Stochastic nonlinear Schrödinger equations.

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The nonlinear Schrödingier equation is a universal model to describe propagation of waves in nonlinear dispersive media. For instance, in the context of optical fibers it writes:

$$\begin{cases} i\dfrac{du}{dz} + \lambda\dfrac{d^2u}{dt^2} + |u|^2u = 0, \\ \\ u(t,0) = u_0(t). \end{cases}$$

- z is the position in the fiber and t is the time.
- The solution is complex valued.
- λ is a parameter of the fiber.

For the mathematical study, we write a more general equation and switch to usual notations for evolution equations:

$$\begin{cases} i\dfrac{du}{dt} + \lambda \Delta u + |u|^{2\sigma} u = 0,\ x \in \mathbb{R}^d,\ t > 0, \\ \\ u(x,0) = u_0(x),\ \ x \in \mathbb{R}^d. \end{cases}$$

- $d \neq 1$: transverse variables are taken into account.
- $\sigma \neq 1$: higher order nonlinear effects are taken into account.
- The solution is complex valued.
- λ is either positive or negative:
  - $\lambda > 0 \rightarrow$ focusing nonlinearity
  - $\lambda < 0 \rightarrow$ defocusing nonlinearity

Dispersion

$$\begin{cases} i\dfrac{du}{dt} + \lambda \Delta u = 0,\ x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = u_0(x),\ x \in \mathbb{R}^d. \end{cases}$$

Take Fourier transform:

$$\frac{d\hat{u}(\xi,t)}{dt} = -i\lambda|\xi|^2\hat{u}(\xi,t)$$

$\longrightarrow$ $\hat{u}(\xi,t) = e^{-i\lambda|\xi|^2 t}\hat{u}_0(\xi)$

$\longrightarrow$ the phase speed depends on the wave length.

Use Plancherel:
$$\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\hat{u}(\xi,t)|^2 d\xi = \int_{\mathbb{R}^d} |\hat{u}_0(\xi)|^2 d\xi = \|u_0(\cdot)\|^2_{L^2(\mathbb{R}^d)}.$$
$\rightarrow$ The $L^2(\mathbb{R}^d)$ norm is conserved, as well as all $H^s(\mathbb{R}^d)$ norms.

Dispersion

$$\begin{cases} i\frac{du}{dt} + \lambda\Delta u = 0,\ x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = u_0(x),\ x \in \mathbb{R}^d. \end{cases}$$

- $\hat{u}(\xi,t) = e^{-i\lambda|\xi|^2 t}\hat{u}_0(\xi)$ and $\|u(\cdot,t)\|_{L^2(\mathbb{R}^d)} = \|u_0(\cdot)\|_{L^2(\mathbb{R}^d)}$.
- inverse Fourier transform:

$$u(x,t) = \frac{1}{(4i\pi t)^{d/2}} \int_{\mathbb{R}^d} \exp\left(i\frac{|x-y|^2}{4t}\right) u_0(y)dy.$$

$$\longrightarrow \sup_{x\in\mathbb{R}^d} |u(x,t)| = |u(\cdot,t)|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(4\pi t)^{d/2}} |u_0|_{L^1(\mathbb{R}^d)}$$

$$\longrightarrow |u(\cdot,t)|_{L^p(\mathbb{R}^d)} \leq \frac{1}{(4\pi t)^{\frac{d}{2}(\frac{1}{2}-\frac{1}{p})}} |u_0|_{L^{p'}(\mathbb{R}^d)}$$

Dispersion + nonlinear effects $\Longrightarrow$ solitary waves.
There exist solution of the form $u(x,t) = e^{i\omega t}\varphi(x)$ with φ real valued and spatially localized.
Dimension $d = 1$:

$$i\frac{du}{dt} + \lambda\frac{\partial^2 u}{\partial x^2} + |u|^{2\sigma}u = 0$$

$$\rightsquigarrow\ -\omega\varphi + \lambda\varphi_{xx} + |\varphi|^{2\sigma+1} = 0$$

$\longrightarrow$ this requires $\lambda > 0$ and $\omega > 0$.

