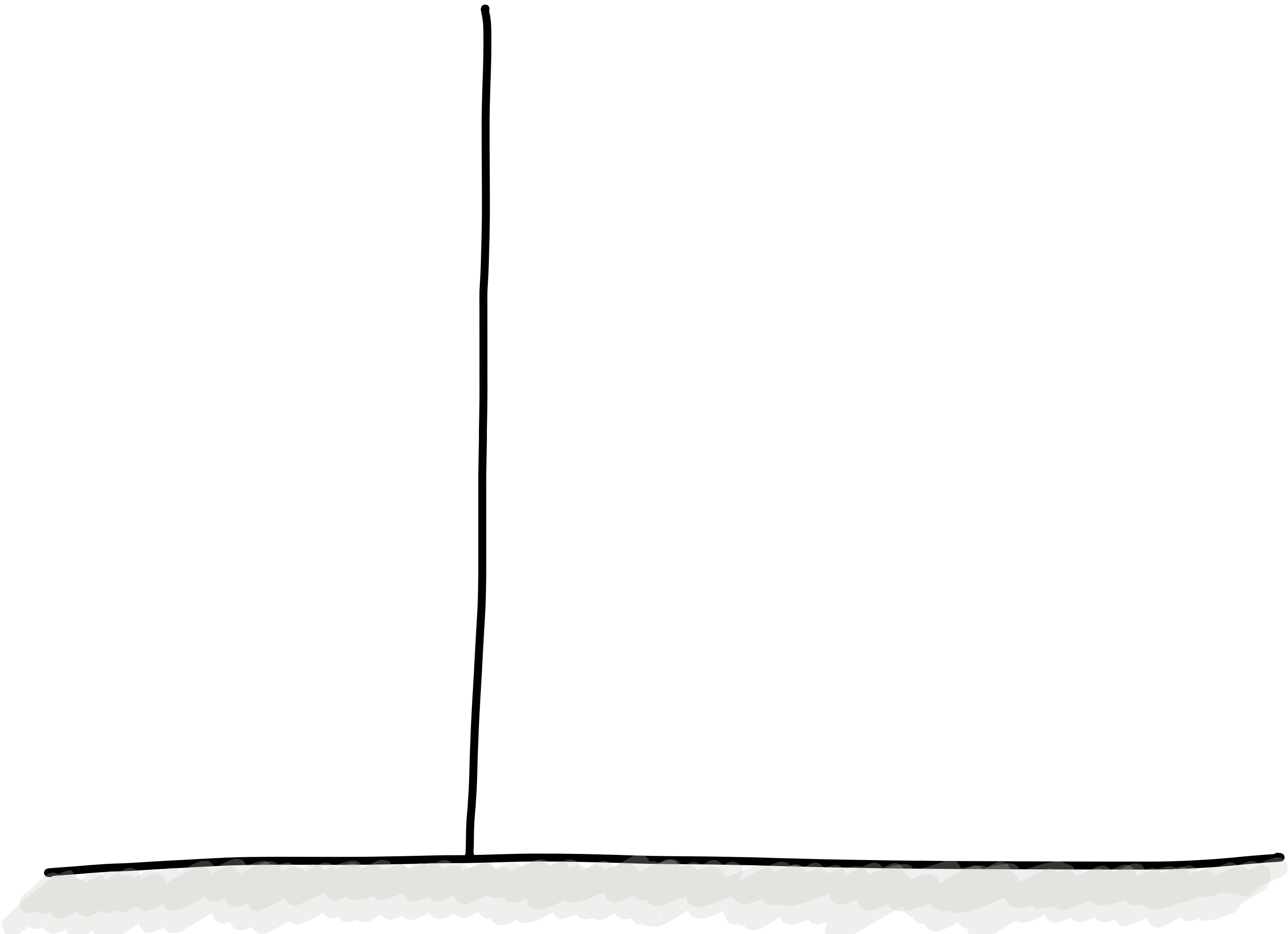
$\pi_1\Big($  $\Big) < \mathrm{Isom}(\mathbb{H}^n)$

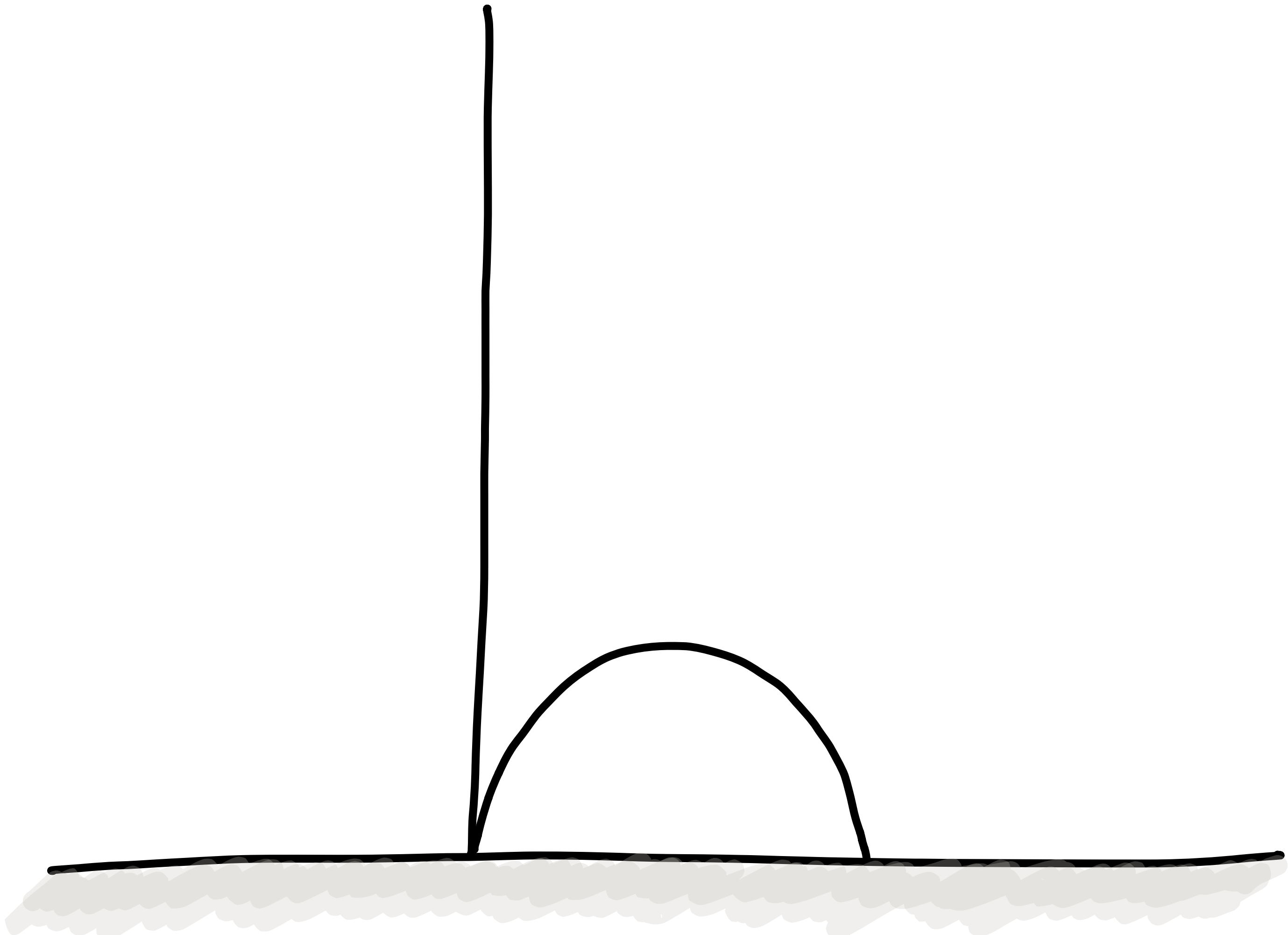


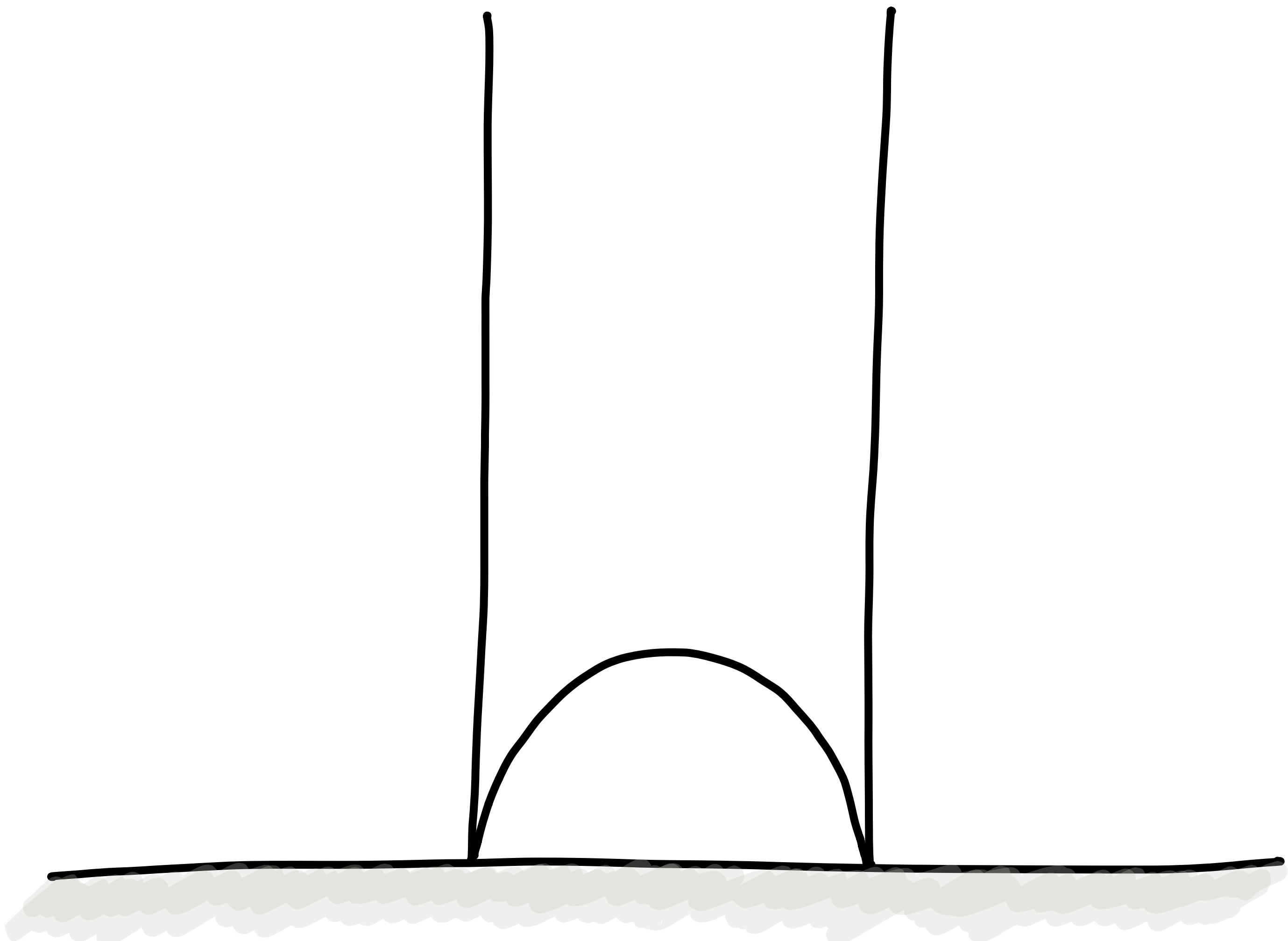
$\pi_1\Big($  $\Big) < \mathrm{Isom}(\mathbb{H}^n)$

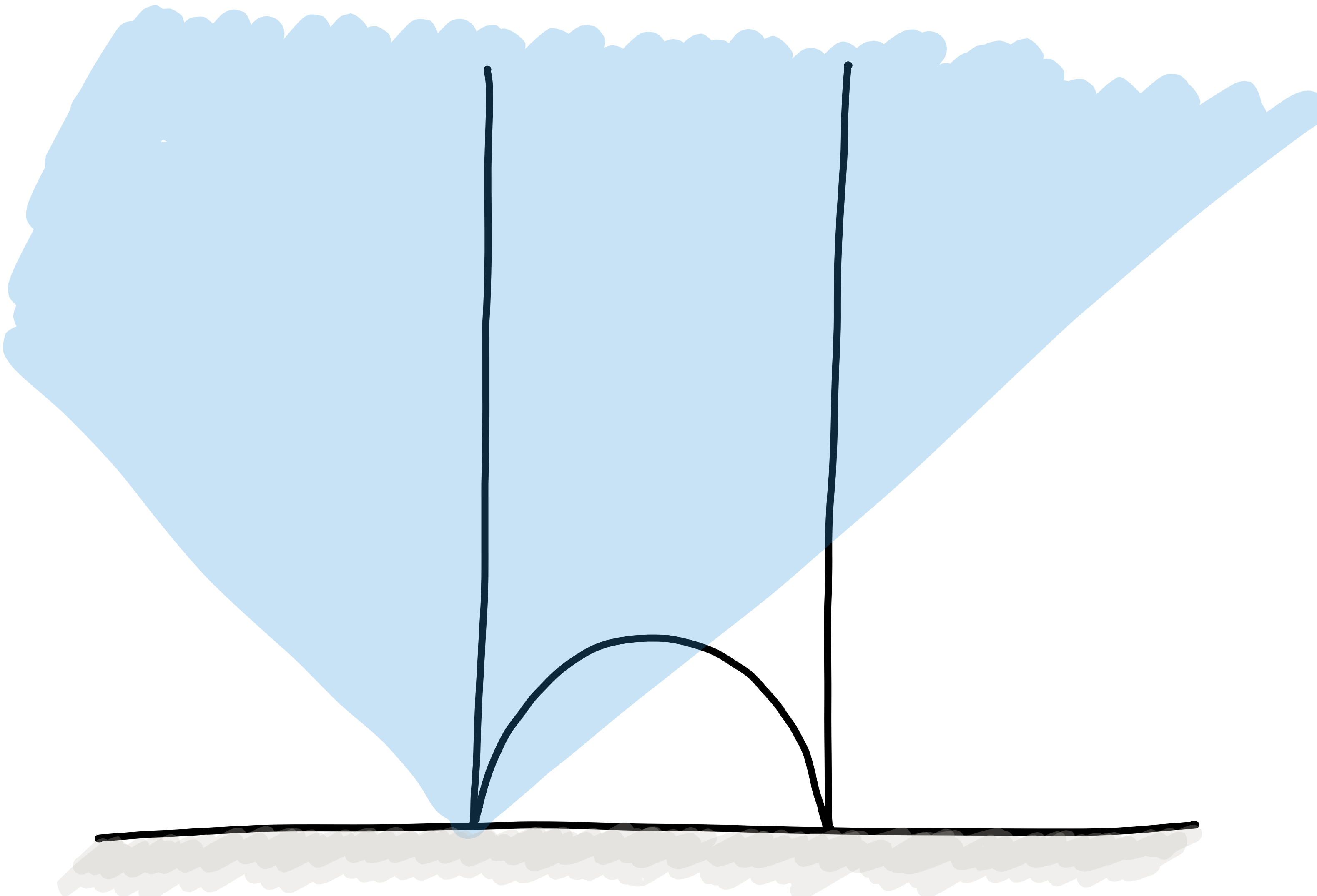


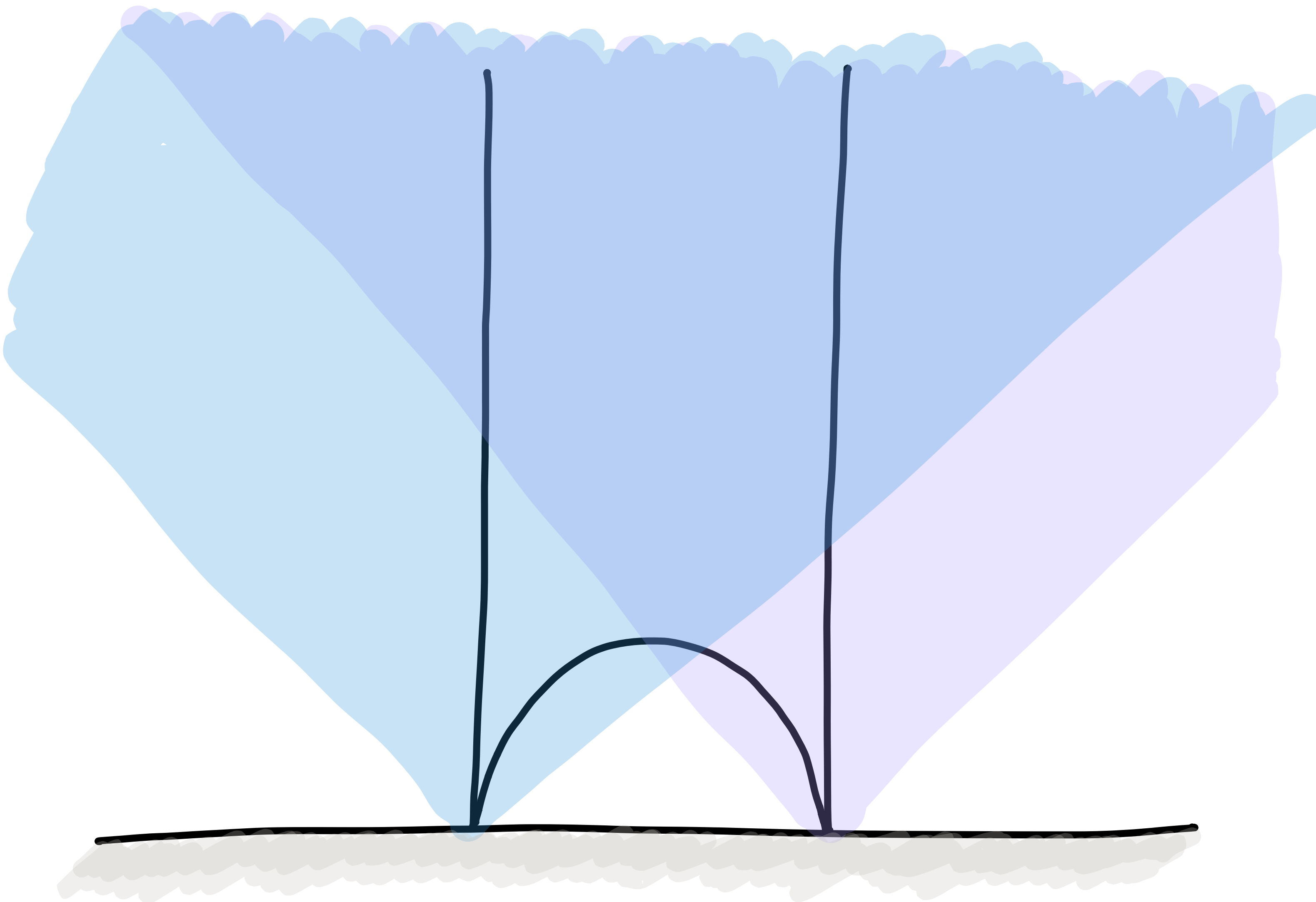












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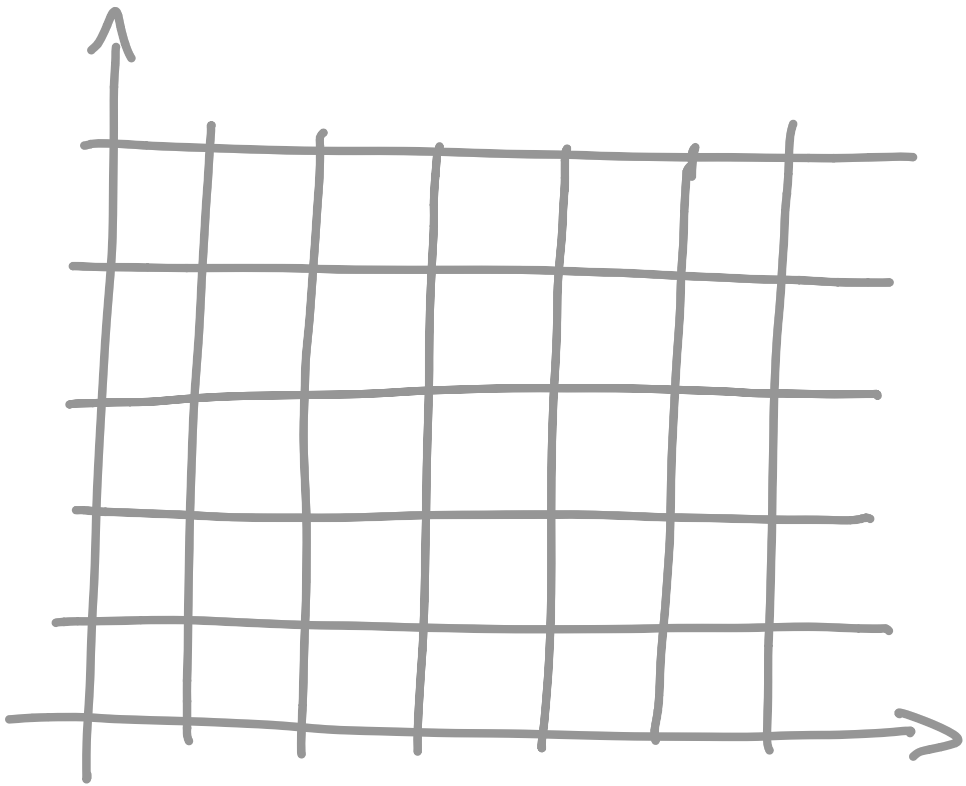

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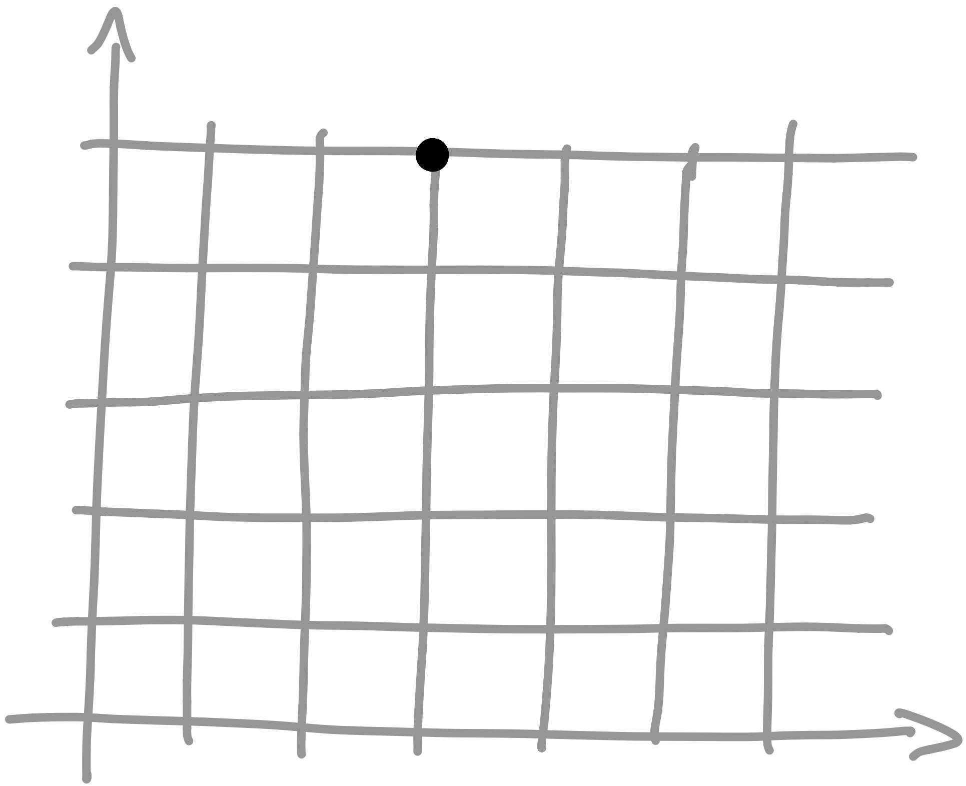