For $\lambda = 1$, $\sigma = 1$, we obtain :

$$u(x,t) = \sqrt{2}A\text{sech}\left(A(x - x_0)\right)\exp(-iA^2 t + i\theta_0).$$

$$\begin{cases} i\frac{du}{dt} + \lambda\Delta u + |u|^{2\sigma} u = 0,\ x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = u_0(x),\ x \in \mathbb{R}^d. \end{cases}$$

Well posed-ness :

$\rightarrow$ local solutions in $L^2(\mathbb{R}^d)$ for $\sigma < 2/d$.

$\rightarrow$ local solutions in $H^1(\mathbb{R}^d)$ for $\sigma < 2/(d-2)$ (∞ if $d = 1, 2$).

There are two important invariant quantities:

• The mass $M(u) = \int_{\mathbb{R}^d} |u|^2 dx$

$\longrightarrow$ global existence in $L^2(\mathbb{R}^d)$ si $\sigma < 2/d$.

- The energy $H(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx - \frac{\lambda}{(2\sigma+2)}\int_{\mathbb{R}^d} |u|^{2\sigma+2} dx$
  - For $\lambda = -1 \rightarrow$ global existence.
  - For $\lambda = 1$ and $\sigma < 2/d$:

$$H(u) \geq c_1 \int_{\mathbb{R}^d} |\nabla u|^2 dx - c_2 \left(\int_{\mathbb{R}^d} |u|^2 dx \right)^k$$

    $\rightarrow$ global existence.
- For $\lambda = 1$ and $\sigma \geq 2/d$, there are solutions which blow-up in finite time and solitary waves are unstable

First random model:

We consider an optical fiber with random dispersion and weak nonlinearity:

$$\begin{cases} i\dfrac{du}{dt} + \dot{\beta} o \Delta u + |u|^{2\sigma} u = 0, \; x \in \mathbb{R}, \; t > 0 \\ \\ u(0) = u_0. \end{cases}$$

The noise $\dot{\beta}$ accounts for a dispersion which varies along the fiber.

Time white noise

- The time white noise is the derivative of the brownian motion
- A brownian motion β, *i.e.* $\beta(t, \omega)$, satisfies:
  - $(\beta(t_1), \beta(t_2), \ldots, \beta(t_n))$ is a gaussian vector for all $t_1, \ldots, t_n$
  - It has independent increments : $\mathbb{E}((\beta(t) - \beta(s))\beta(t)) = 0$, $t \geq s$.
  - $\mathbb{E}(\beta(t)^2) = t$

    $\rightarrow$ $\mathbb{E}((\beta(t) - \beta(s))^2) = t - s$, β is never differentiable ...

    $\rightarrow$ $\mathbb{E}(\beta(t)\beta(s)) = \min\{t, s\}$.

    $\rightarrow$ Formally : $\mathbb{E}(\dot{\beta}(t)\dot{\beta}(s)) = \delta_{t=s}$

Time white noise

- Formally : $\dot\beta(t,\omega) = \sum_{k\in\mathbb{N}} \chi_k(\omega) f_k(t)$ with (f_k) a complete orthonormal system in $L^2([0,T])$ and χ_k a squence of independent $\mathcal{N}(0,1)$ random variables.

$\rightarrow$ $\dot\beta$ is the time white noise.

- $\dot\beta$ has regularity less than $1/2$. The solution of a stochastic equation $dx = f(x)dt + g(x)d\beta$ is not more regular than β. $\rightarrow$ The product $g(x)d\beta$ is not well defined.
- Two possibilities:
  - Ito : $f(t)\dot\beta(t) \approx f(t)\dfrac{\beta(t+\delta)-\beta(t)}{\delta}$
    $\rightarrow$ it has good mathematical properties.
  - Stratonovitch : $f(t)\dot\beta(t) \approx f(t)\dfrac{\beta(t+\delta)-\beta(t-\delta)}{2\delta}$
    $\rightarrow$ it appears naturally and the chain rule can be used.

$$
\begin{cases} i\dfrac{du}{dt} + \dot{\beta} o \Delta u + |u|^{2\sigma} u = 0, \, x \in \mathbb{R}, \, t > 0 \\ \\ u(0) = u_0. \end{cases}
$$

- This is a Stratonovich product.
- The mass is still conserved but the equation is not Hamiltonian anymore. The evolution of the energy is complicated.