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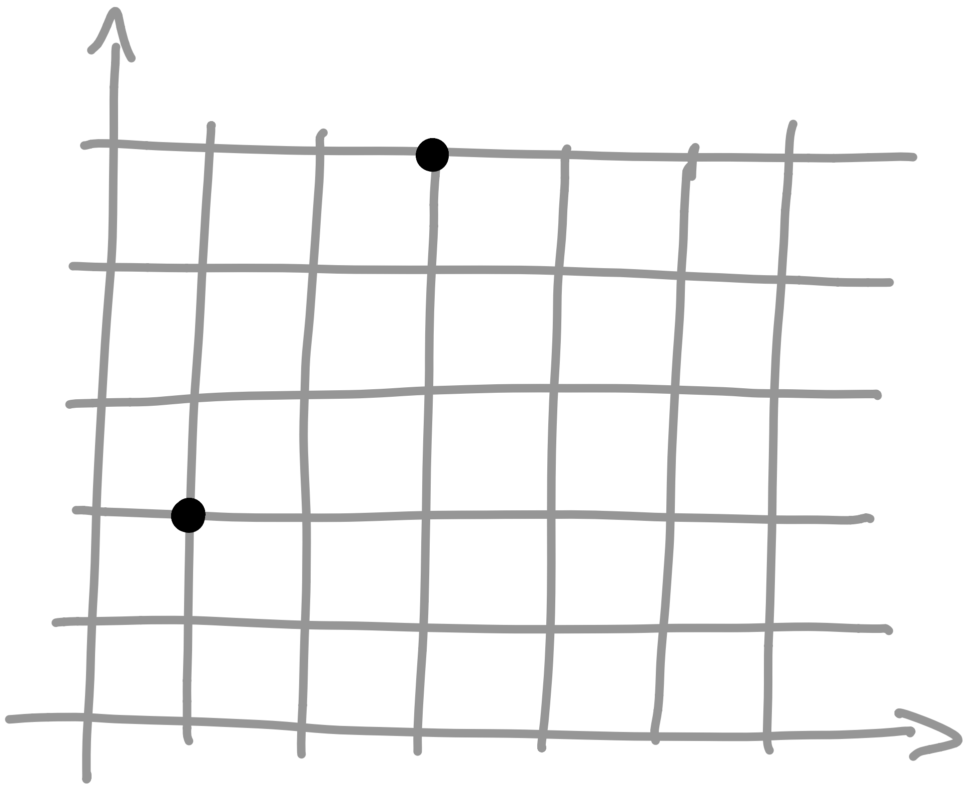

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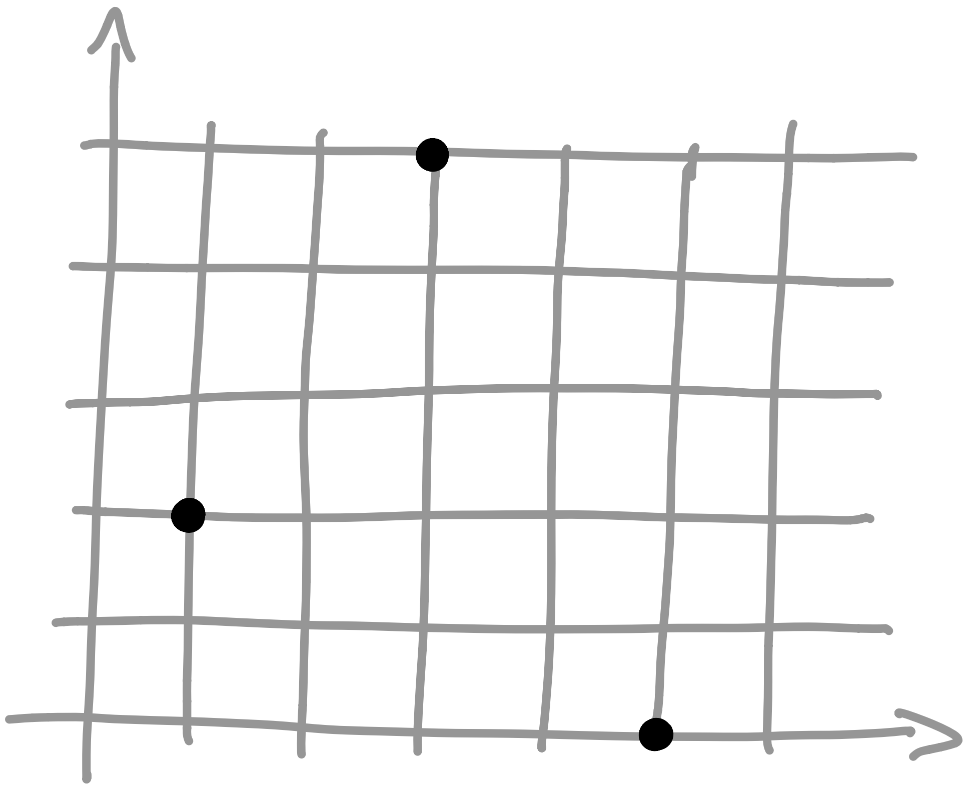

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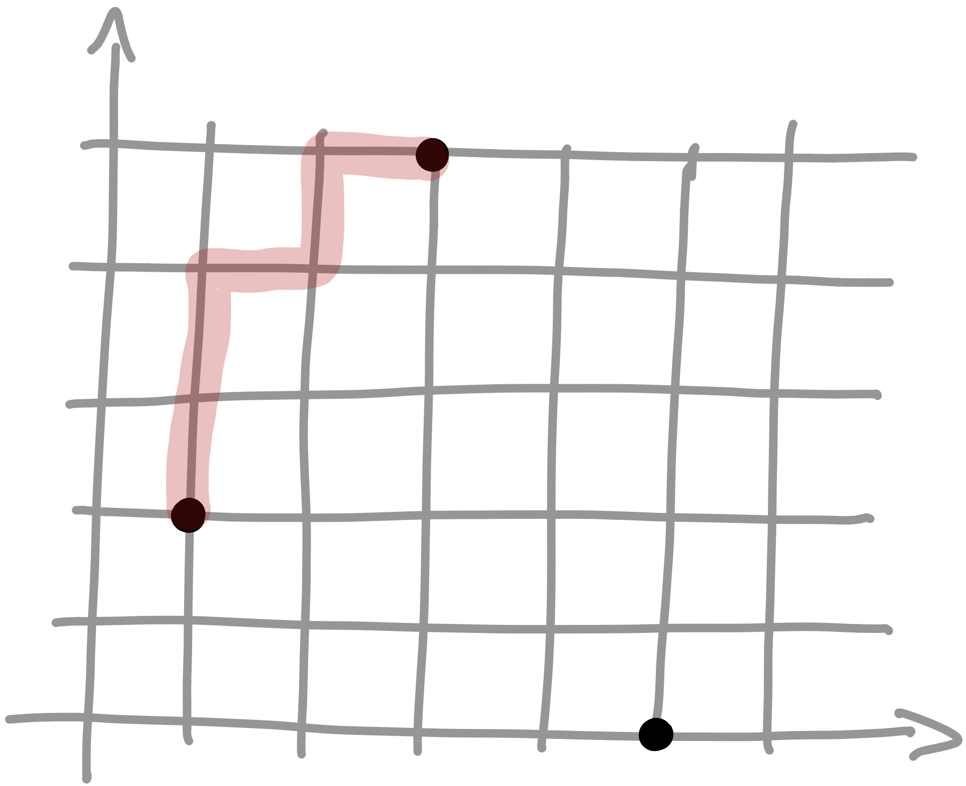

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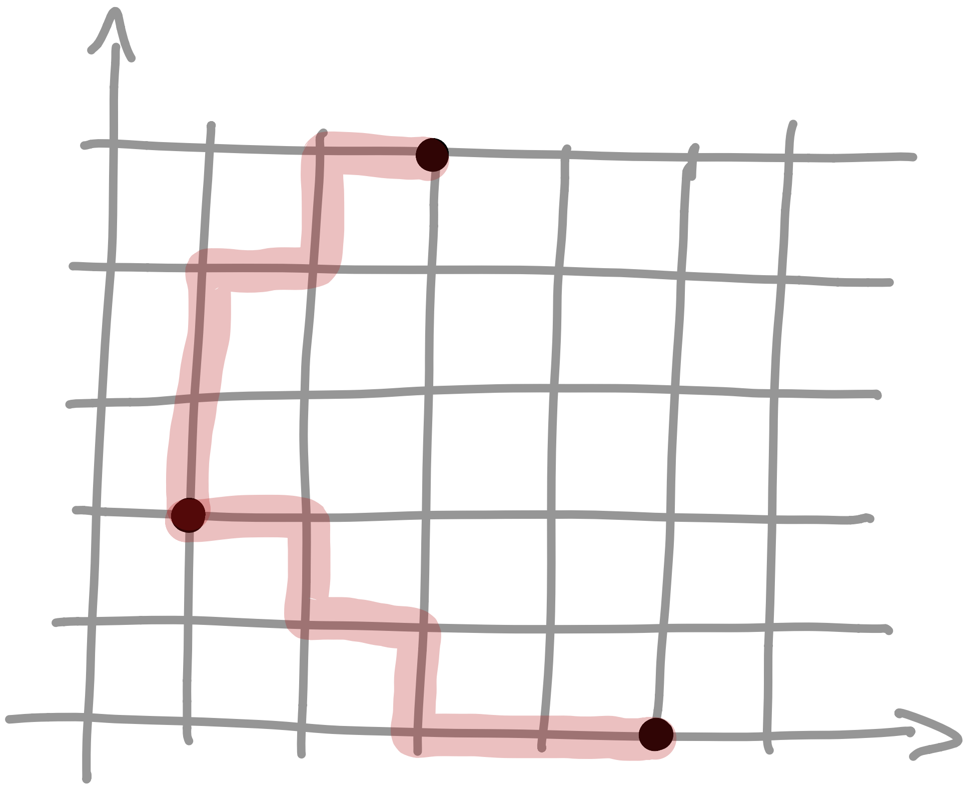

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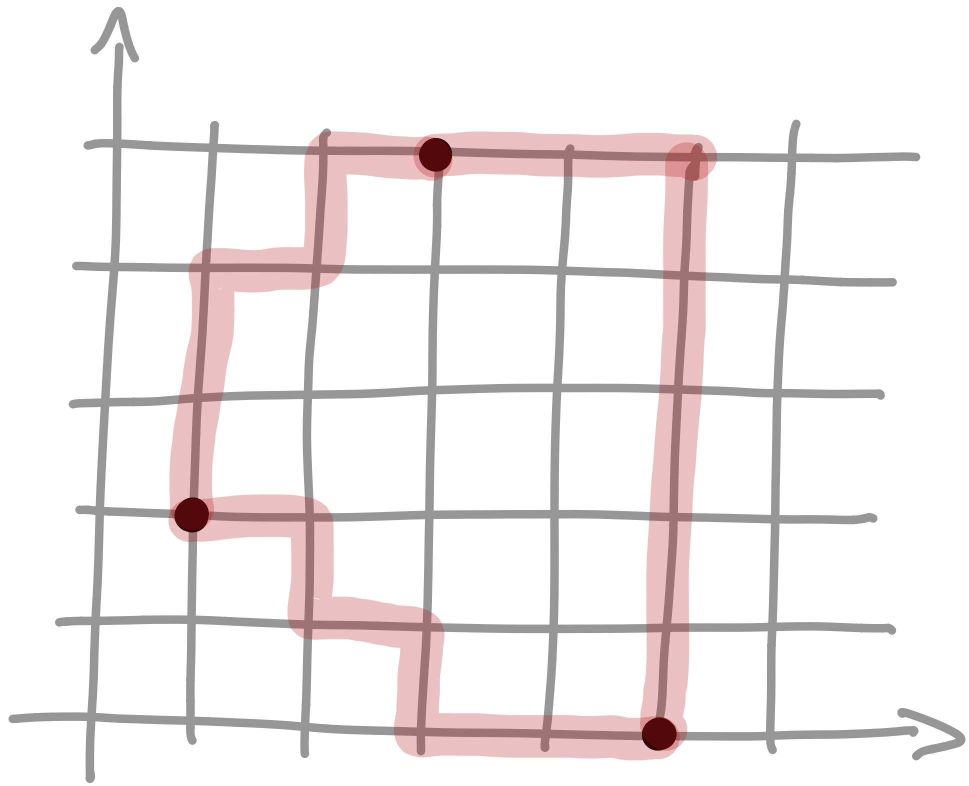

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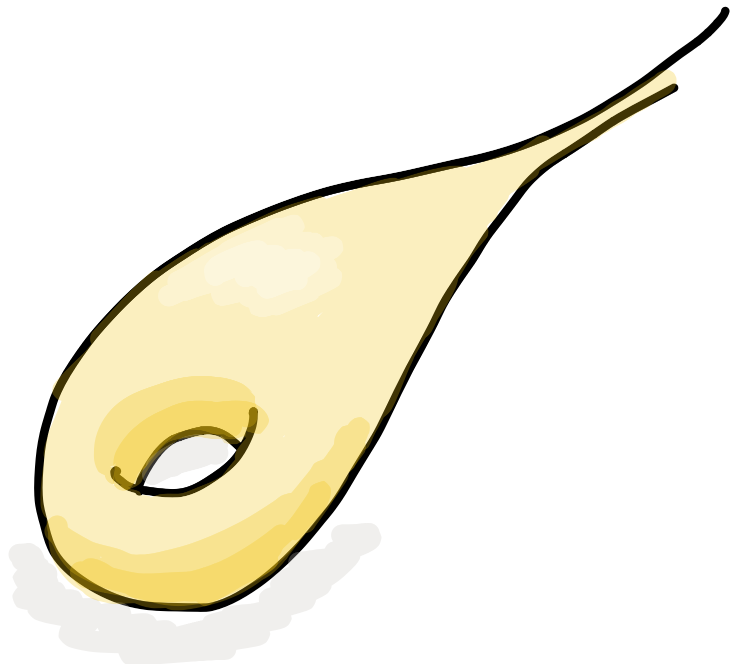
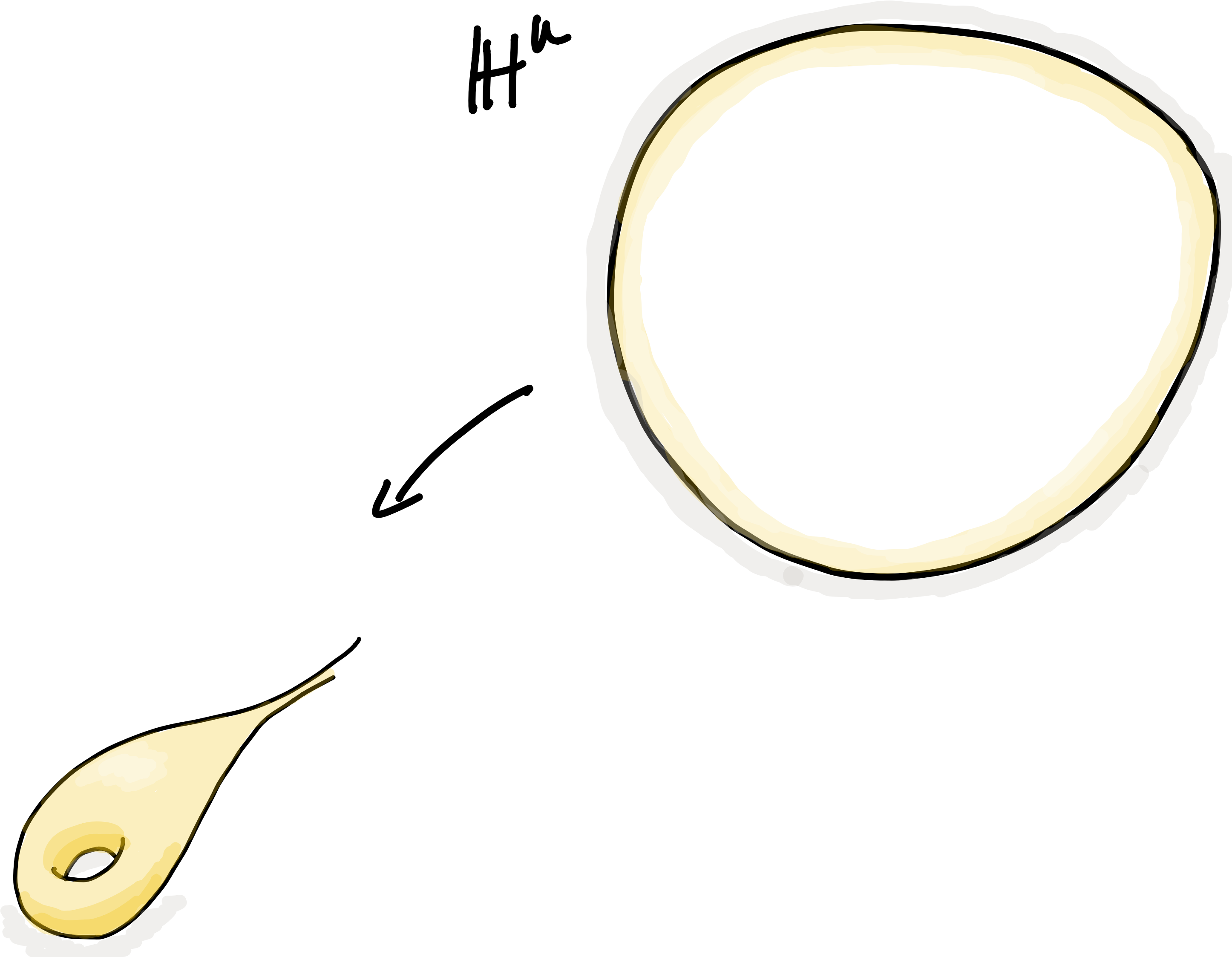
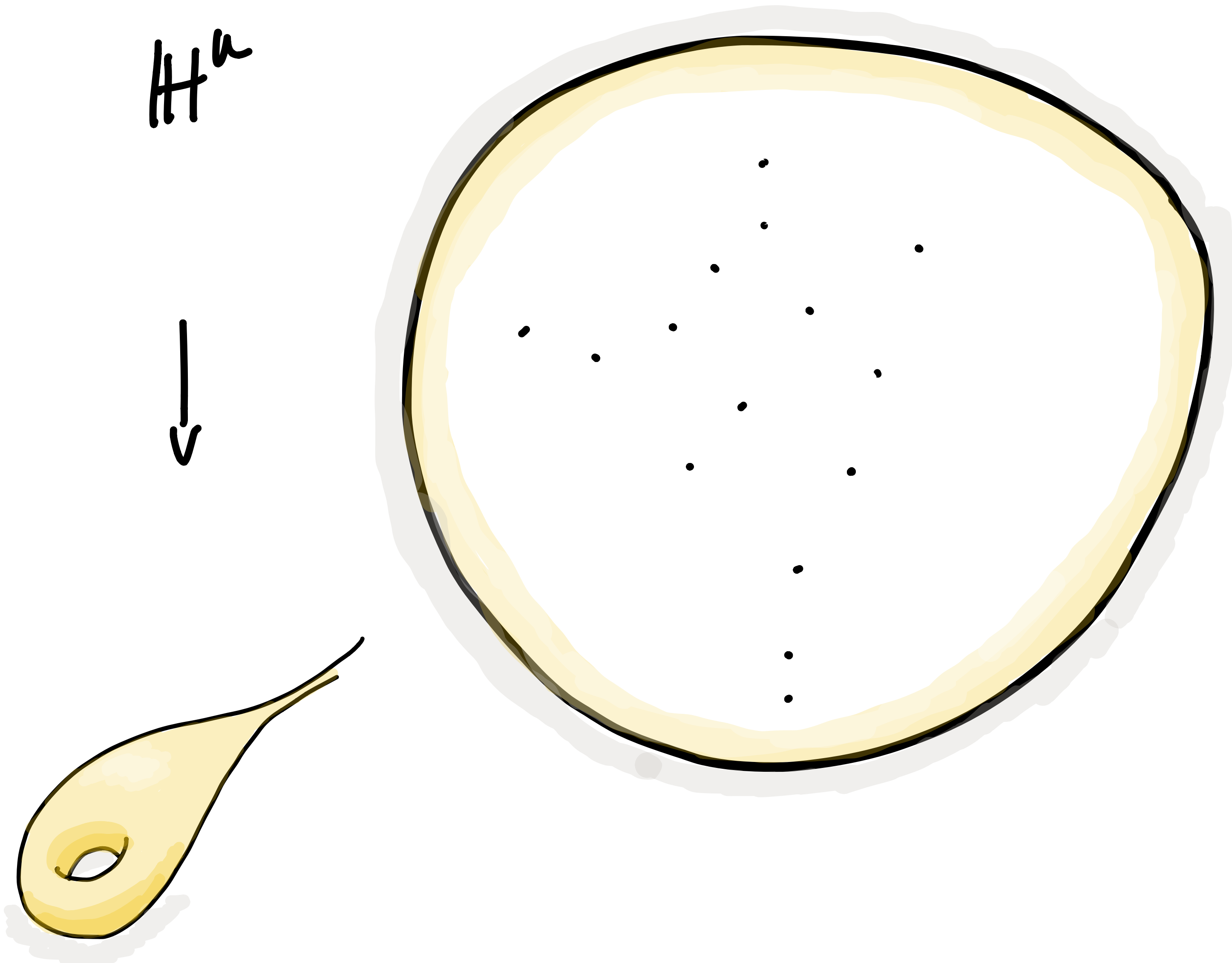
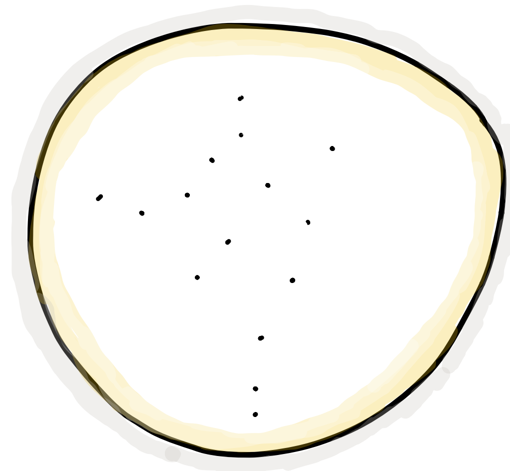
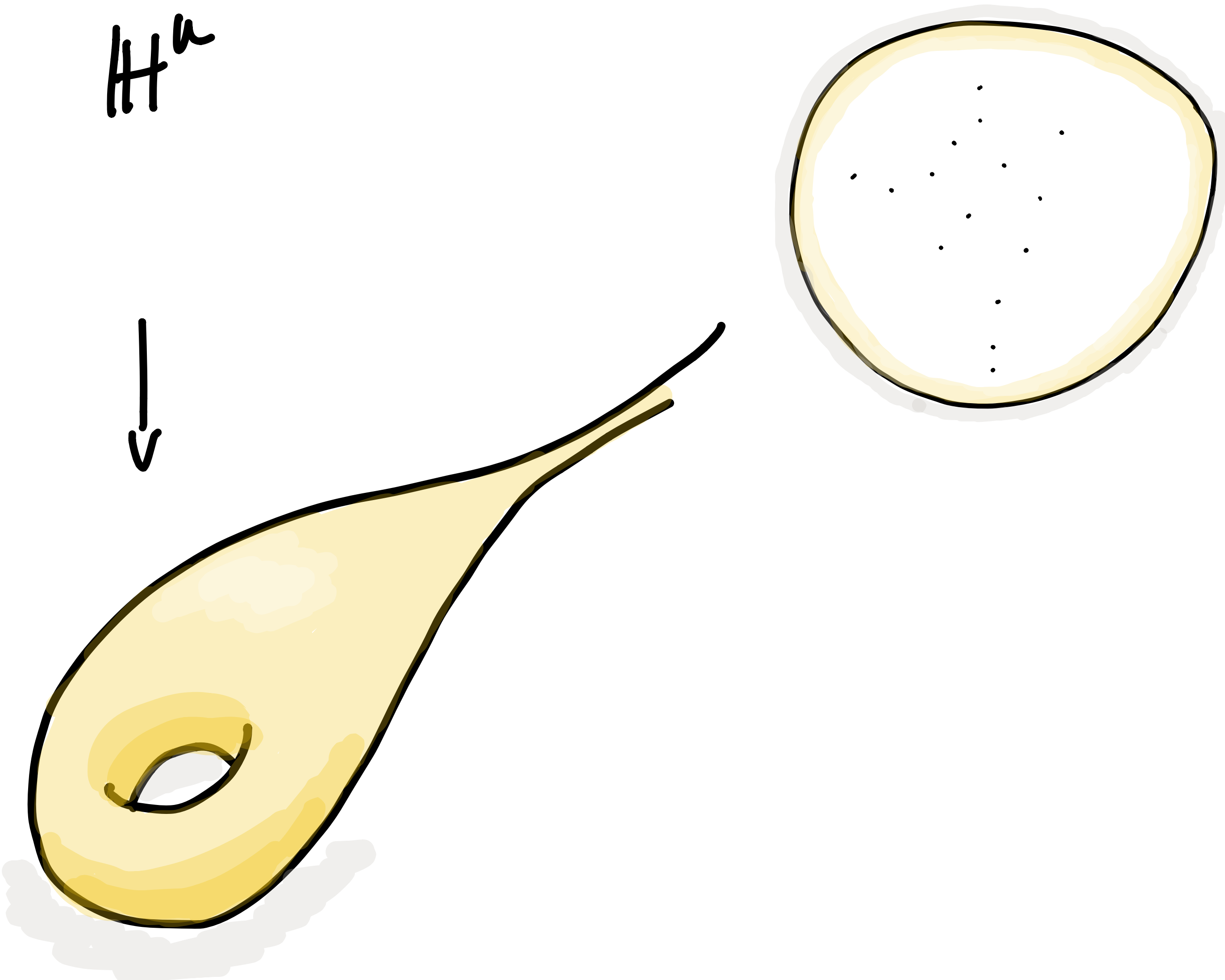