We study a more general equation with $\sigma > 0$ and $d \geq 1$,

$$\begin{cases} idu + \Delta u o d\beta + |u|^{2\sigma} u dt = 0, \; x \in \mathbb{R}^d, \; t > 0 \\ \\ u(0) = u_0. \end{cases}$$

$\dot{\beta}$ is a time white noise.
We consider this equation as a semilinear one and, as usual, study first the linear evolution $S(t,s)$ associated to:

$$\begin{cases} idv + \Delta v o d\beta = 0, \; x \in \mathbb{R}^d, \; t > s \\ \\ v(s) = v_s. \end{cases}$$

$\rightarrow v(t) = S(t,s)v_s$
Then rewrite the equation in the mild form:

$$u(t) = S(t,0)u_0 + i \int_0^t S(t,s)|u(s)|^{2\sigma} u(s) ds.$$

The deterministic case:

$$\begin{cases} i\dfrac{du}{dt} + \Delta u + |u|^{2\sigma}u = 0,\ x \in \mathbb{R}^d,\ t > 0 \\ \\ u(0) = u_0. \end{cases}$$

$$U(t) = e^{it\Delta} \longrightarrow u(t) = U(t)u_0 + i\int_0^t U(t-s)|u(s)|^{2\sigma}u(s)ds.$$

- We use a fixed point argument in $L^r(0,T;L^p(\mathbb{R}^d))$, we need that $U(t)$ maps $L^{p/(2\sigma+1)}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ with an integrable norm.
- We already know:

$$|U(t)v|_{L^p(\mathbb{R}^d)} \leq \frac{1}{4\pi}|t|^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})}|v|_{L^{p'}(\mathbb{R}^d)}$$

- It follows

$$\left|\int_0^t U(t-s)v(s)ds\right|_{L^r(0,T;L^p(\mathbb{R}^d))} \leq \frac{1}{4\pi}\left|\int_0^t |t-s|^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})}|v(s)|_{L^{p'}(\mathbb{R}^d)}ds\right|_{L^r(0,T)}$$

The deterministic case:

$$\begin{cases} i\dfrac{du}{dt} + \Delta u + |u|^{2\sigma} u = 0,\ x \in \mathbb{R}^d,\ t > 0 \\ \\ u(0) = u_0. \end{cases}$$

$$U(t) = e^{it\Delta} \longrightarrow u(t) = U(t)u_0 + i \int_0^t U(t-s)|u(s)|^{2\sigma} u(s) ds.$$

- We use a fixed point argument in $L^r(0, T; L^p(\mathbb{R}^d))$, we need that $U(t)$ maps $L^{p/(2\sigma+1)}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ with an integrable norm.
- We already know:

$$|U(t)v|_{L^p(\mathbb{R}^d)} \leq \frac{1}{4\pi} |t|^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})} |v|_{L^{p'}(\mathbb{R}^d)}$$

- It follows

$$\left| \int_0^t U(t-s)v(s)ds \right|_{L^r(0,T;L^p(\mathbb{R}^d))} \leq c|v|_{L^{r'}(0,T:L^{p'}(\mathbb{R}^d))}$$

$$U(t) = e^{it\Delta} \longrightarrow u(t) = U(t)u_0 + i\int_0^t U(t-s)|u(s)|^{2\sigma}u(s)ds.$$

- If $\frac{1}{r} = \frac{d}{2}(\frac{1}{2} - \frac{1}{p})$:

$$\left|\int_0^{\cdot} U(\cdot - s)v(s)ds\right|_{L^r(0,T;L^p(\mathbb{R}^d))} \leq c|v|_{L^{r'}(0,T;L^{p'}(\mathbb{R}^d))}$$

$$|U(\cdot)u_0|_{L^r(0,T;L^p(\mathbb{R}^d))} \leq c|u_0|_{L^2(\mathbb{R}^d)}$$

(Strichartz estimates)

- For $p = 2\sigma + 2$, $p' = \dfrac{2\sigma+2}{2\sigma+1}$ and $\left||u(s)|^{2\sigma}u(s)\right|_{L^{p'}} = |u(s)|_{L^p}^{2\sigma+1}$

$\rightarrow$ Fixed point in $C([0,T];L^2(\mathbb{R}^d)) \cap L^r(0,T;L^p(\mathbb{R}^d))$ if $\sigma < 2/d$

(Kato, Tsutsumi, Ginibre, Velo,...)

The stochastic case:

$$v(t) = S(t,s)v_s \text{ solution of } \begin{cases} idv + \Delta v o d\beta = 0, \; x \in \mathbb{R}^d, \; t \geq s, \\ \\ v(s) = v_s. \end{cases}$$

- $\hat{v}(t,\xi) = e^{-i|\xi|^2(\beta(t)-\beta(s))}\hat{v}_s(\xi), \; t \geq s, \; \xi \in \mathbb{R}^d.$
- If $v_s \in L^2(\mathbb{R}^d)$, then $v(\cdot) \in C([s,T]; L^2(\mathbb{R}^d))$ a.s. and $|v(t)|_{L^2} = |v_s|_{L^2}$, a.s. for $t \geq s$.
- If $v_s \in L^1(\mathbb{R}^d)$,

$$v(t) = \frac{1}{(4i\pi\,(\beta(t) - \beta(s)))^{d/2}} \int_{\mathbb{R}^d} \exp\left(i \frac{|x-y|^2}{4(\beta(t) - \beta(s))} \right) v_s(y) dy.$$

- $|v(t)|_{L^\infty(\mathbb{R}^d)} \leq c \dfrac{1}{|\beta(t) - \beta(s)|^{d/2}} |v_s|_{L^1(\mathbb{R}^d)}.$

$$|S(t,s)v_s|_{L^2(\mathbb{R}^d)} = |v_s|_{L^2(\mathbb{R}^d)}$$
$$|S(t,s)v_s|_{L^\infty(\mathbb{R}^d)} \leq c\frac{1}{|\beta(t)-\beta(s)|^{d/2}}|v_s|_{L^1(\mathbb{R}^d)}.$$

$\rightarrow$ For $p \geq 2$ and $s \leq t$, $S(t,s)$ maps $L^{p'}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ and

$$|S(t,s)u_s|_{L^p(\mathbb{R}^d)} \leq \frac{C_p}{|\beta(t)-\beta(s)|^{\frac{d}{2}(\frac{1}{2}-\frac{1}{p})}}|u_s|_{L^{p'}},\ u_s \in L^{p'}.$$

We need an inequality of the type

$$\left|\int_0^{\cdot} S(\cdot,s)f(s)ds\right|_{L^r(0,T;L^p(\mathbb{R}^d))} \leq c|f|_{L^{r'}(0,T;L^{p'}(\mathbb{R}^d))}$$

but we cannot use convolution inequalities ...

Proposition Let $\alpha \in [0,1)$, $2 \leq r < \frac{2}{\alpha}$ et ρ satisfying $r' \leq \rho \leq r$, there exists $C_{\alpha,\rho,r}$ such that for all $T \geq 0$, $g \in L^\rho_{\mathcal{P}}(\Omega; L^{r'}(0,T))$:

$$\left| \int_0^t |\beta(t) - \beta(s)|^{-\alpha} |g(s)| ds \right|_{L^\rho_\omega(L^r(0,T))} \leq C_{\alpha,\rho,r} T^{\frac{2}{r} - \frac{\alpha}{2}} |g|_{L^\rho(\Omega; L^{r'}(0,T))}$$

Proposition Soient $2 \leq r \leq \infty$, $2 \leq p \leq \infty$ tels que $\frac{2}{r} > d\left(\frac{1}{2} - \frac{1}{p}\right)$ et ρ tel que $r' \leq \rho \leq r$, il existe $c_{\rho,r,p} > 0$ tel que pour tout $s \in \mathbb{R}$, $T \geq 0$ et $f \in L^\rho_{\mathcal{P}}(\Omega; L^{r'}(s, s+T; L^{p'}(\mathbb{R}^d)))$

$$\left| \int_s^\cdot S(\cdot, s) f(s) ds \right|_{L^\rho(\Omega; L^r(s,s+T; L^p))} \leq c_{\rho,r,p} T^\beta |f|_{L^\rho(\Omega; L^{r'}(s,s+T; L^{p'}))}$$

où $\beta = \frac{2}{r} - \frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$.