$\mathbb{H}^n$


$\mathbb{H}^n$

$\mathbb{H}^u$

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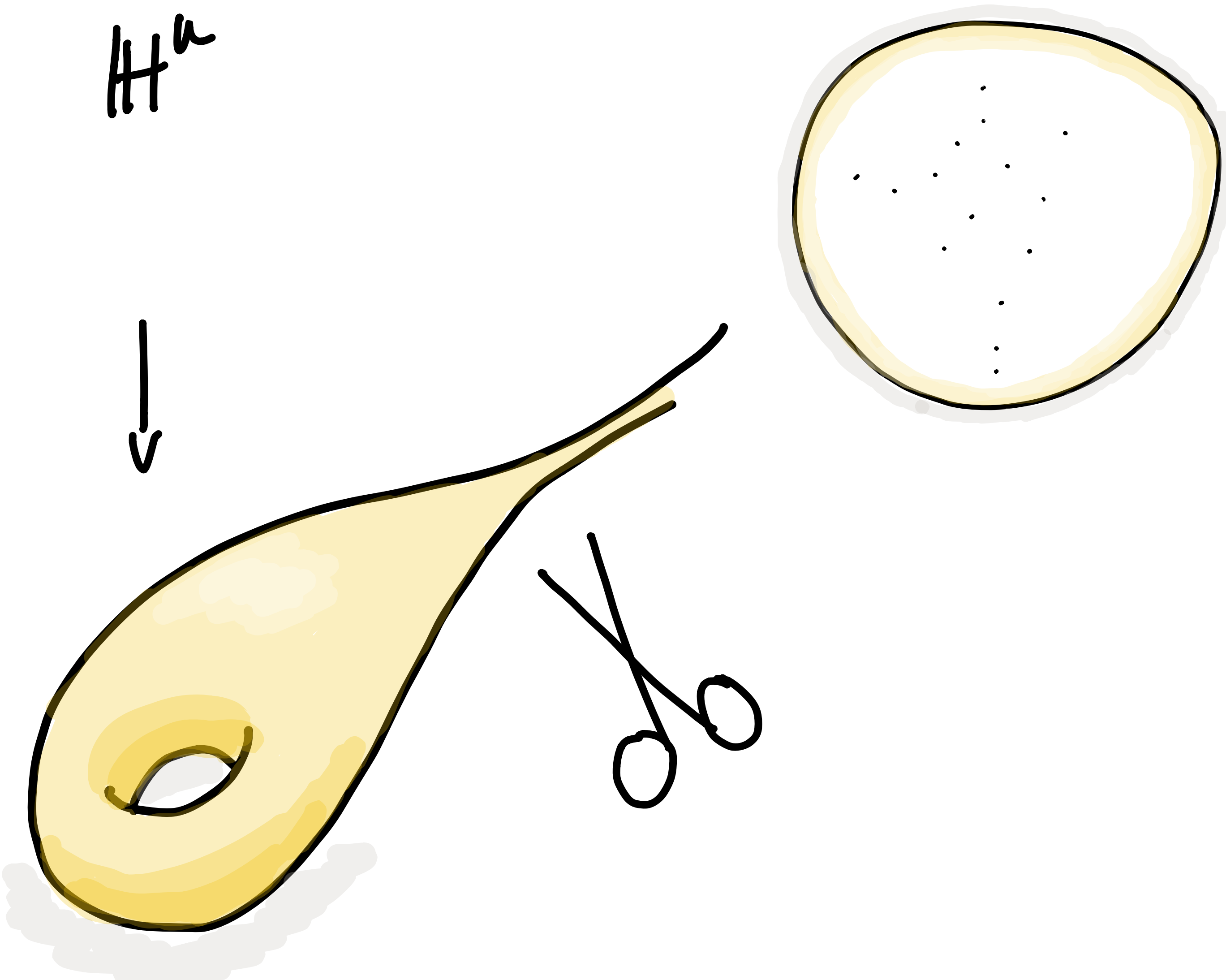
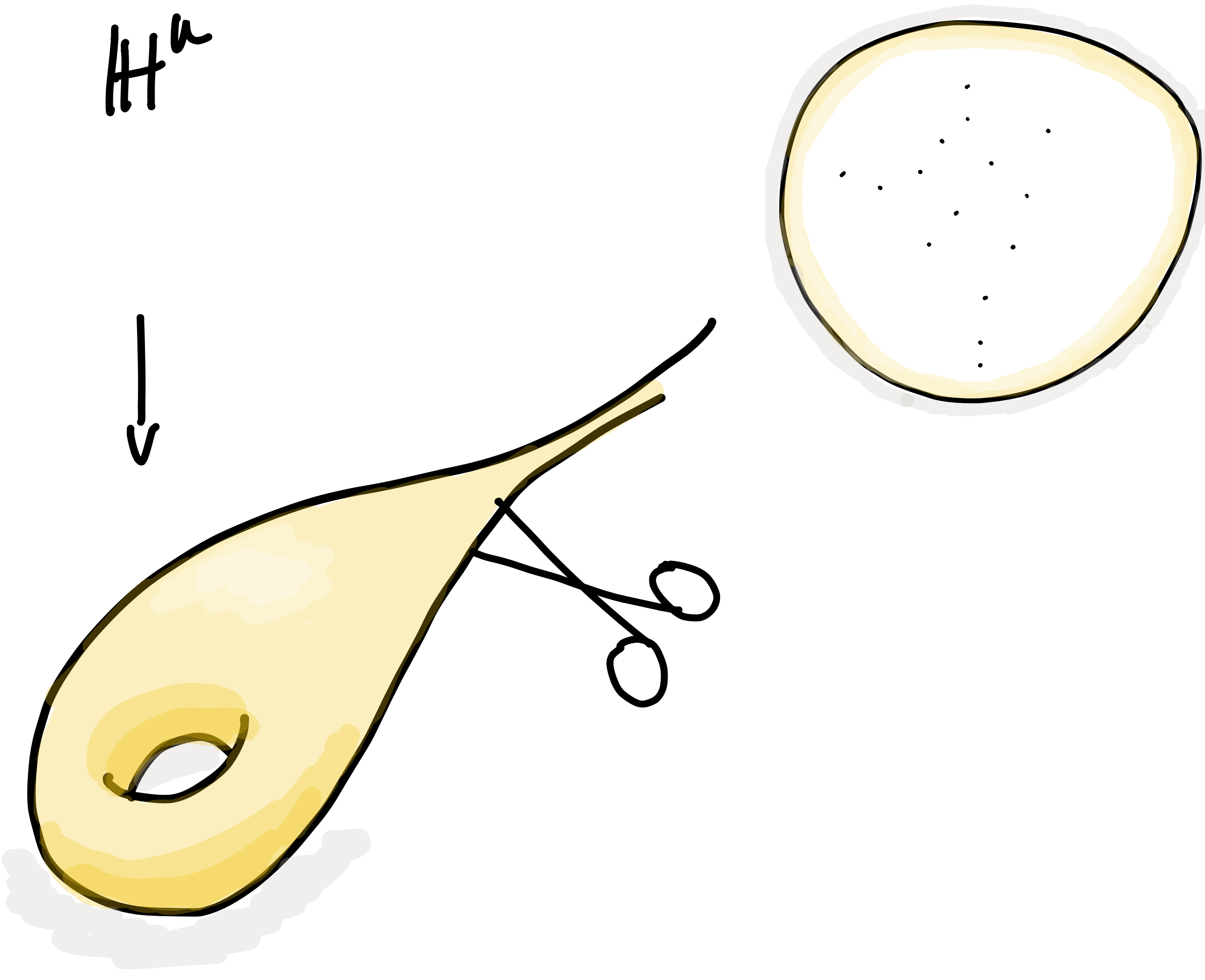

$\mathbb{H}^n$

III^{a}



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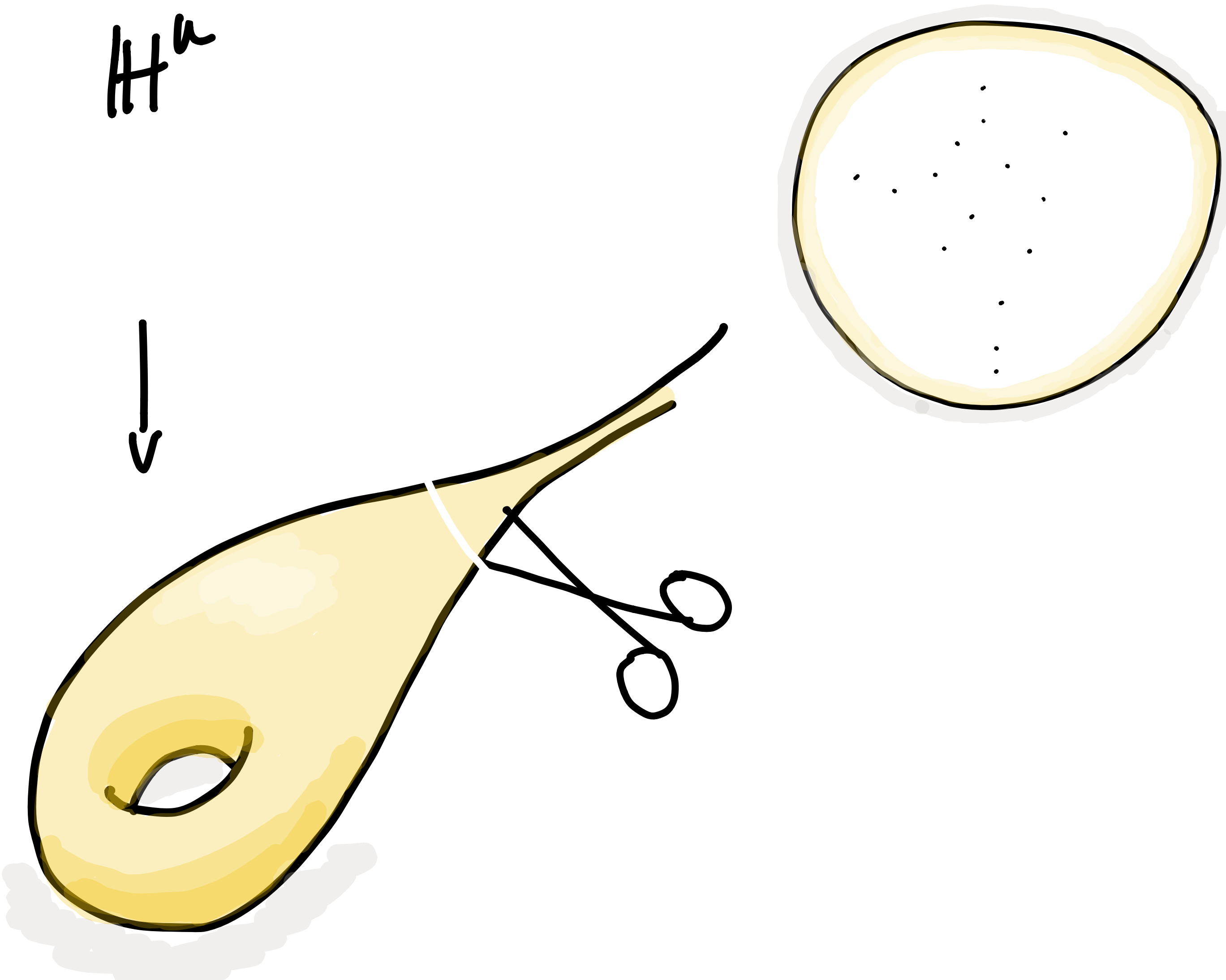

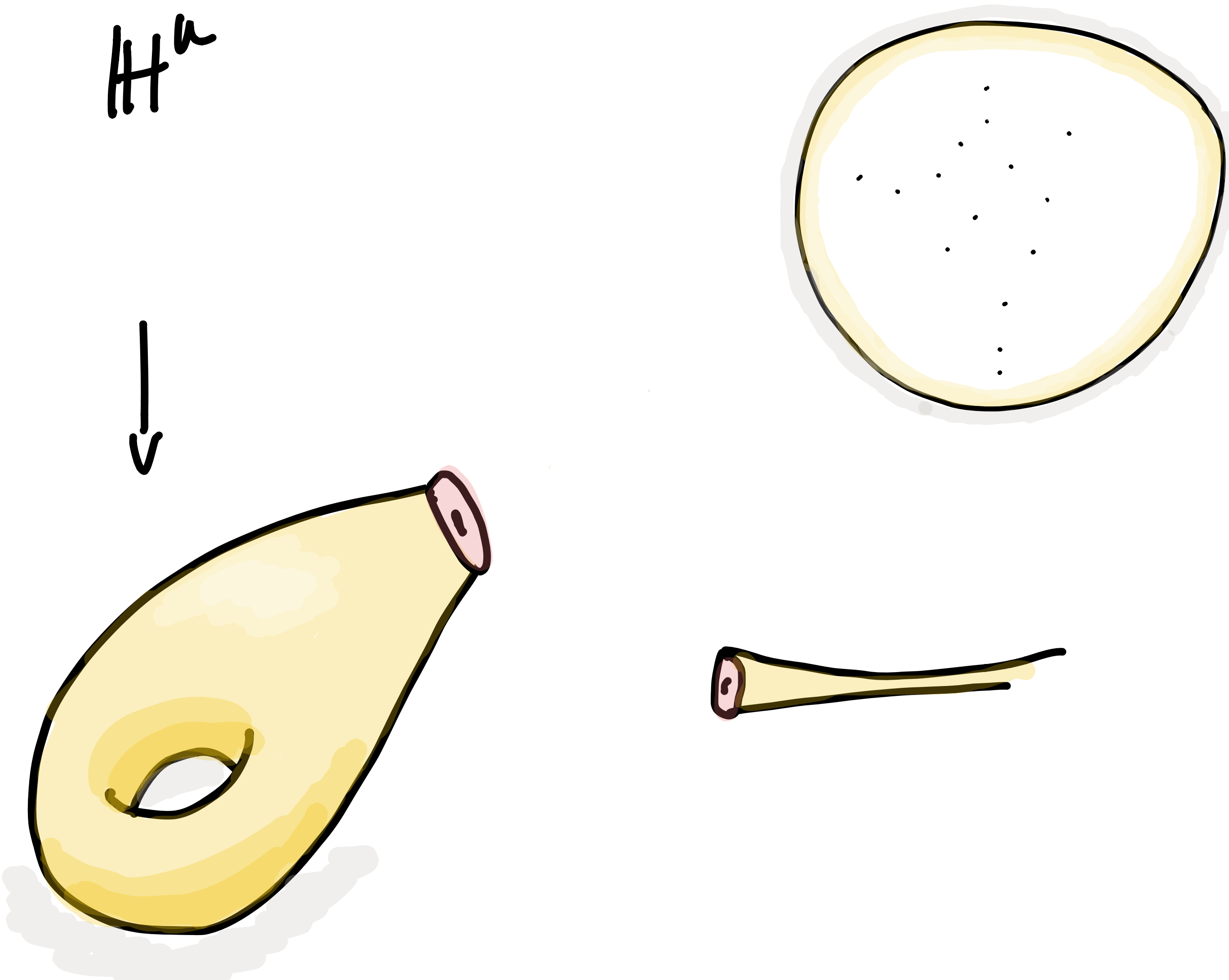
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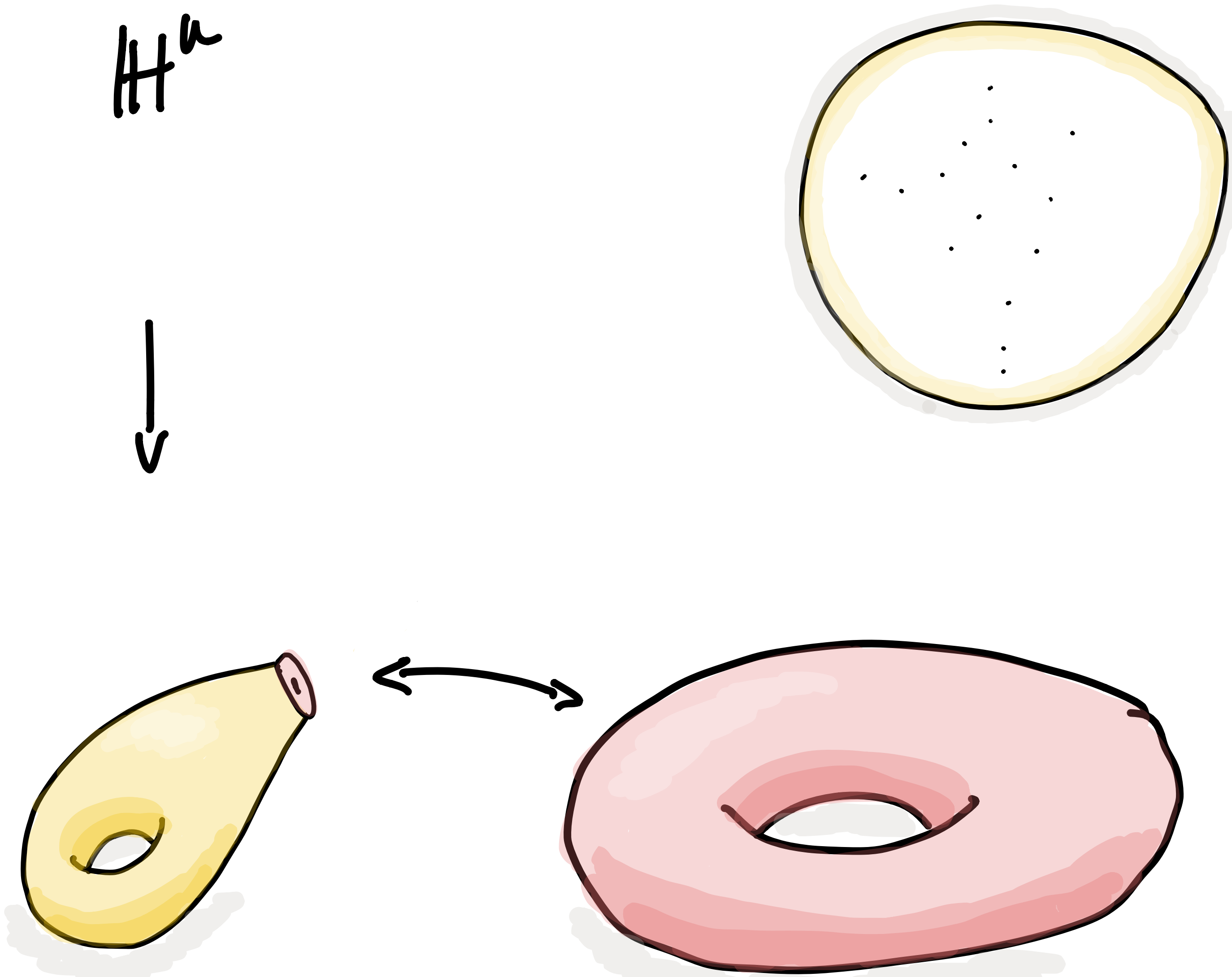
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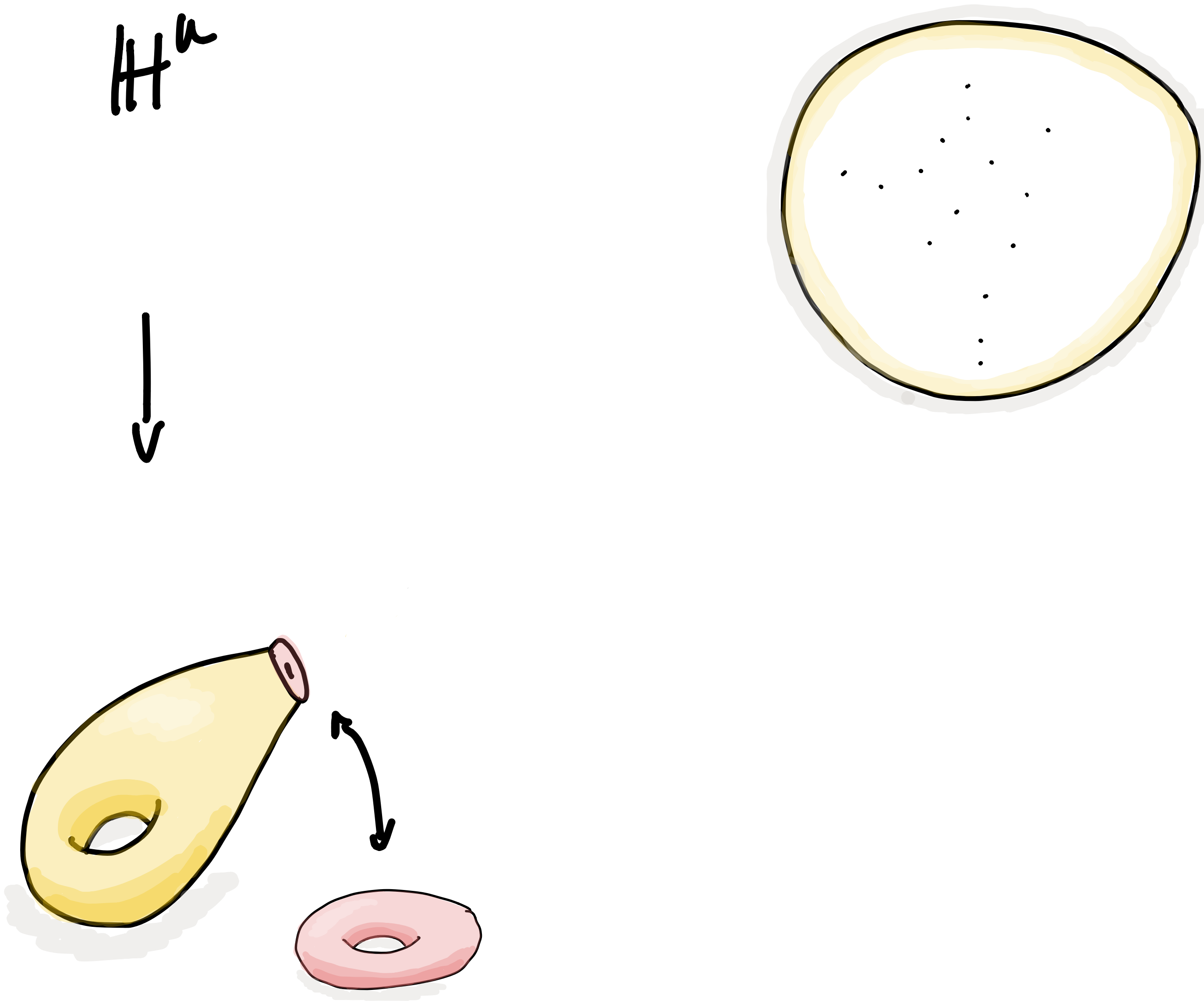
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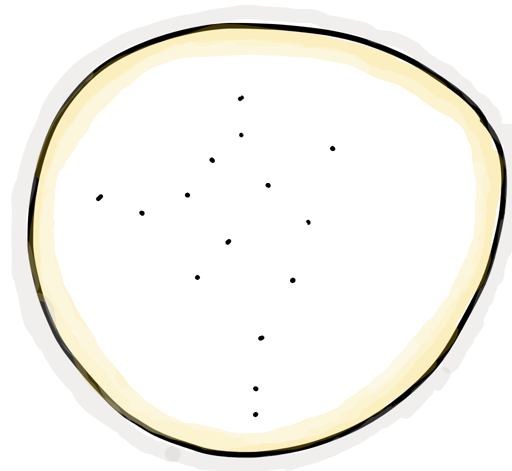