$$\begin{cases} idu + \Delta u o d\beta + |u|^{2\sigma} u dt = 0, \; x \in \mathbb{R}^d, \; t > 0 \\ \\ u(0) = u_0. \end{cases}$$

$$u(t) = S(t,0)u_0 + i \int_0^t S(t,s)|u(s)|^{2\sigma} u(s) ds.$$

Theorem (AD, A. de bouard) Let $\sigma < \dfrac{2}{d}$ and $u_0 \in L^2_x$ a.s., $\mathcal{F}_0$-measurable, there exists a unique solution in $L^r_{loc}(0,\infty; L^p(\mathbb{R}^d))$ a.s. with $p = 2\sigma + 2 \le r < \dfrac{4(\sigma+1)}{d\sigma}$; and $u \in C(\mathbb{R}^+; L^2(\mathbb{R}^d))$ a.s..
If $u_0 \in H^1(\mathbb{R}^d)$, then $u \in C(\mathbb{R}^+; H^1(\mathbb{R}^d))$,

If $d = 1$, we can improve the Strichartz inequality to $p = 10$ et $r = 5$
It follows from

$$\mathbb{E}\int_0^T \left\| D^{1/2} \left| \int_0^t S(t,s)f(s)ds \right|^2 \right\|_{L^2(\mathbb{R})}^2 dt \leq cT^{1/2}\mathbb{E}\Big(\|f\|_{L^1(0,T:L^2)}^4 \Big)$$

($D^{1/2}$ is the Fourier multiplier by $|\xi|^{1/2}$). This implies:

$$\left\| \int_s^{\cdot} S(.,\sigma)f(\sigma)d\sigma \right\|_{L^4(\Omega,L^5(s,s+T;L^{10}(\mathbb{R})))} \leq cT^{1/10}\|f\|_{L^4(\Omega,L^1(s,s+T;L^2))}$$

$\longrightarrow$ This gives global existence for $\sigma = 2$ (AD, Y. Tsutsumi).
Whereas in the deterministic case, there are solutions which form singularity in finite time.

Numerical simulations

(AD, A. de Bouard, R. Belaouar)

- Splitting + spectral
- Finite differences:

$\longrightarrow$ Crank-Nicolson :

$$\frac{i}{\delta t}(u_j^{n+1} - u_j^n) + \frac{\chi_n}{\sqrt{\delta t}}(u_{j+1}^{n+1/2} - 2u_j^{n+1/2} + u_{j-1}^{n+1/2})$$
$$= \frac{1}{2}(|u_j^n|^2 + |u_j^{n+1}|^2)u_j^{n+1/2}$$

χ_n : family of independent $\mathcal{N}(0,1)$.

$\longrightarrow$ Relaxation scheme:

$$\frac{i}{\delta t}(u_j^{n+1} - u_j^n) + \frac{\chi_n}{\sqrt{\delta t}}(u_{j+1}^{n+1/2} - 2u_j^{n+1/2} + u_{j-1}^{n+1/2}) = \phi_j^{n+1/2} u_j^{n+1/2}$$
$$\frac{1}{2}(\phi_j^{n-1/2} + \phi_j^{n+1/2}) = |u_j^n|^2$$

Conjecture

The problem is well posed for $\sigma < 4/d$.
This is the critical exponent one can guess from a scaling argument:
Itô form of the equation:

$$\begin{cases} idu + \Delta u d\beta + i\Delta^2 u dt + |u|^{2\sigma} u dt = 0,\ x \in \mathbb{R}^d,\ t > 0 \\ u(0) = u_0. \end{cases}$$

Two models with a noise depending on space and time:

- $idu + (\Delta u + |u|^{2\sigma} u)dt = d\tilde{W}$, (additive noise).
- $idu + (\Delta u + |u|^{2\sigma} u)dt = u \circ d\tilde{W}$, (multiplicative noise).
- The noise is due to amplifiers used for the transmission in very long optical fibers.

The space-time white noise:

- $W(t,x,\omega) = \sum_{i\in\mathbb{N}} \beta_i(t,\omega)e_i(x)$ with (β_i) independent brownian motions and (e_i) is a complete orthonormal system in $L^2(\mathbb{R}^d)$.
- Formally: $\dfrac{dW}{dt} = \sum_i \dfrac{d\beta_i}{dt} e_i = \sum_{i,k} \chi_{i,k} e_i f_k$.
- Formally : $\mathbb{E}\left(\dfrac{dW}{dt}(t,x)\dfrac{dW}{dt}(s,y)\right) = \delta_{t=s}\delta_{x=y}$.

Spatial correlation:

We take a kernel k and define:

$$\tilde{W}(x,t) = \int_{\mathbb{R}^d} k(x,y)W(y,t)dy = \sum_i \Phi e_i \beta_i$$

where $\Phi e_i(x) = \displaystyle\int_{\mathbb{R}^d} k(x,y)e_i(y)dy$.

- $\mathbb{E}\left(\tilde{W}(t,x)\tilde{W}(s,y)\right) = \min\{t,s\} \displaystyle\int_{\mathbb{R}^d} k(x,z)k(y,z)dz.$
- $c(x,y) = \int_{\mathbb{R}^d} k(x,z)k(y,z)dz$ is the spatial correlation.
- The spatial regularity of $\tilde{W}$ is linked to the regularity of k. For $k = \delta_{x=y}$, we recover the space time white noise..
- $\tilde{W}$ has trajectories in $L^2(\mathbb{R}^d)$ for $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$
- We can prove that the equations are well-posed under assumptions of this type. (AD, A. de Bouard, A. Millet, Z. Bzrezniak, ...)

$$idu + (\Delta u + |u|^{2\sigma} u)dt = u \circ d\tilde{W}$$

- The mass is preserved.
- Evolution of the energy:

$$H(u(t)) = H(u_0) - Im \int_{\mathbb{R}^d} \int_0^t \bar{u} \nabla u \cdot \nabla dW dx + \frac{1}{2} \sum_i \int_0^t \int_{\mathbb{R}^d} |u|^2 |\nabla \Phi e_i|^2 dxds$$

$$\mathbb{E}(H(u(t))) = \mathbb{E}(H(u_0)) + \frac{1}{2} \sum_i \int_0^t \int_{\mathbb{R}^d} \mathbb{E}(|u|^2) |\nabla \Phi e_i|^2 dxds.$$

$\rightarrow$ It grows linearly in time.

Blow-up, $\sigma \geq 2/d$

The deterministic case:

$$V(u) = \int_{\mathbb{R}^d} |x|^2 |u(x)|^2 dx, \quad G(u) = Im \int_{\mathbb{R}^d} u(x) x \nabla \bar{u}(x) dx.$$

$$\frac{dV(u)}{dt} = 4G(u), \quad \frac{dG(u)}{dt} = 4H(u) + \frac{2 - \sigma d}{\sigma + 1} \int_0^t \int_{\mathbb{R}^d} |u|^{2\sigma+2} dx ds.$$

- $V(u(t)) \leq V(u_0) + 4tG(u_0) + 8t^2 H(u_0)$
- impossible if $H(u_0) < 0$.
- If this condition holds, the solution develops a singularity in finite time.

Blow-up, $\sigma \geq 2/d$, multiplicative noise:

$$V(u(\tau))$$

$$= V(u_0) + 4G(u_0)\tau + 8H(u_0)\tau^2$$

$$+4\frac{2-\sigma d}{\sigma+1}\int_0^\tau \int_0^s |u(s_1)|_{L^{2\sigma+2}}^{2\sigma+2} ds_1 ds$$

$$+8\int_0^\tau \int_0^s \int_0^{s_1} \int_{\mathbb{R}^d} |u(s_2,x)|^2 f_\phi^1(x) dx ds_2 ds_1 ds$$

$$+4\sum_{k\in\mathbb{N}} \int_0^\tau \int_0^s \int_{\mathbb{R}^d} |u(s_1,x)|^2 x\cdot\nabla(\phi e_k)(x) dx\, d\beta_k(s_1) ds$$