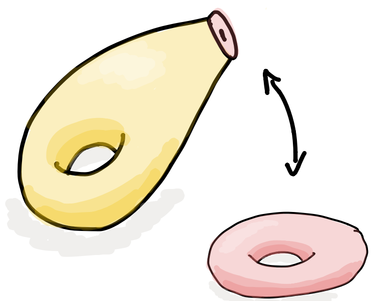

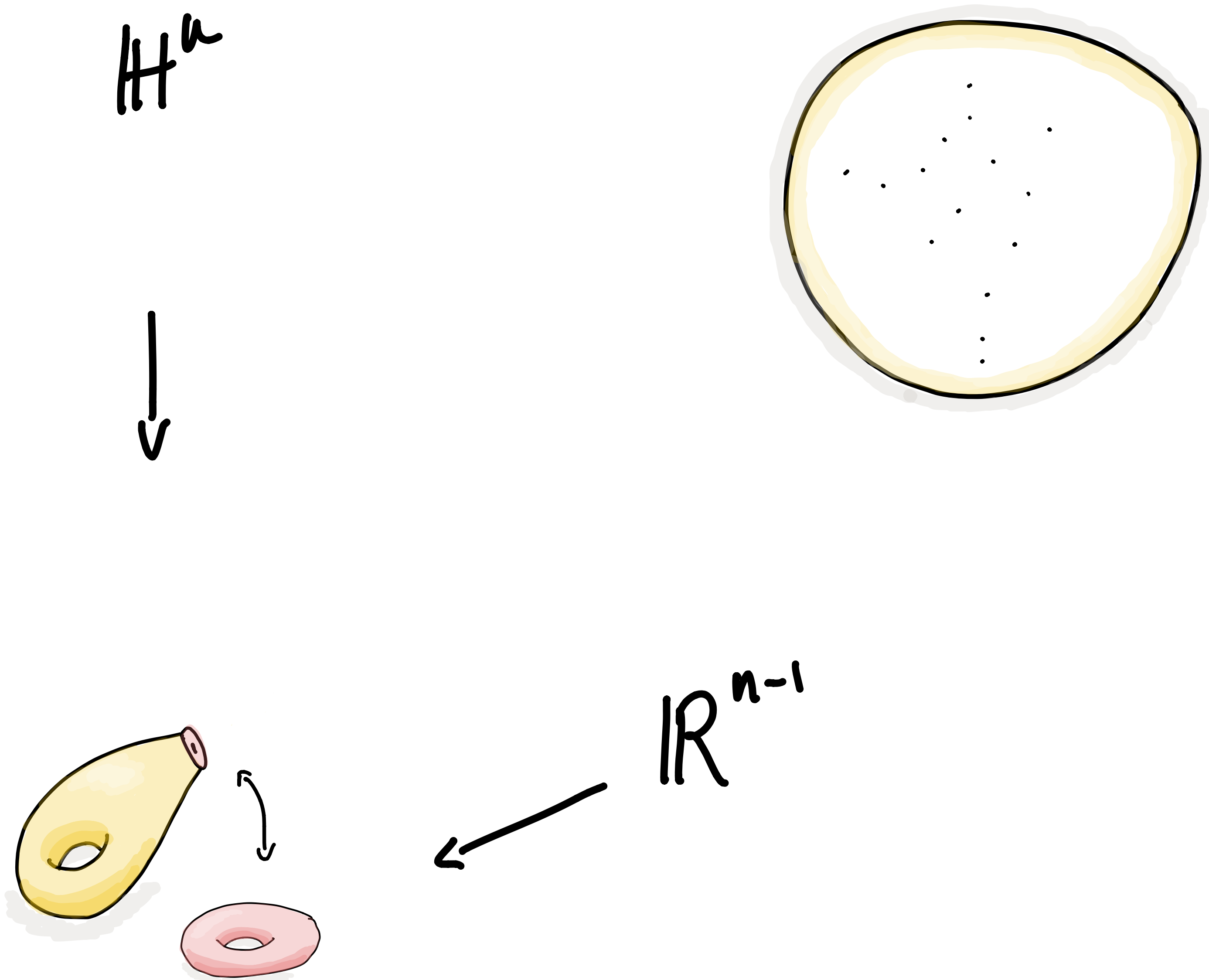
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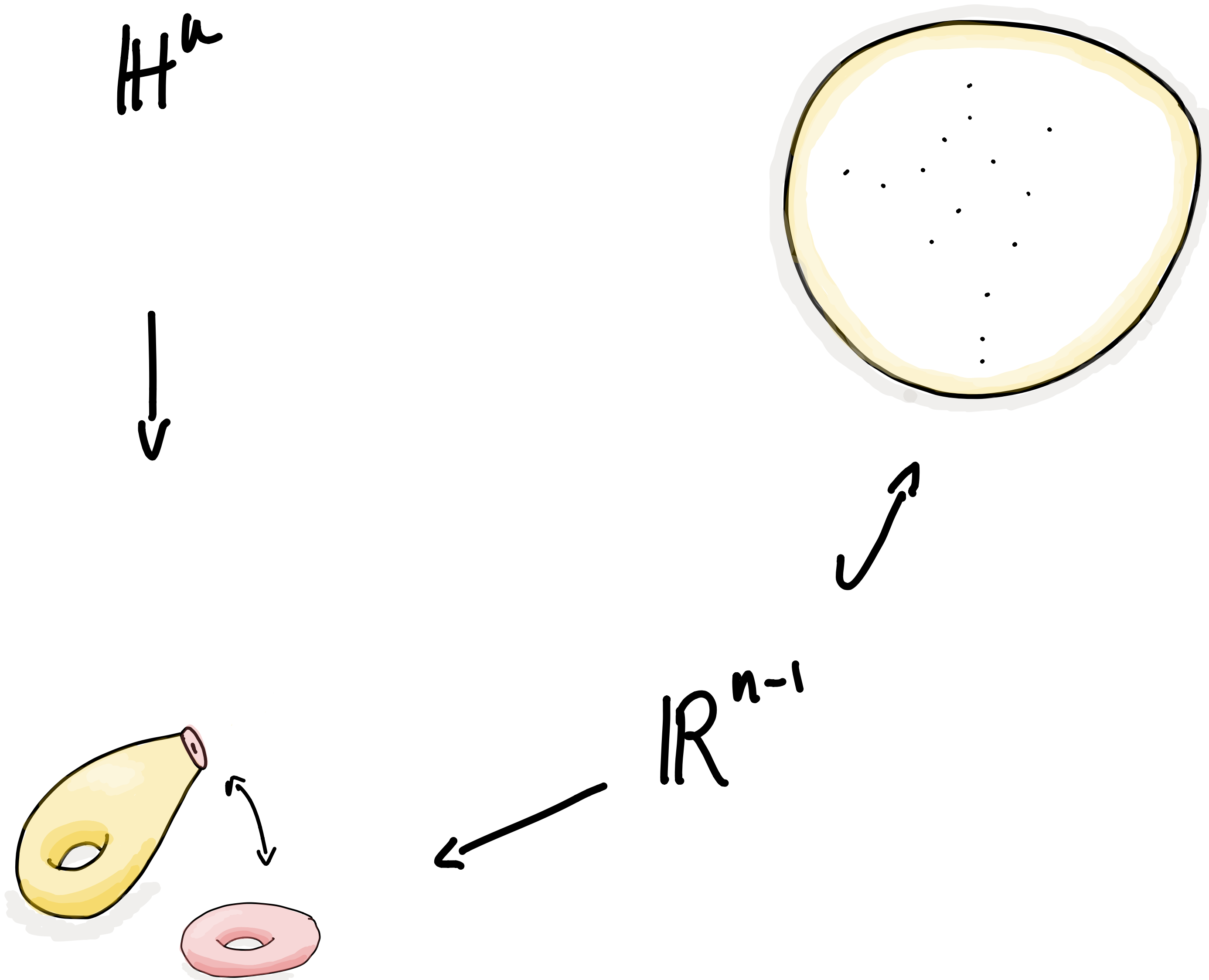
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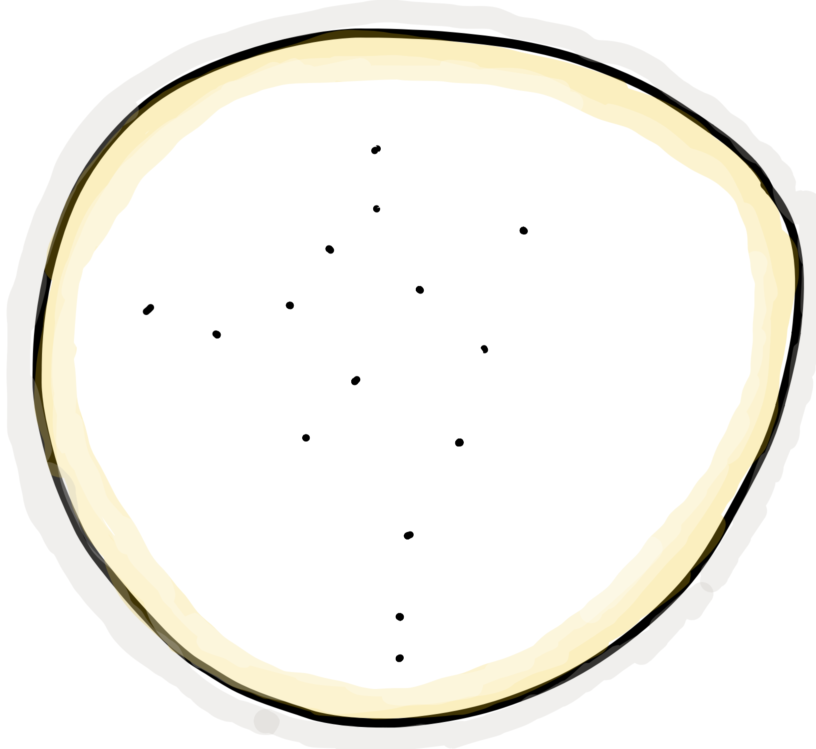

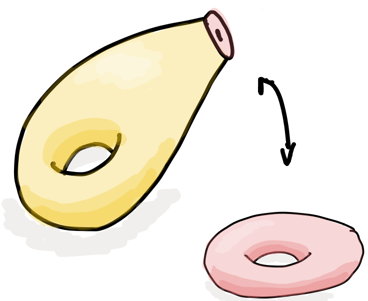

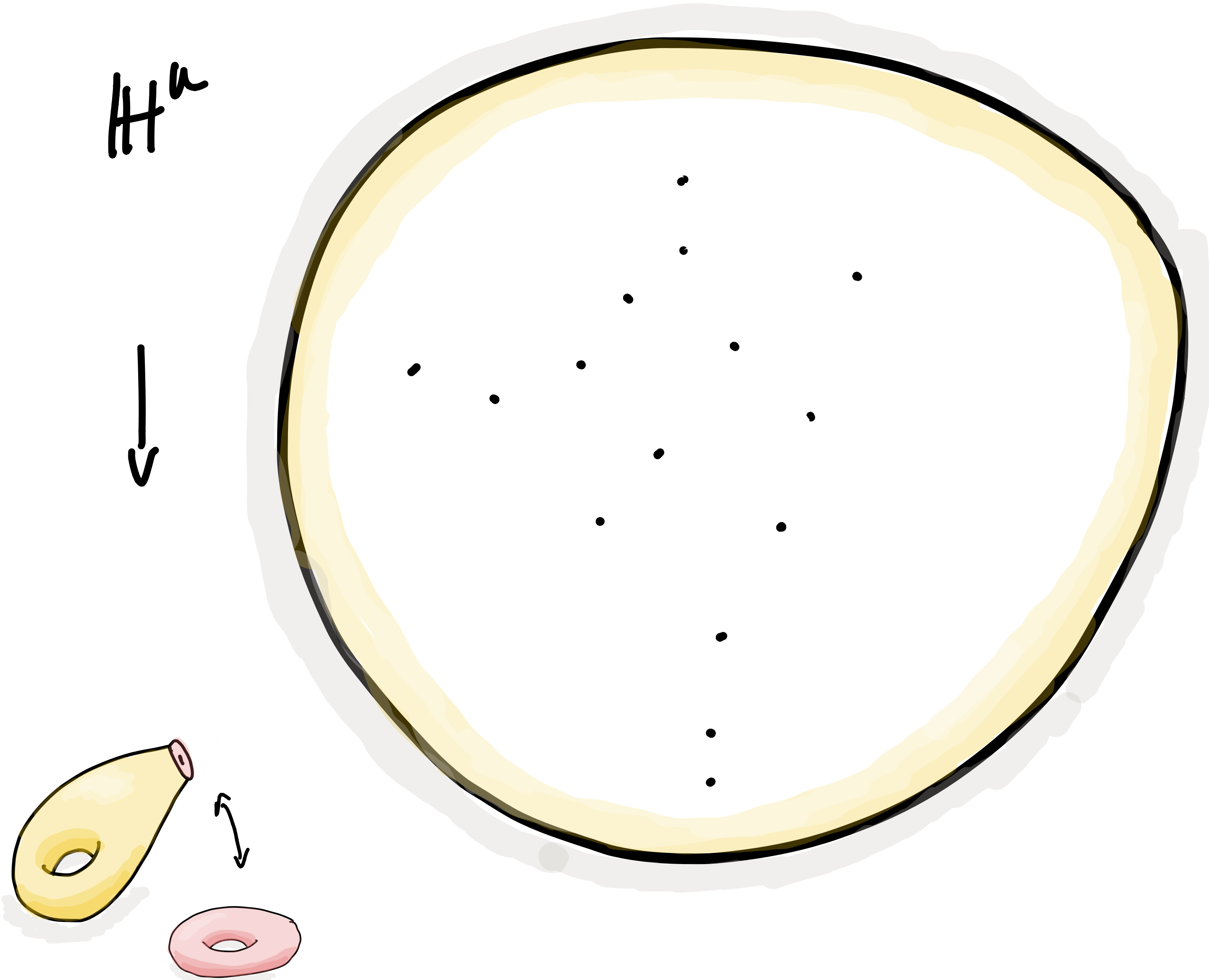
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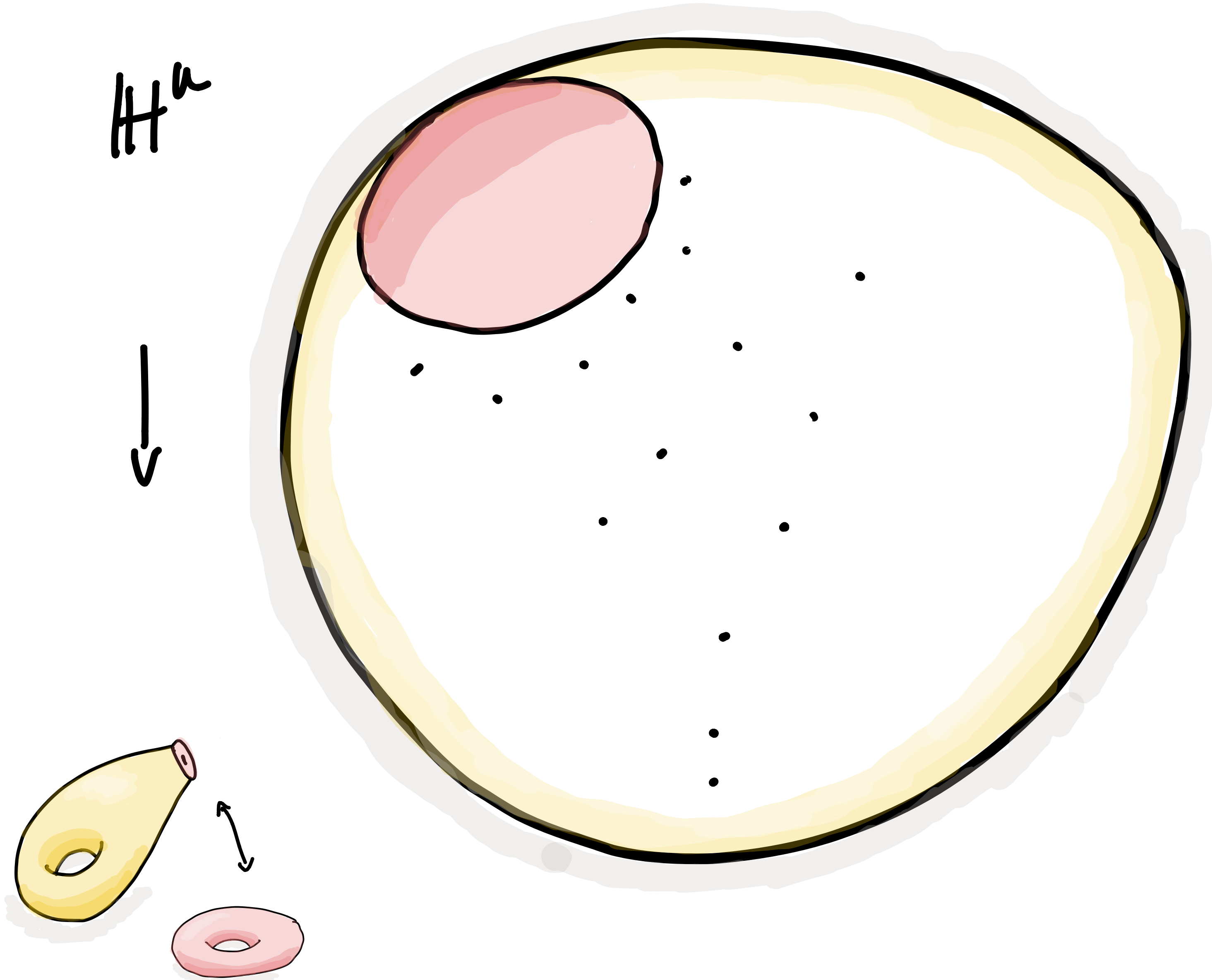
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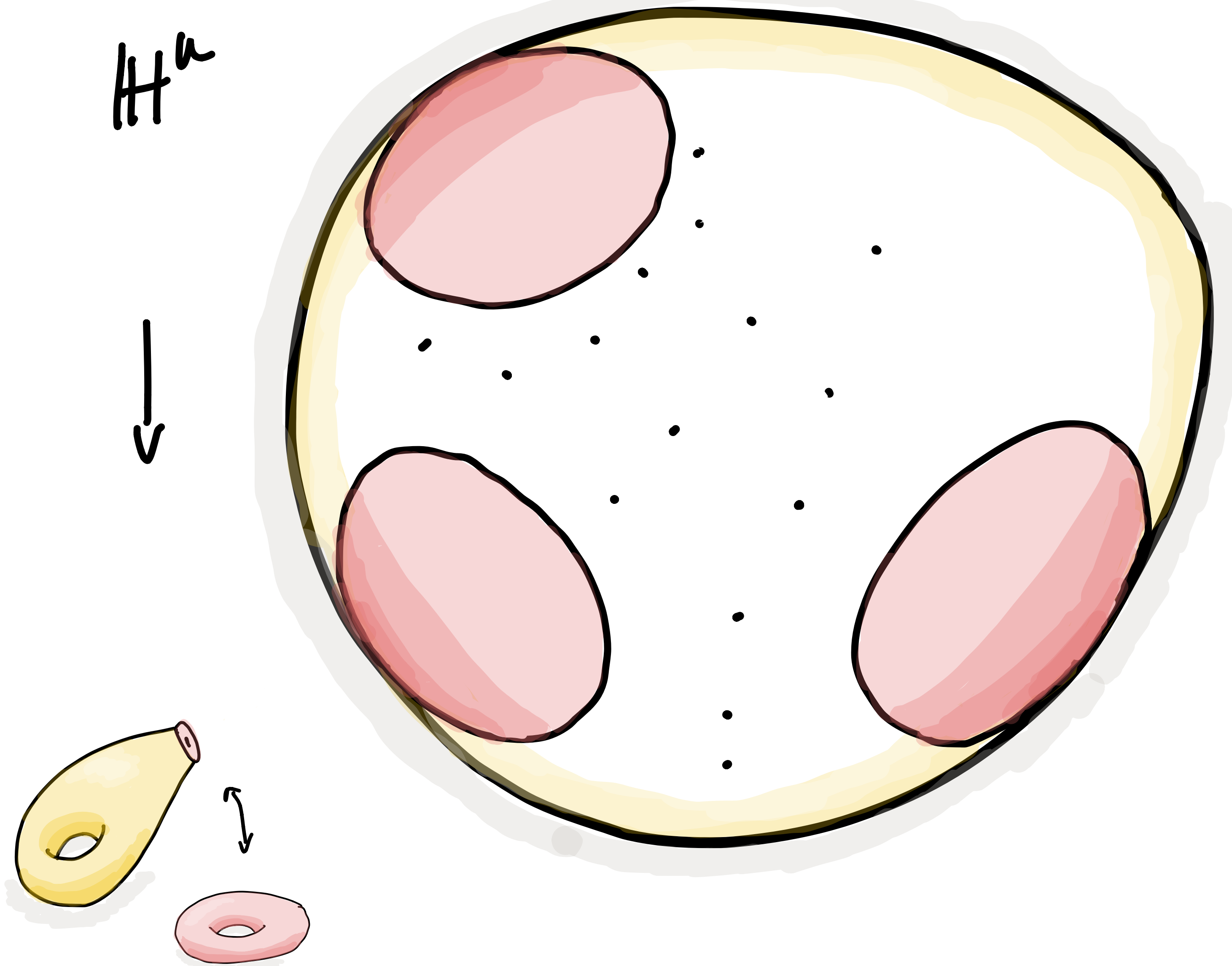
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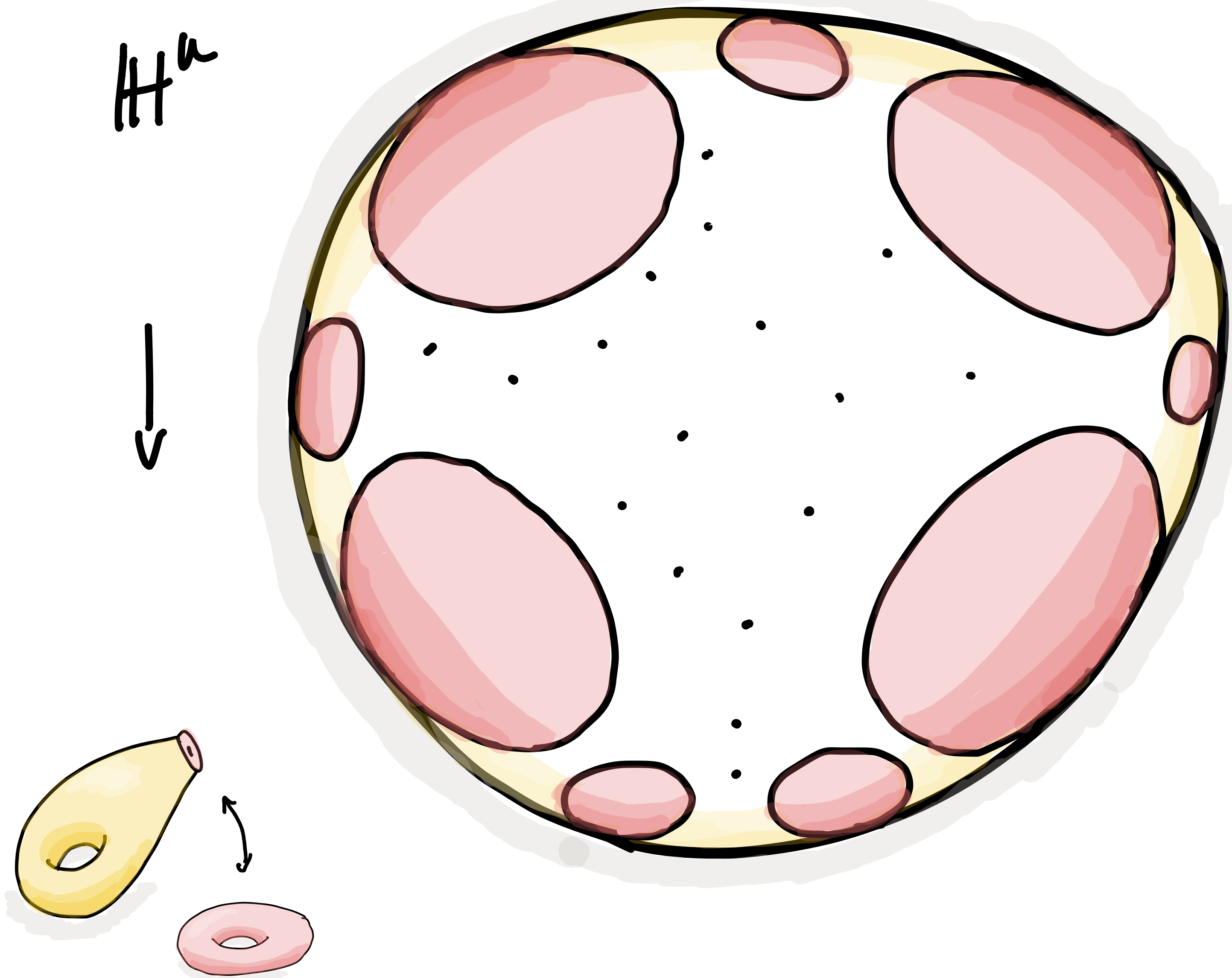

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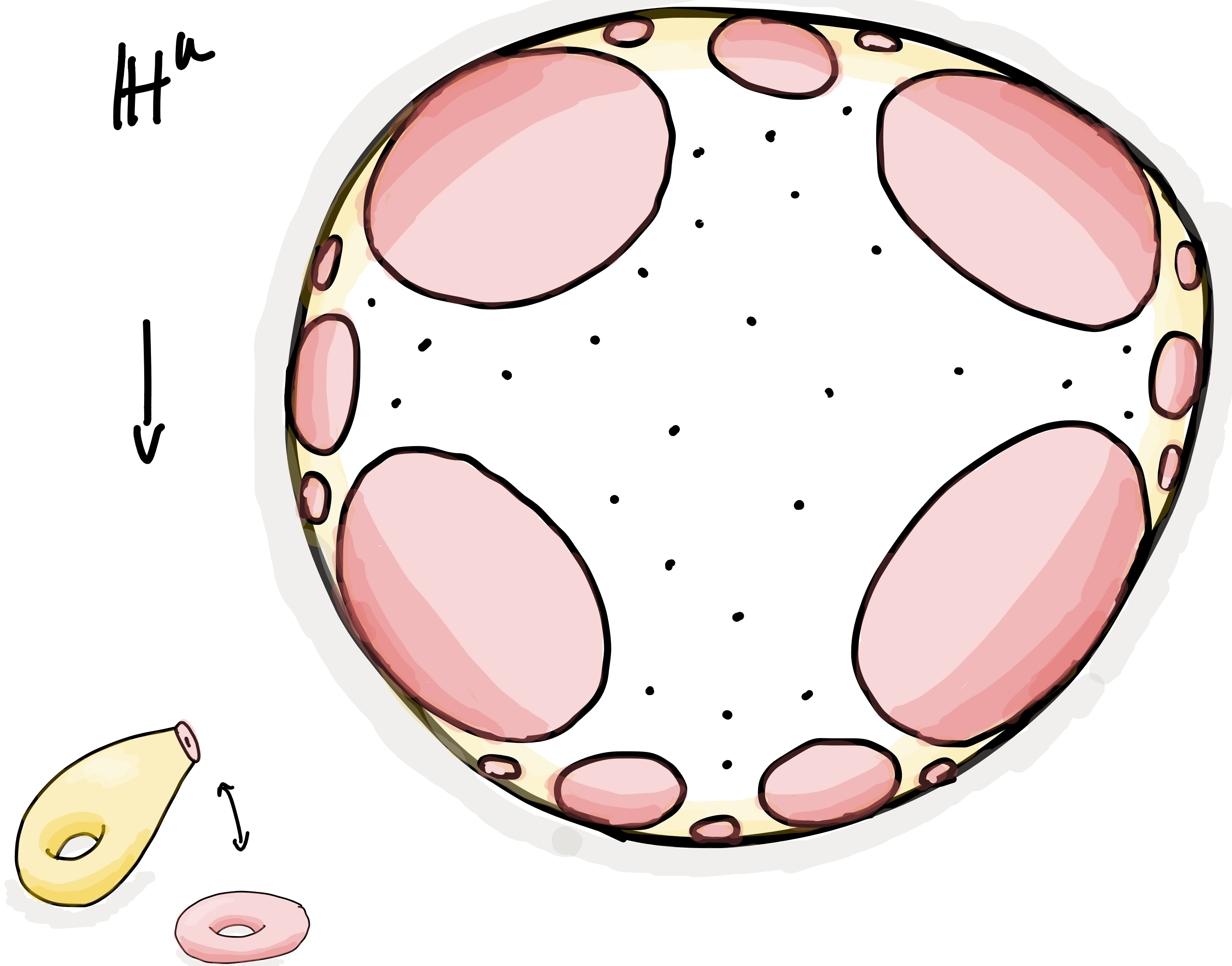
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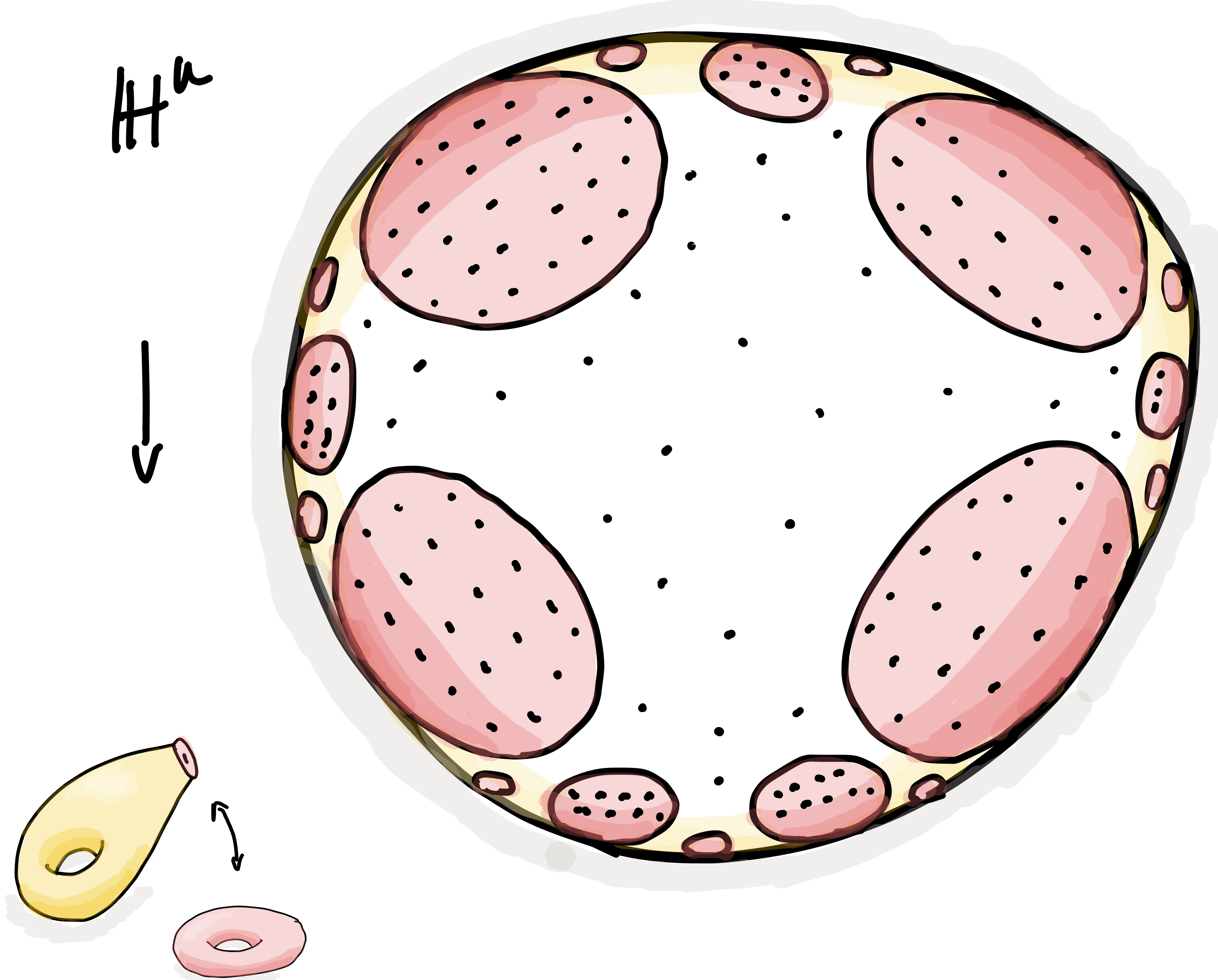
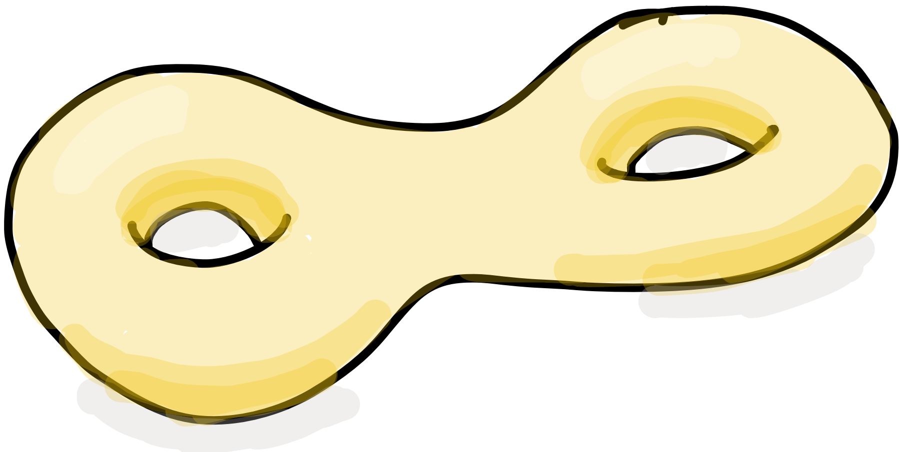
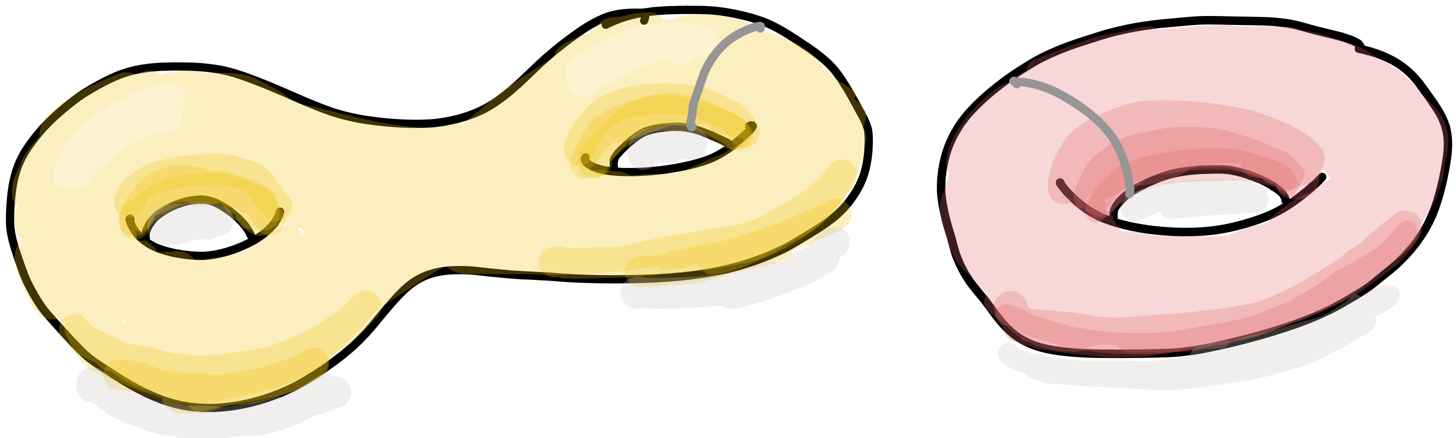
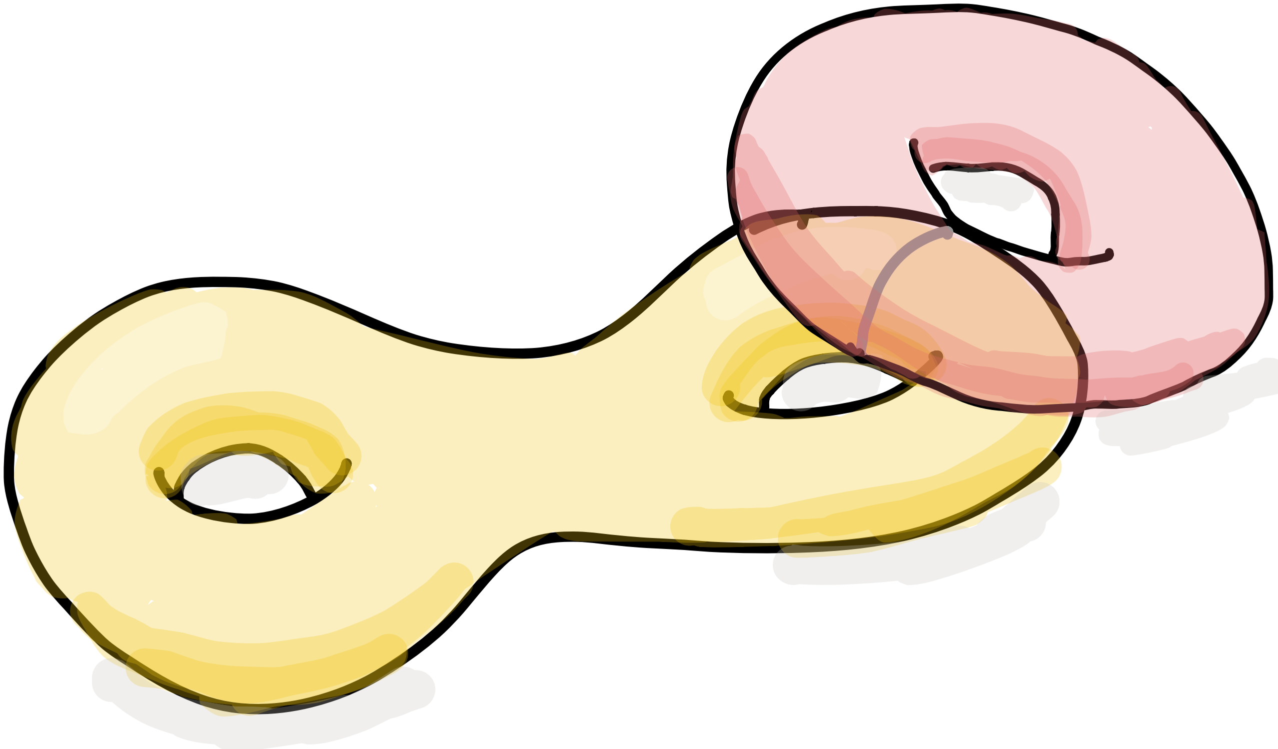
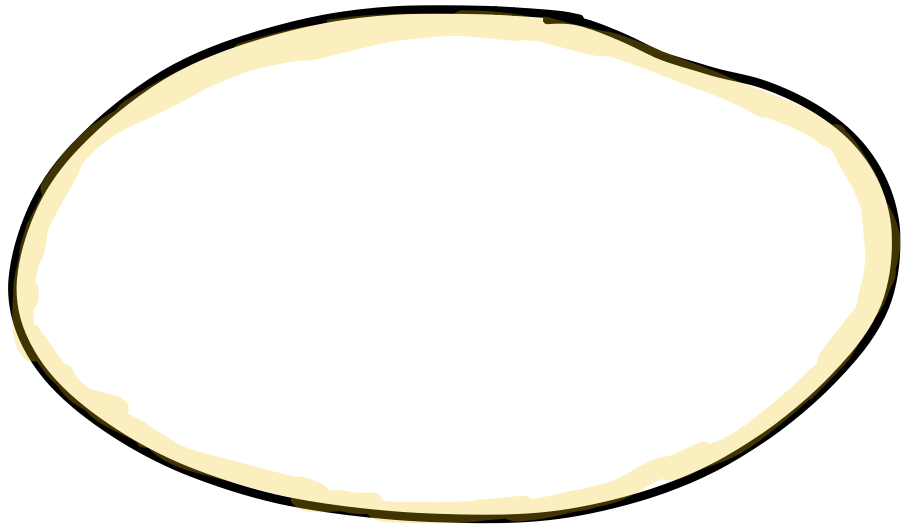
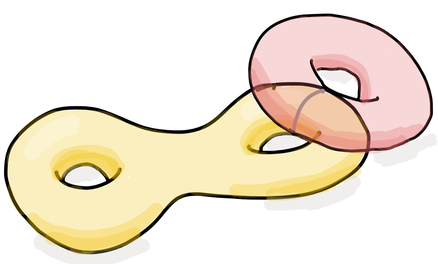