$$-16\,\mathrm{Im}\sum_{k\in\mathbb{N}} \int_0^\tau \int_0^s \int_0^{s_1} \int_{\mathbb{R}^d} \bar{u}(s_2,x)\nabla u(s_2,x)\cdot\nabla(\phi e_k)(x) dx\, d\beta_k(s_2) ds_1 d$$

$$idu + (\Delta u + |u|^{2\sigma} u)dt = u \circ d\tilde{W}$$

Theorem: Let $\sigma \geq \frac{2}{d}$, $u_0 \in L^2(\Omega; \Sigma) \cap L^{2\sigma+2}(\Omega; L^{2\sigma+2}(\mathbb{R}^d))$, k sufficiently smooth. If there exists $\bar{t} > 0$ such that

$$\mathbb{E}(V(u_0)) + 4\mathbb{E}(G(u_0))\bar{t} + 8\mathbb{E}(H(u_0))\bar{t}^2 + \frac{4}{3}\bar{t}^3 m_\phi \mathbb{E}(M(u_0)) < 0$$

then

$$\mathbb{P}(\tau^*(u_0) \leq \bar{t}) > 0.$$

Using the noise as a control, we then prove that **in fact this holds for all initial data**. (AD, A. de Bouard)

Numerical simulations.

AD, M. Barton-Smith, L. Di Menza

It seems that a multiplicative space-time white noise does not destroy propagation.
On the contrary, it improves it since it prevents collapse of the wave.
The study of the equation

$$idu + (\Delta u + |u|^{2\sigma} u)dt = u \circ dW$$

with $\frac{dW}{dt}$ being a space-time white noise seems out of reach.

A singular equation with spatial noise

The nonlinear Schrödinger equation with a random potential:

$$i\partial_t u = \Delta u + \lambda |u|^{2\sigma} u + Vu,$$

- $u = u(x,t) \in \mathbb{C}$, $x \in O$, $t \geq 0$. O is either the whole space $\mathbb{R}^d$, $\mathbb{T}^d$, a manifold or a open subset.
- It describes the evolution of a wave in a disordered medium.
- When the medium is totally disordered, it is natural to consider $V = \xi$ a white noise in space.
- For $\lambda = 0$, it is a complex version to the Parabolic Anderson Model.

The case of a smooth potential:

$$i\partial_t u = \Delta u + \lambda |u|^{2\sigma} u + Vu.$$

- The mass and energy are again conserved: $M(u) = \int_O |u(x)|^2 dx$,
$$H(u) = \int_O |\nabla u(x)|^2 - \frac{\lambda}{\sigma+1} |u(x)|^{2\sigma+2} - V(x)|u(x)|^2 dx.$$
- It is locally well posed in $L^2(\mathbb{R}^d)$ for $\sigma < \frac{2}{d}$ and in $H^1(\mathbb{R}^d)$ for $\sigma < \frac{2}{d-2}$.
- For $\sigma < \frac{2}{d-2}$ and $\lambda \leq 0$, we have global well posedness in $H^1(\mathbb{R}^d)$. For $\lambda \geq 0$, we have global well posedness for $\sigma < \frac{2}{d}$ in $L^2(\mathbb{R}^d)$ and in $H^1(\mathbb{R}^d)$:
- The smoothing effect are not strong enough to smooth a white noise.
- It seems difficult to obtain Strichartz estimate for the linear part including the potential.

White noise potential: $i\partial_t u = \Delta u + \lambda|u|^{2\sigma} u + \xi u.$

We now consider a white noise in space ξ on the torus $\mathbb{T}^d$. The one dimensional case $d = 1$ is easy.