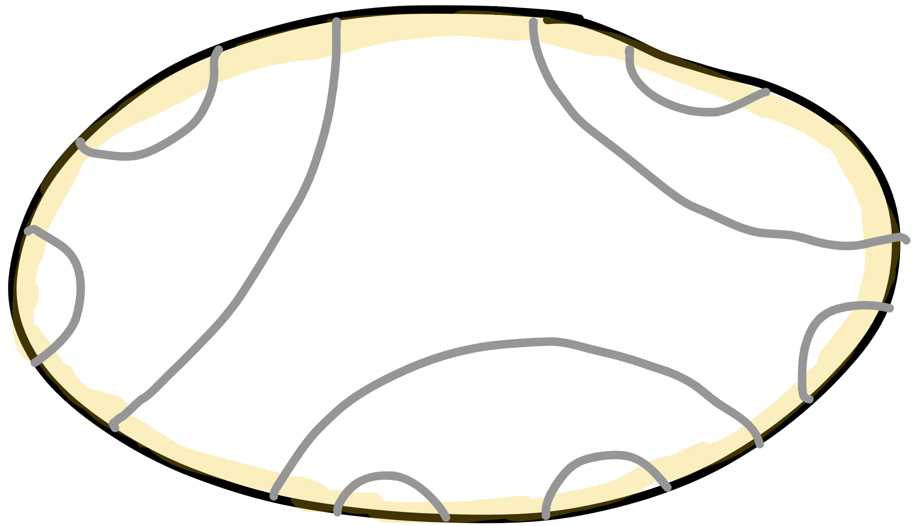


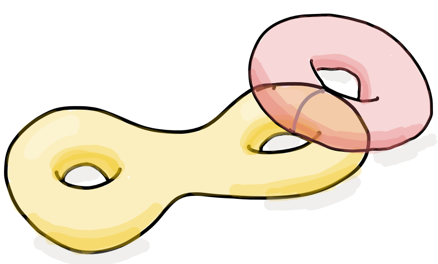


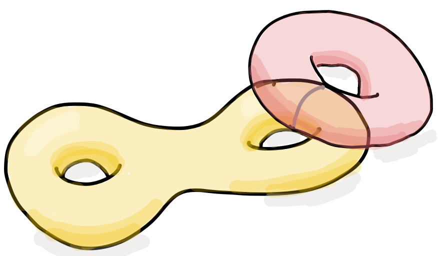


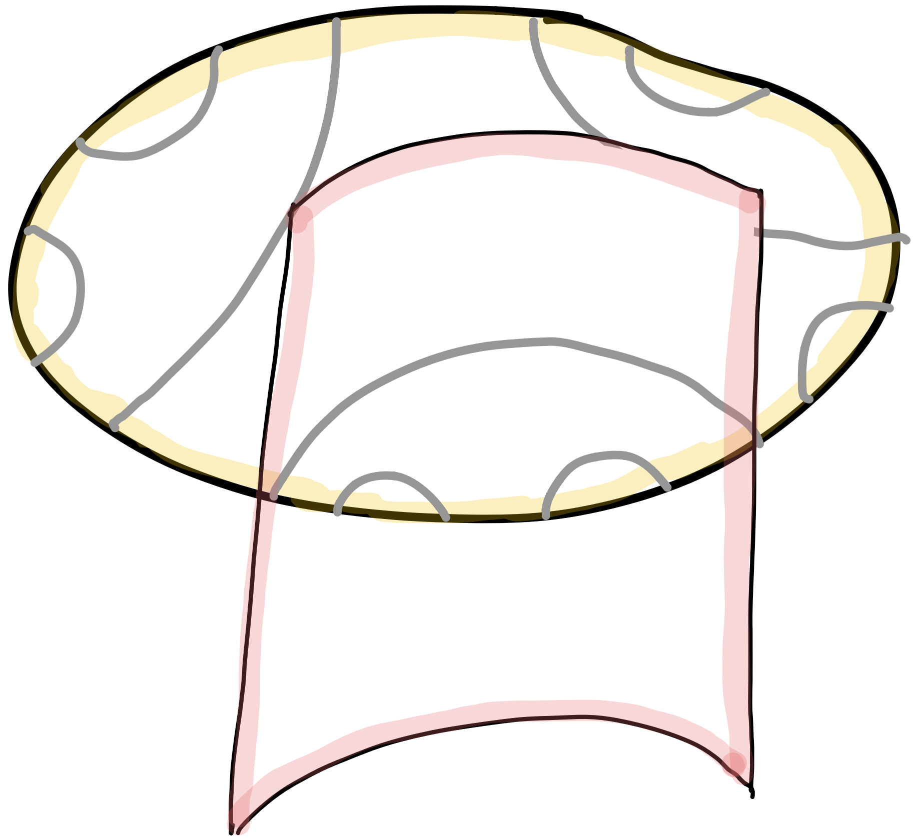


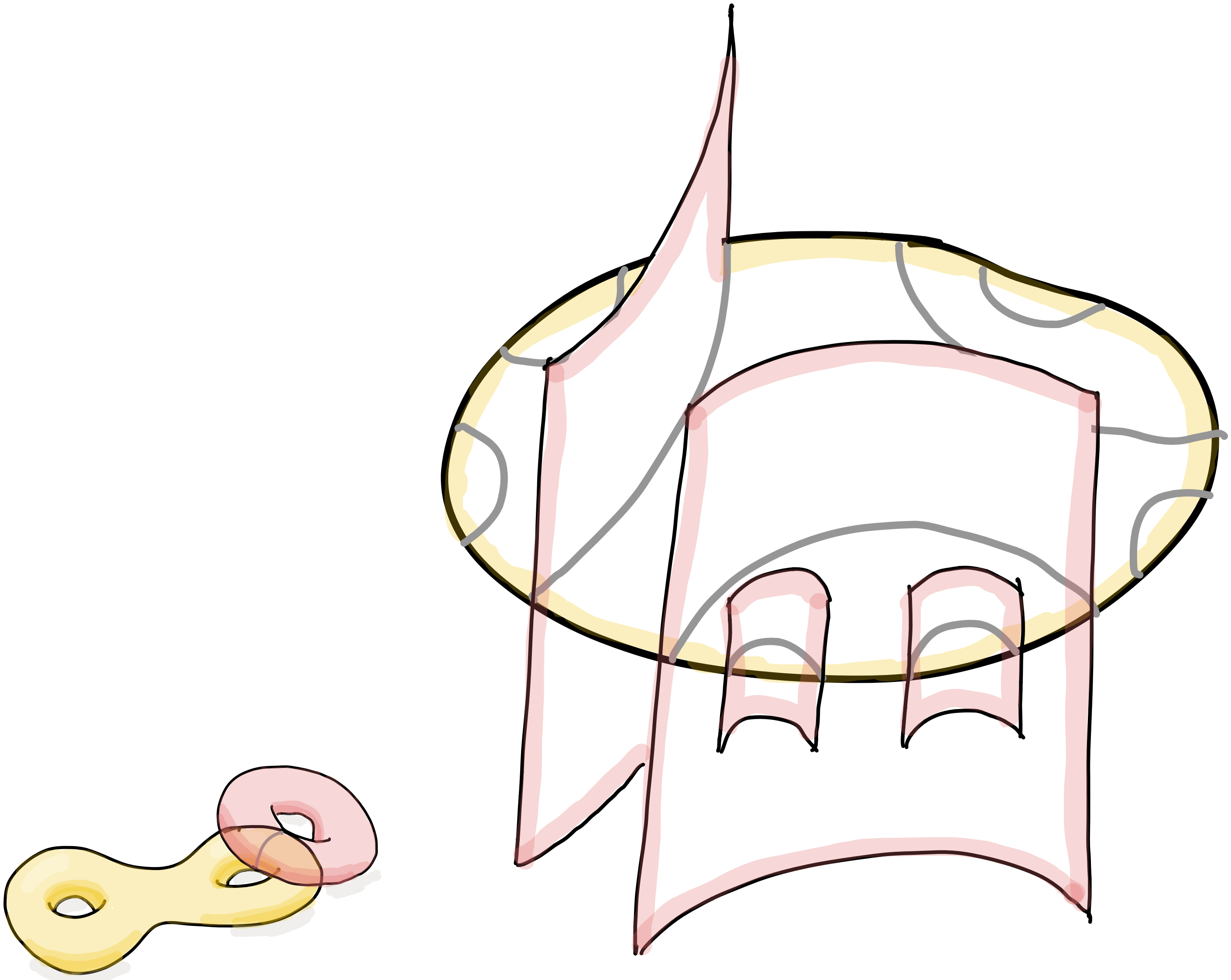


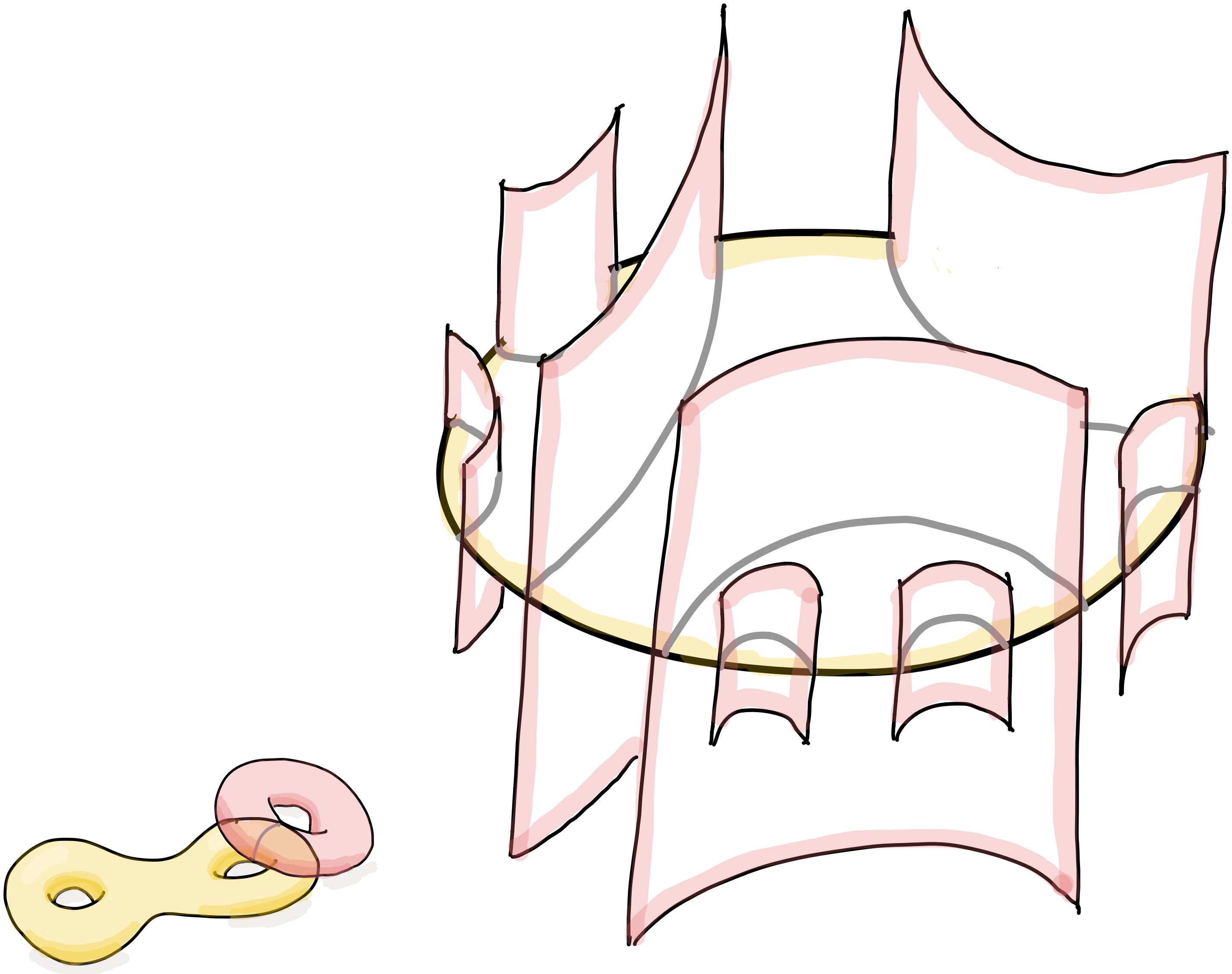


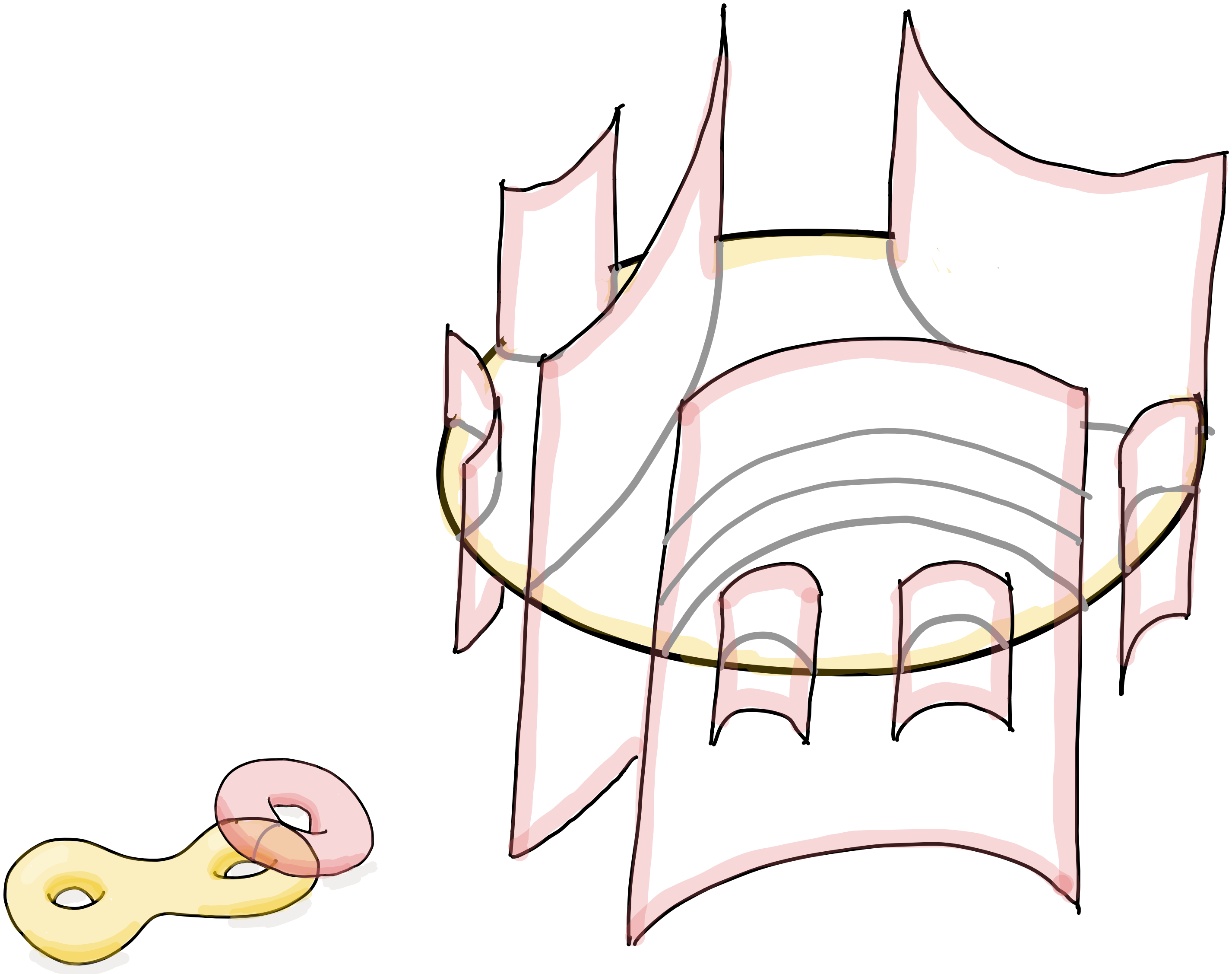


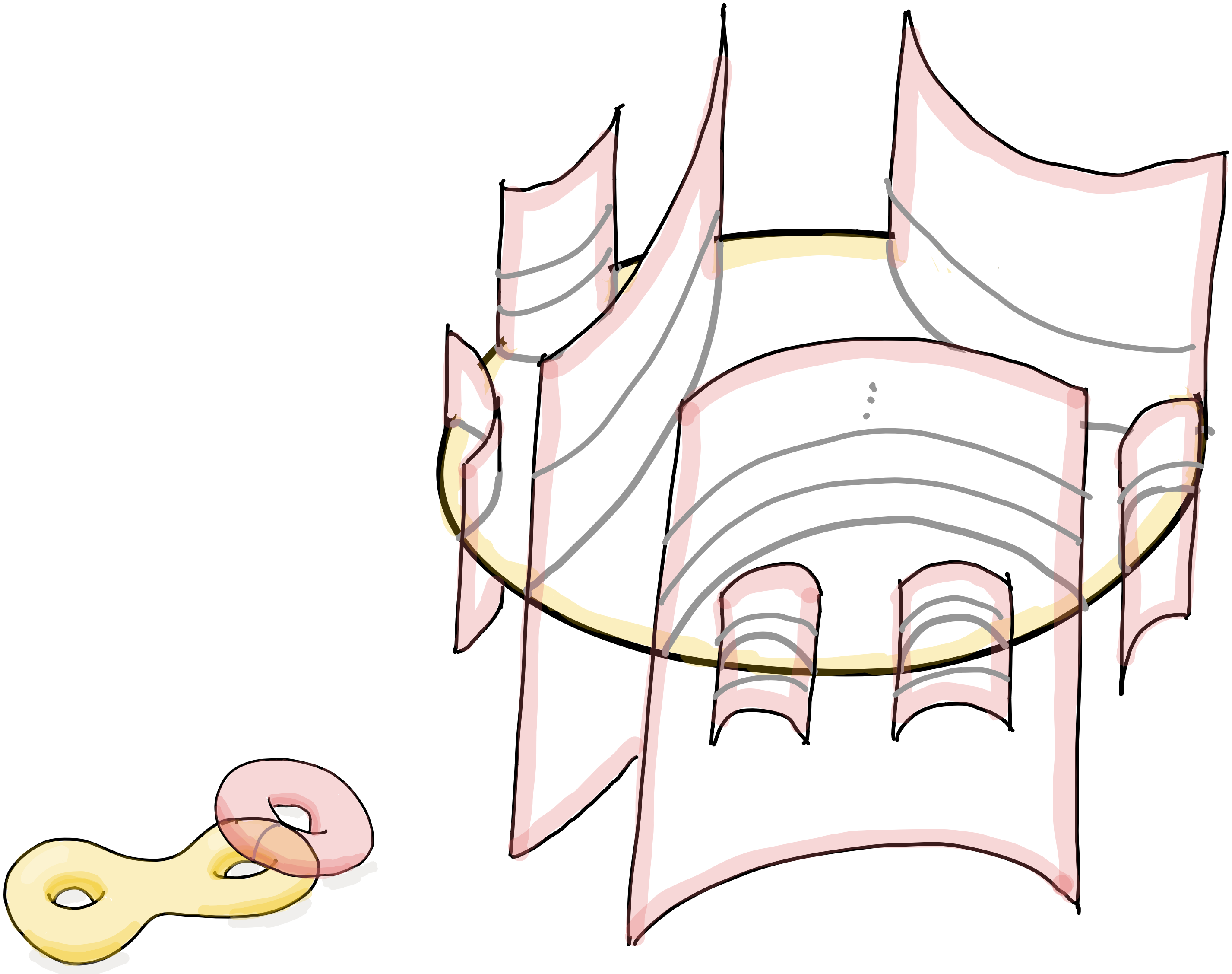


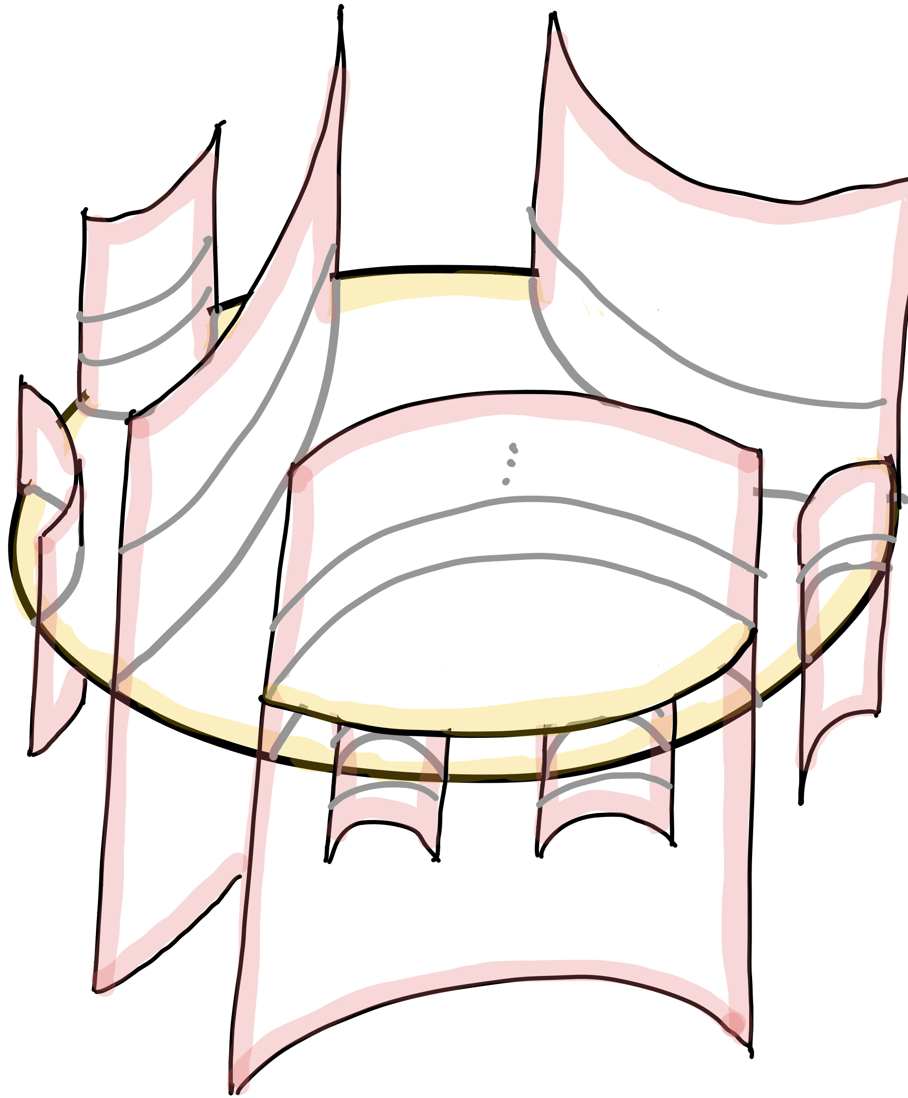
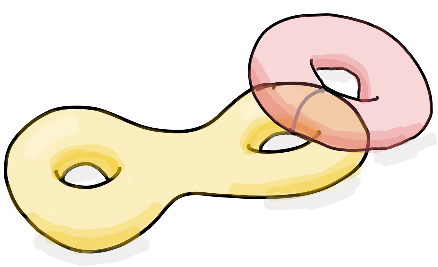


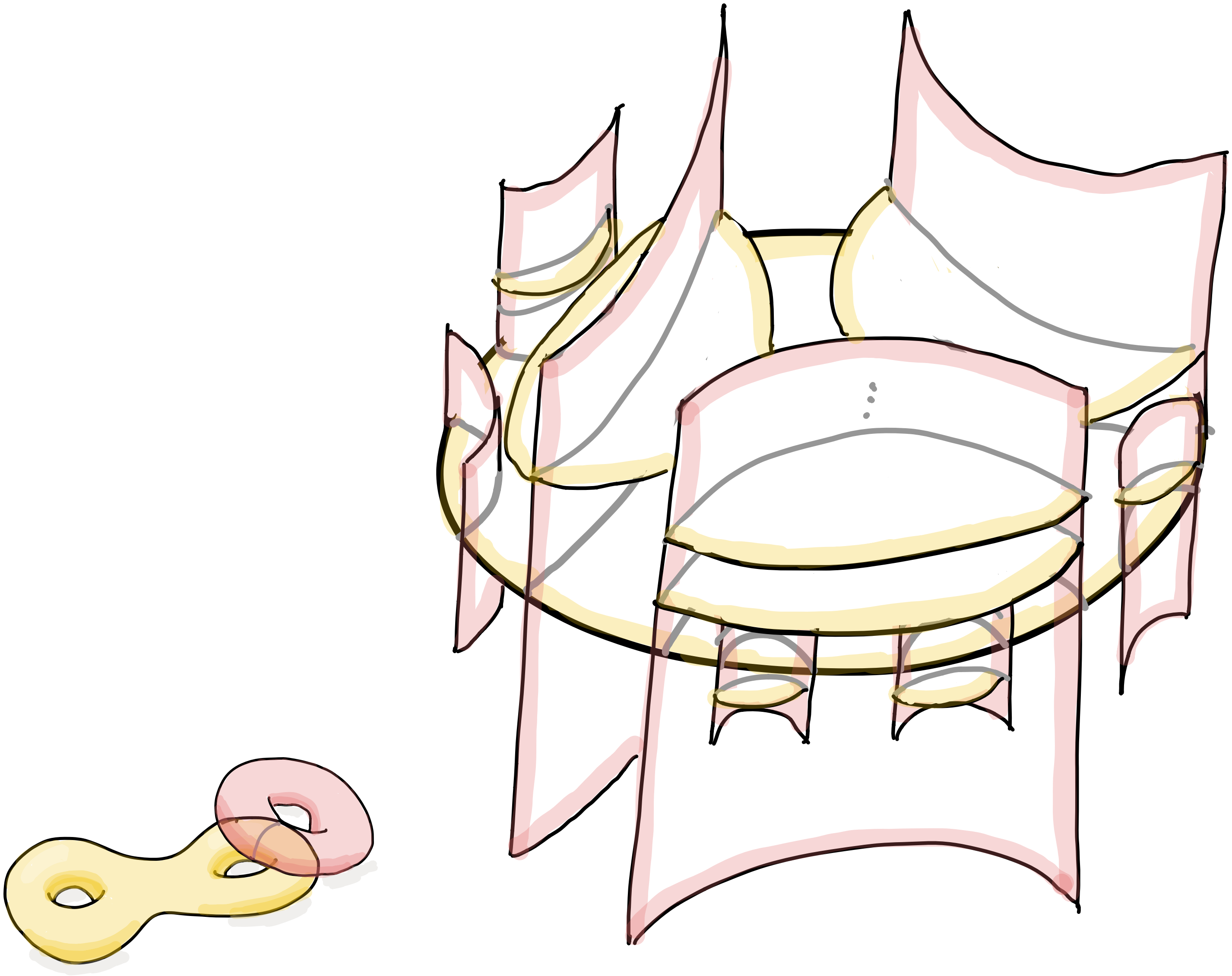


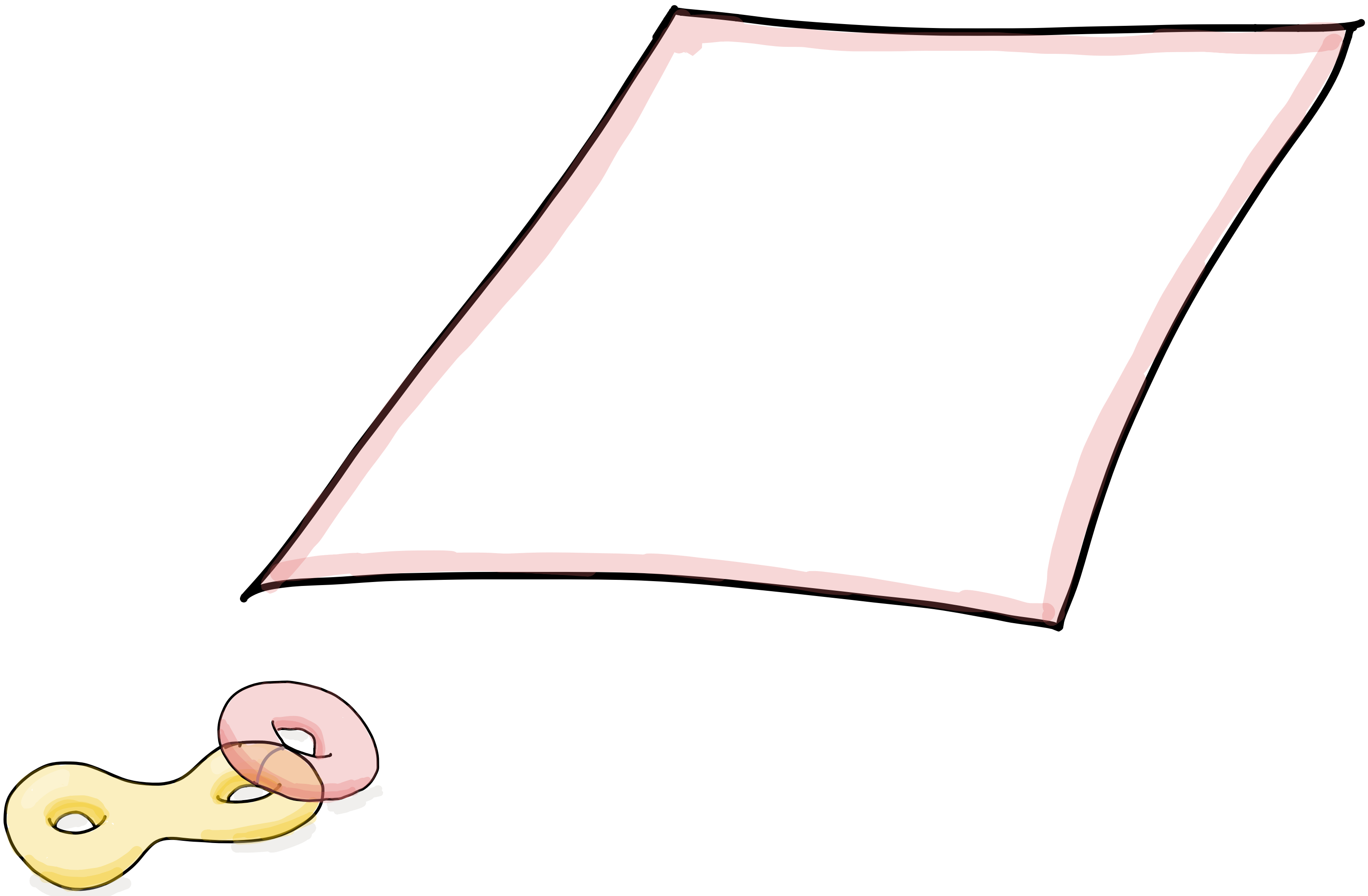


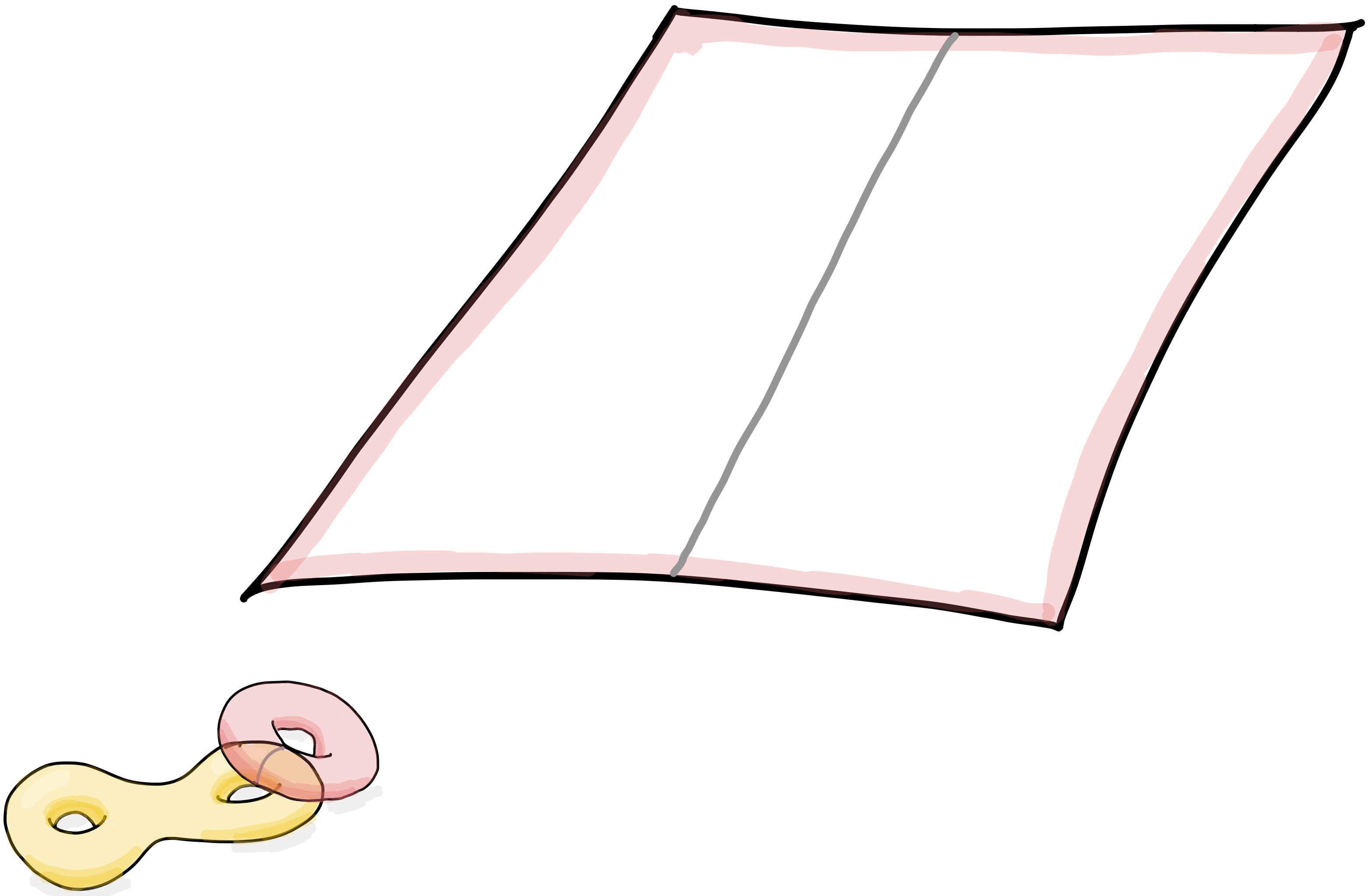


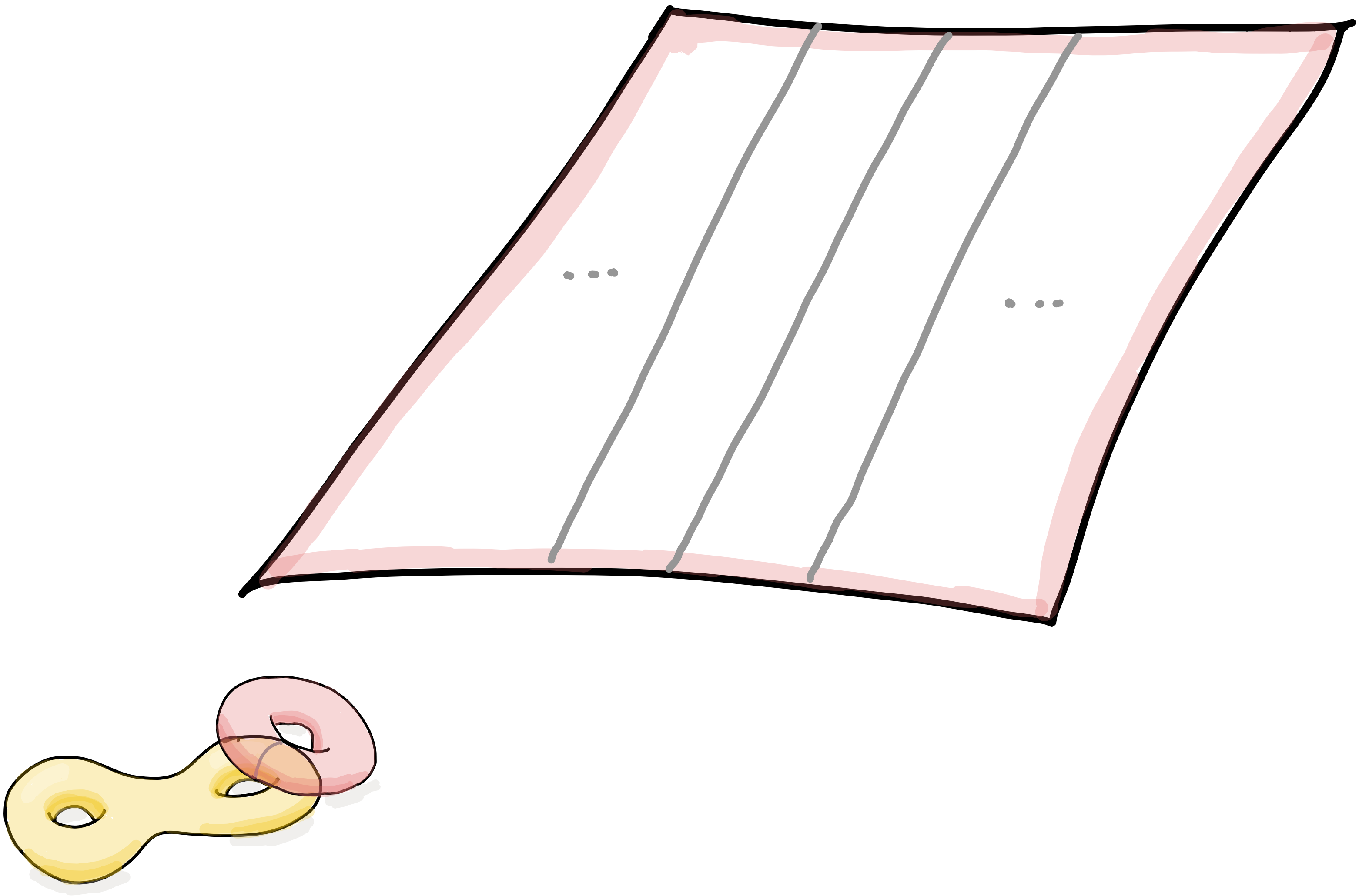


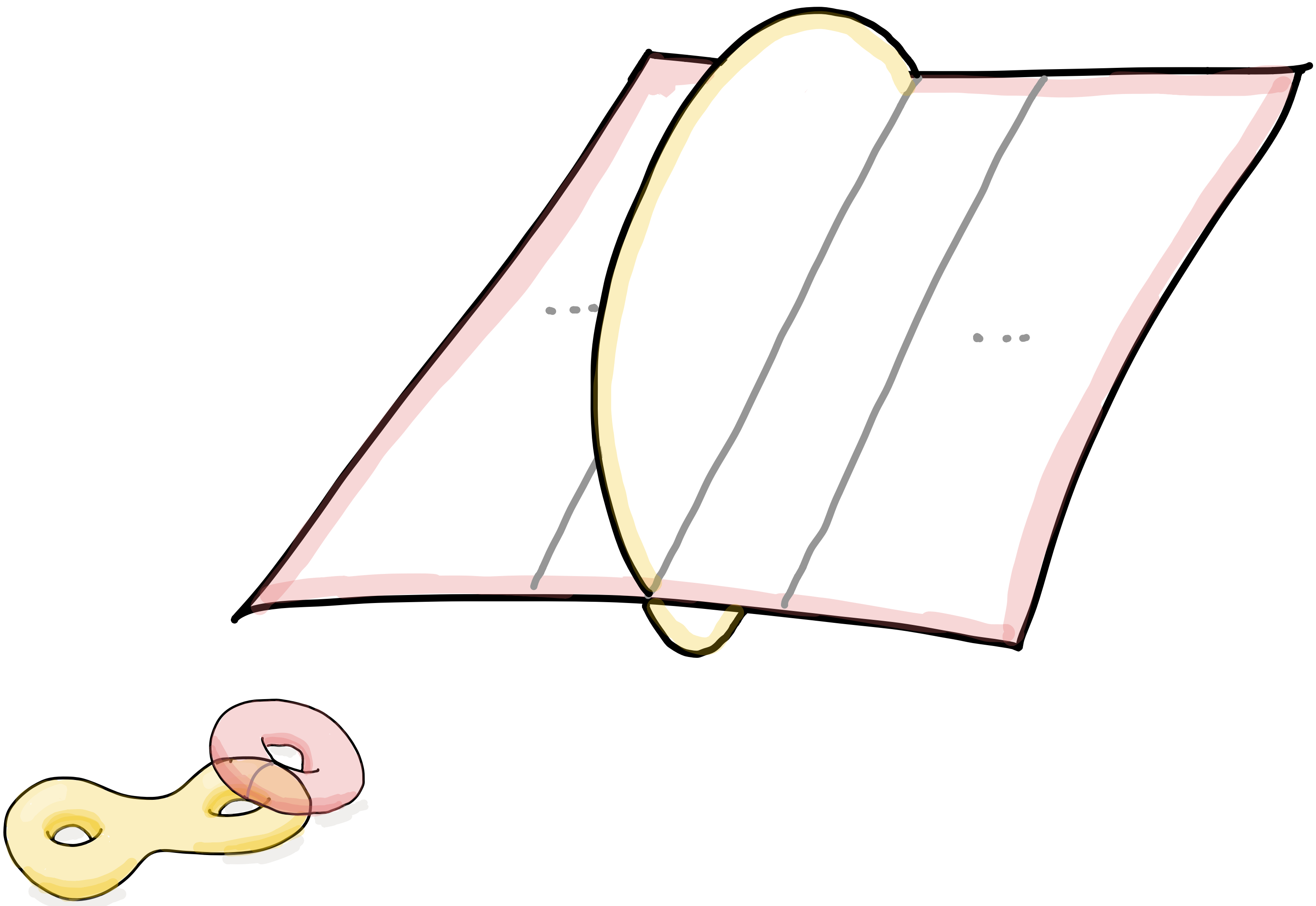



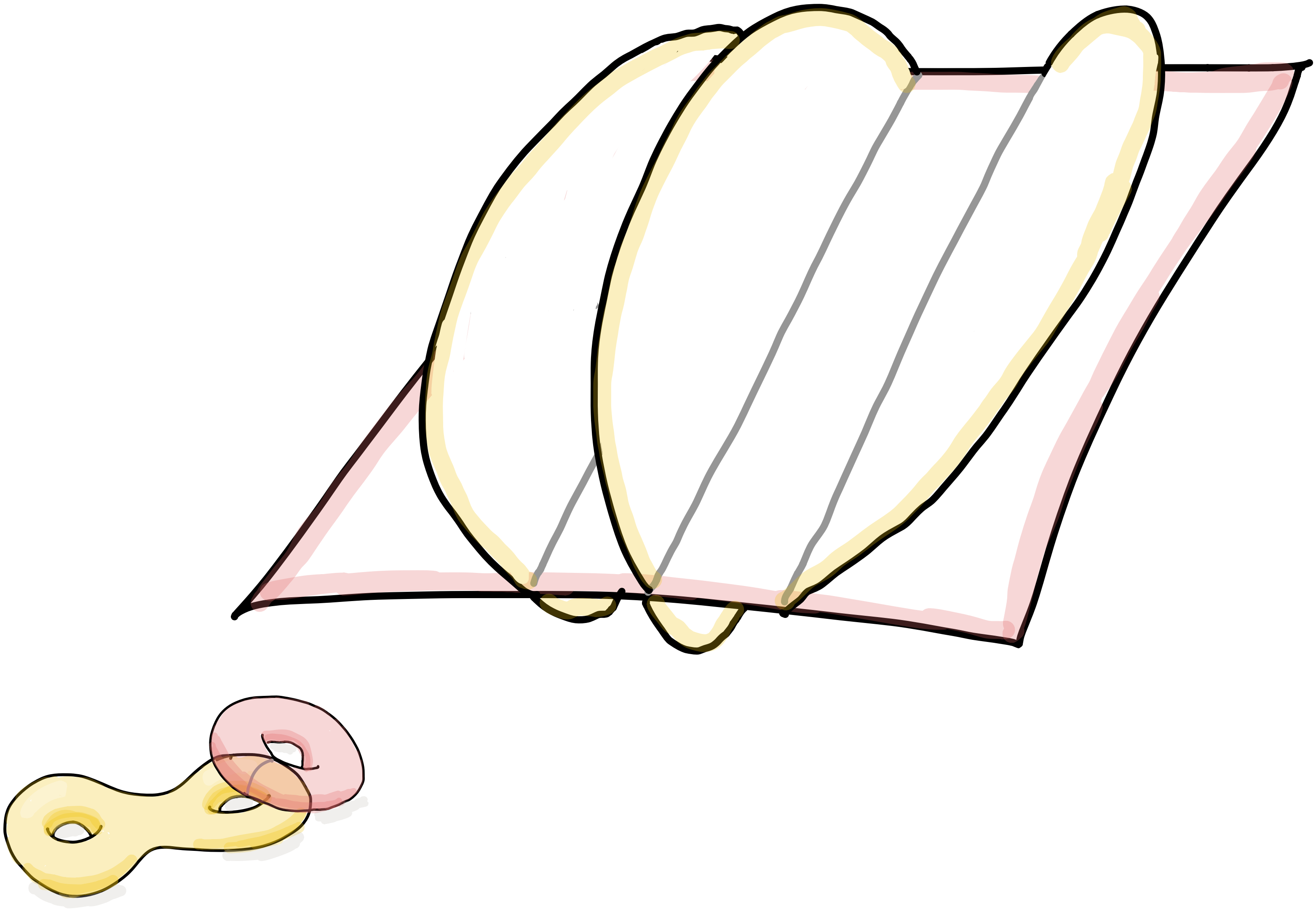


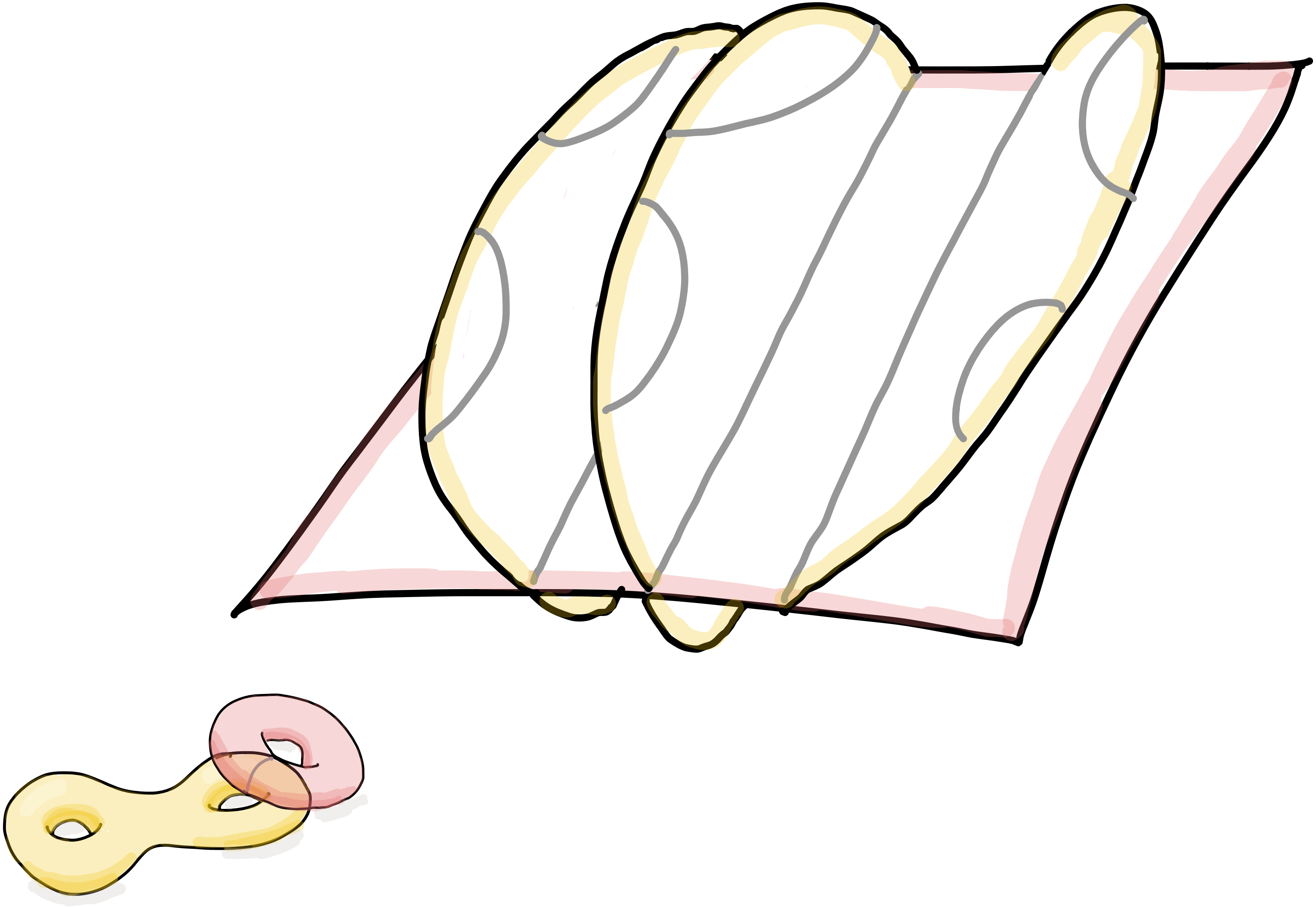


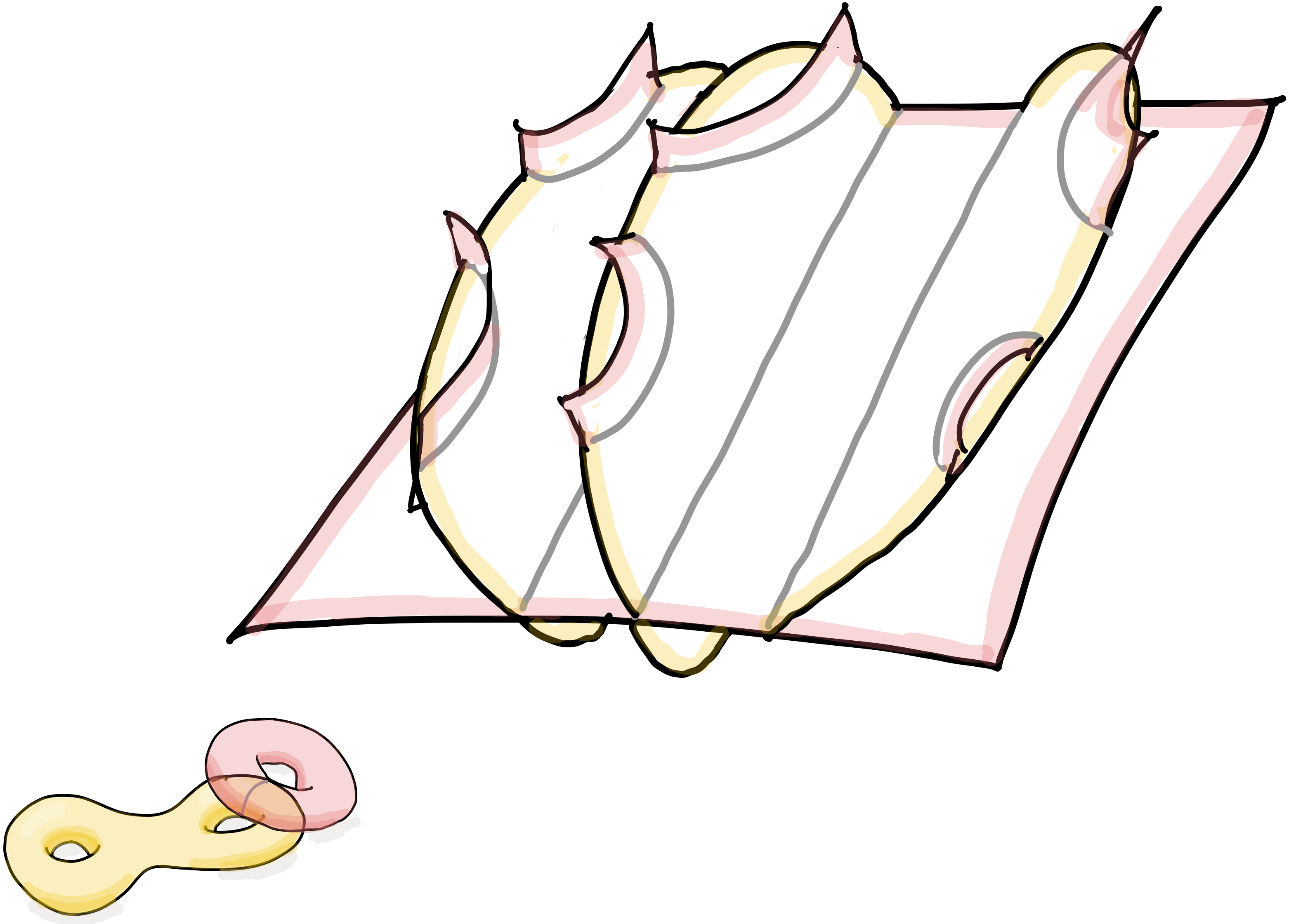


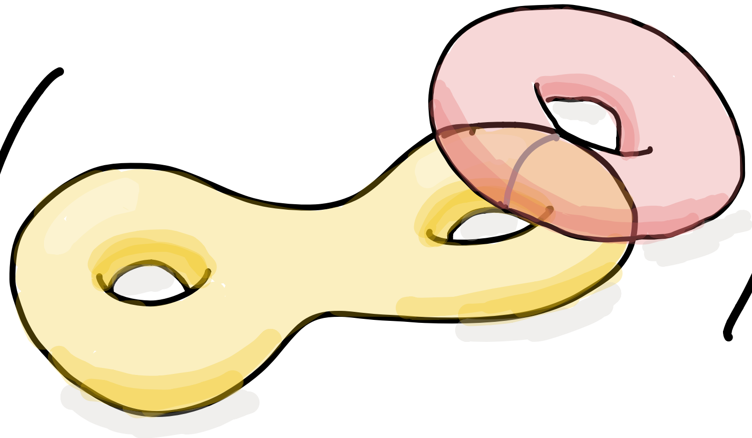


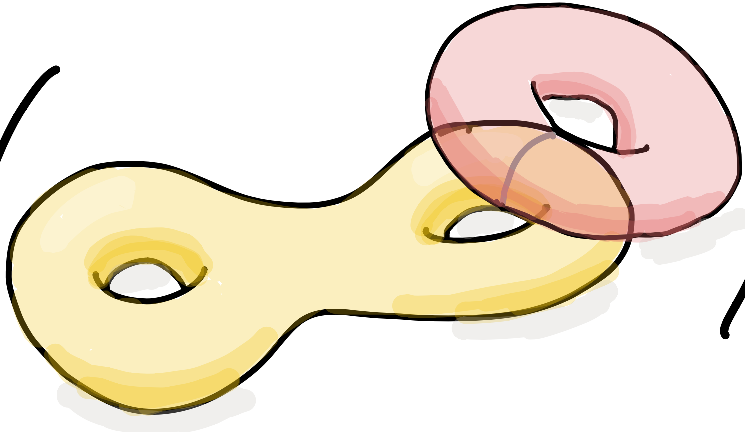


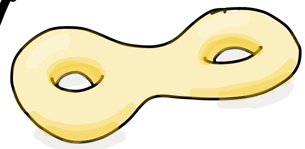


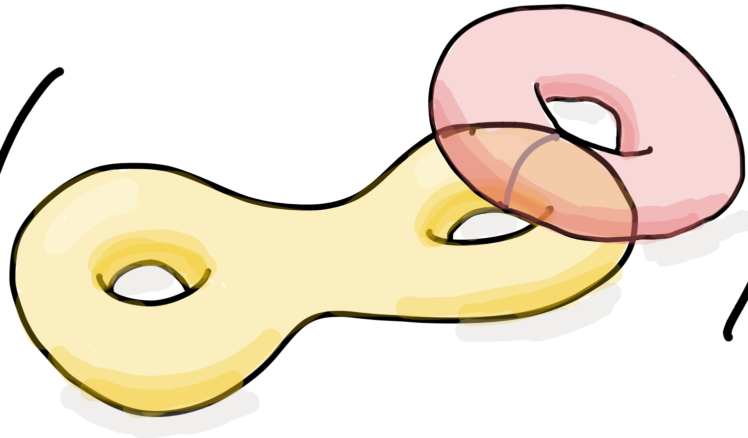


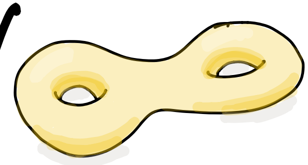


π_1 ()

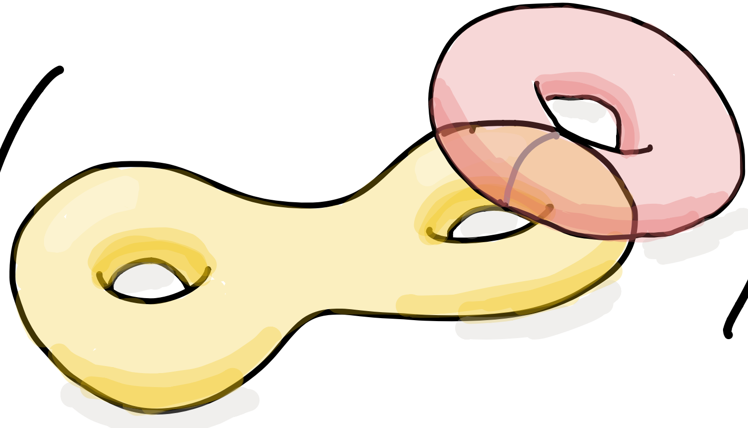
$\pi_1\Big($  $\Big) =$

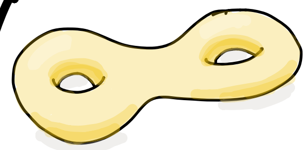
$= \pi_1\Big($  $\Big) *_{\mathbb{Z}} \mathbb{Z}^2$

$\pi_1\Big($  $\Big) =$

$= \pi_1\Big($  $\Big) *_{\mathbb{Z}} \mathbb{Z}^2$

pas hyperbolique à la Gromov

$\pi_1\Big($  $\Big) =$

$= \pi_1\Big($  $\Big) *_{\mathbb{Z}} \mathbb{Z}^2$

pas hyperbolique à la Gromov

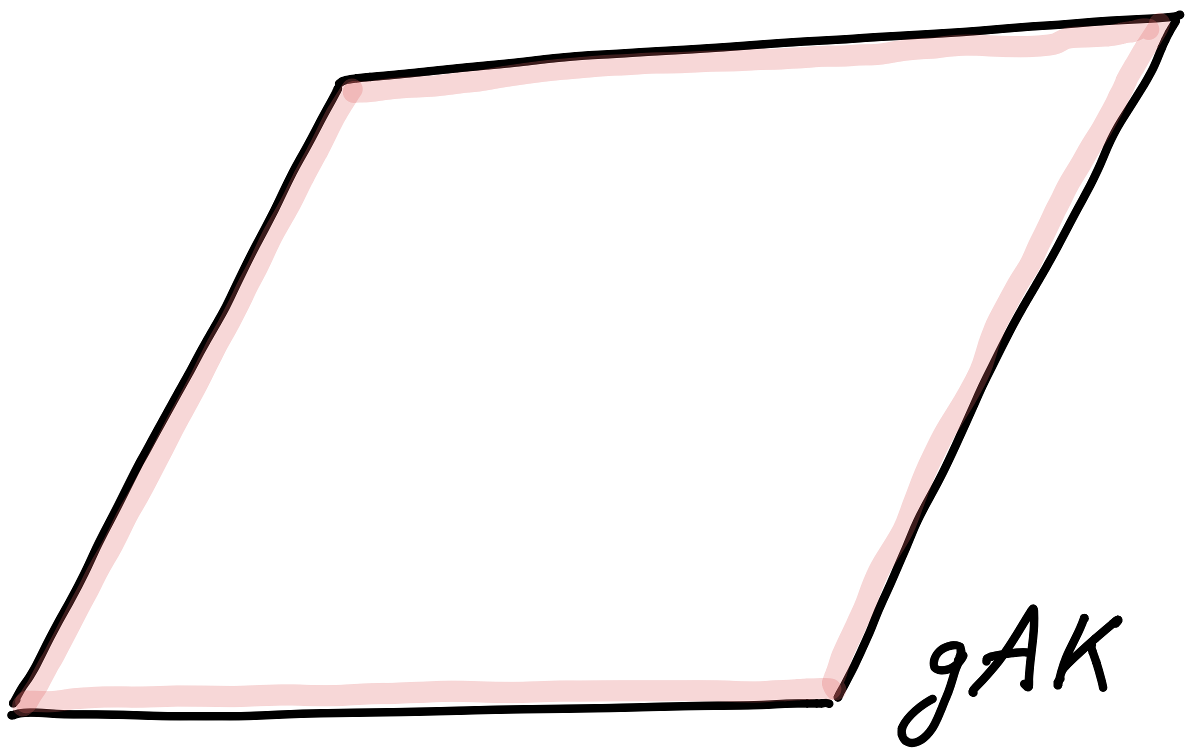
mais hyperbolique relativement à $\mathbb{Z}^2$

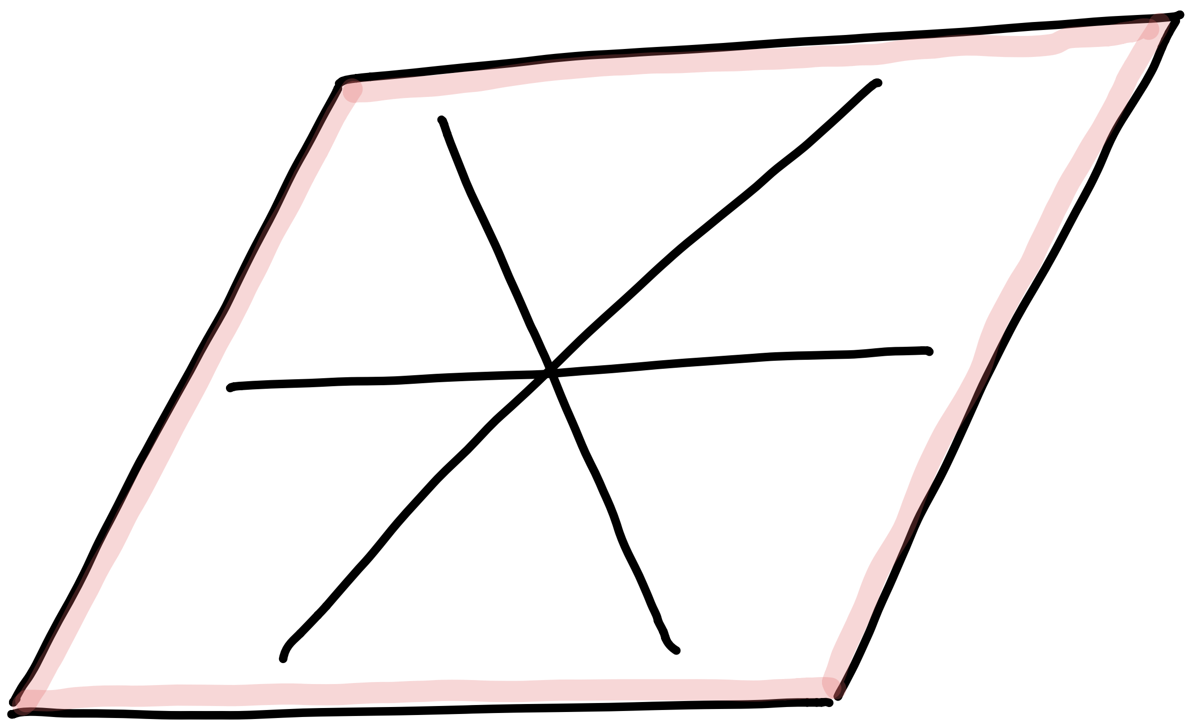
$SL_n(\mathbb{Z})$: loin de l'hyperbolicité

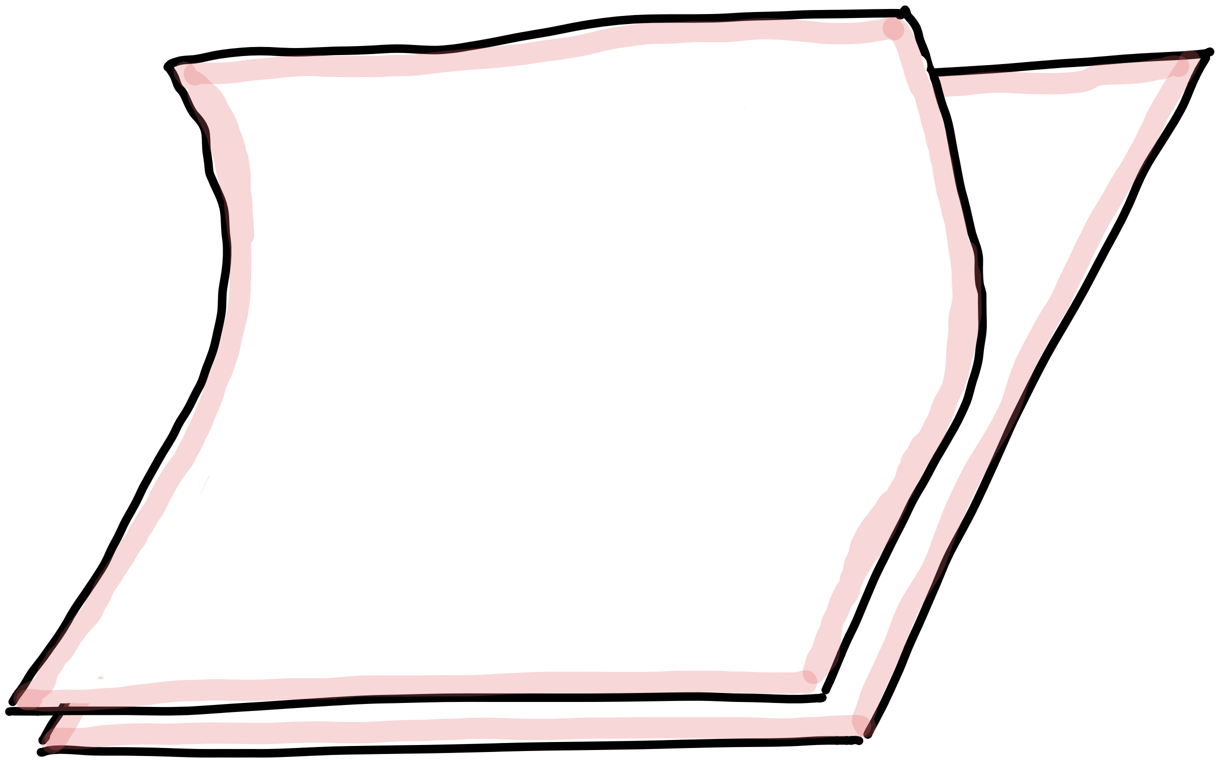
Cas $SL_3(\mathbb{Z}) < SL_3(\mathbb{R})$

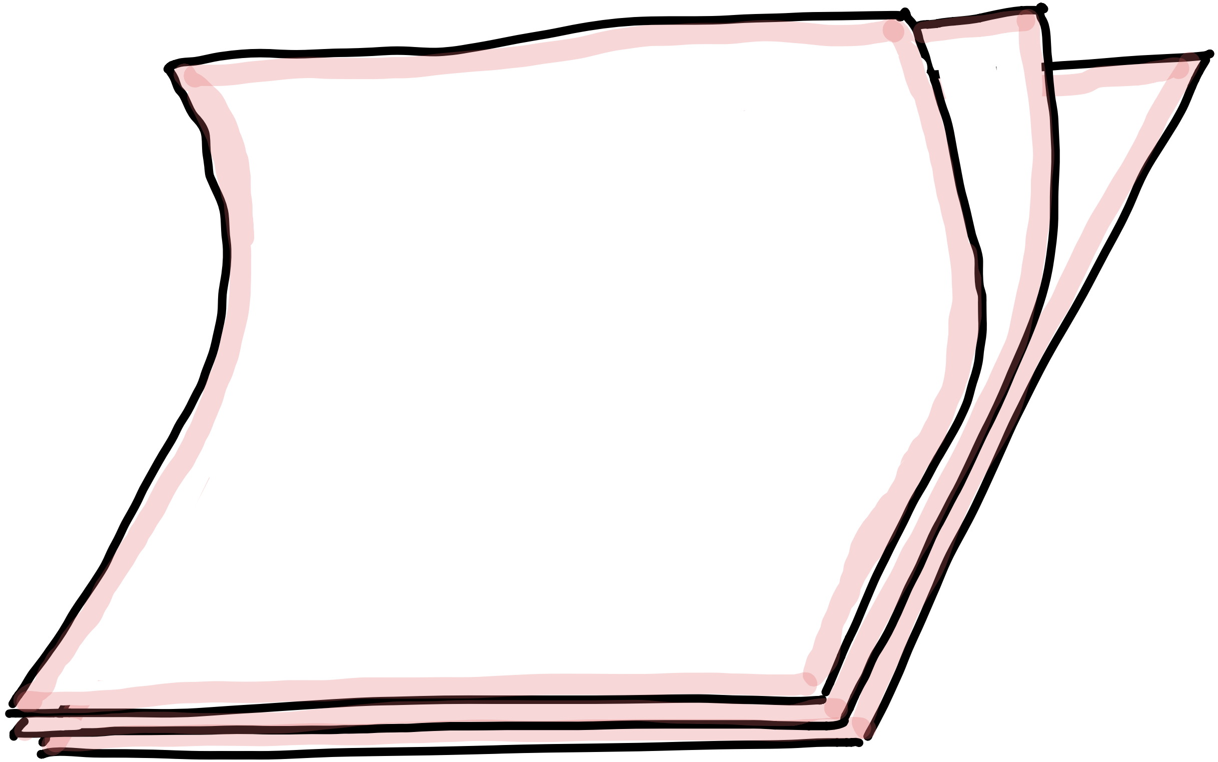
géométrie de $SL_3(\mathbb{R}) / SO_3(\mathbb{R})$

assez bien comprise


gAK







mais $SL_3(\mathbb{R}) / SL_3(\mathbb{Z})$ n'est

pas compact

mais $SL_3(\mathbb{R}) / SL_3(\mathbb{Z})$ n'est pas compact, les cusps ont dimension 2 et s'intersectent mais $SL_3(\mathbb{R}) / SL_3(\mathbb{Z})$ n'est

pas compact, les cusps ont

dimension 2 et s'intersectent

la géométrie de $SL_n(\mathbb{Z})$ $n \geq 3$

nous échappe.

Fin