- We again have preservation of the mass and energy:

$$M(u) = \int_O |u(x)|^2 dx, \; H(u) = \int_O |\nabla u(x)|^2 - \frac{\lambda}{\sigma+1}|u(x)|^{2\sigma+2} - \xi(x)|u(x)|^2 dx.$$

- Recall that $\xi \in C^{-\alpha}(\mathbb{T})$ for any $\alpha > \frac{1}{2}$ and:

$$\int_{\mathbb{T}} \xi(x)|u(x)|^2 dx = \left(\xi, |u|^2\right)_{L^2(\mathbb{T})} \leq \|\xi\|_{H^{-\alpha}} \left\||u|^2\right\|_{H^\alpha}$$

In dimension one $H^\alpha(\mathbb{T})$ is an agebra for $\alpha > \frac{1}{2}$:

$$\int_{\mathbb{T}} \xi(x)|u(x)|^2 dx \leq c\|\xi\|_{H^{-\alpha}} \|u\|_{H^\alpha}^2 \leq c\|\xi\|_{H^{-\alpha}} \|u\|_{L^2}^{2(1-\alpha)} \|\nabla u\|_{L^2}^{2\alpha}$$

- This gives a uniform bound in $H^1(\mathbb{T})$ (for $\sigma < 2$ if $\lambda > 0$) and by compactness a global solution. Uniqueness is easy.

White noise potential: $i\partial_t u = \Delta u + \lambda |u|^{2\sigma} u + \xi u$.

The dimension two is a limit case. The noise $\xi \in C^{-\alpha}(\mathbb{T})$ for any $\alpha > 1$ and $H^{\alpha}(\mathbb{T})$ is an agebra for $\alpha > 1$

- We first consider the linear case $i\partial_t u = \Delta u + \xi u$ and recall the transformation used for the heat equation (Hairer, Labbé):

$$\Delta Y = \xi, \quad v = ue^{Y}$$

(Assume for simplicity that we work with zero spatial average). The equation is transformed into:

$$i\partial_t v = \Delta v - 2\nabla v \cdot \nabla Y + v|\nabla Y|^2.$$

- $Y \in C^{\gamma}$ for any $\gamma < 1$, $|\nabla Y|^2$ is not well defined. A renormalization is necessary.
- Let ξ_ε be a smooth noise, $\Delta Y_\varepsilon = \xi_\varepsilon$ then $C_\varepsilon = \mathbb{E}(|\nabla Y_\varepsilon(x)|^2) \sim -c \ln \varepsilon$ and $|\nabla Y_\varepsilon|^2 - C_\varepsilon$ converges in $C^{-\kappa}(\mathbb{T}^2)$ for any $\kappa > 0$. Denote the limit by $: |\nabla Y| :^2$.

Renormalized equation

- We thus study:

$$i\partial_t v_\varepsilon = \Delta v_\varepsilon - 2\nabla v_\varepsilon \cdot \nabla Y_\varepsilon + v_\varepsilon :|\nabla Y_\varepsilon|:^2$$

  and try to get bounds on v_ε. Note that this corresponds to:

$$i\partial_t u_\varepsilon = \Delta u_\varepsilon - u_\varepsilon(\xi_\varepsilon - C_\varepsilon)$$

  The phase of the unknown is renormalized: the new unknown is $e^{iC_\varepsilon t}$ times the original one.
- We have the invariant quantities:

$$M(u_\varepsilon) = \int_{\mathbb{T}^2} |u_\varepsilon(x)|^2 dx,\ H(u_\varepsilon) = \int_{\mathbb{T}^2} |\nabla u_\varepsilon(x)|^2 + (\xi_\varepsilon - C_\varepsilon)|u_\varepsilon(x)|^2 dx.$$

- On the transformed equation:

$$\widetilde{M}(v_\varepsilon) = \int_{\mathbb{T}^2} |v_\varepsilon(x)|^2 e^{-2Y_\varepsilon} dx,\ \widetilde{H}(v_\varepsilon) = \int_{\mathbb{T}^2} (|\nabla v_\varepsilon(x)|^2 + |v_\varepsilon|^2 :|\nabla Y_\varepsilon(x)|:^2) e^{-2Y_\varepsilon} dx$$

H^1 bound

$$\widetilde{M}(v_\varepsilon)=\int_{\mathbb{T}^2}|v_\varepsilon(x)|^2 e^{-2Y_\varepsilon}dx,\ \widetilde{H}(v_\varepsilon)=\int_{\mathbb{T}^2}(|\nabla v_\varepsilon(x)|^2+|v_\varepsilon|^2 :|\nabla Y_\varepsilon(x)|:^2)e^{-2Y_\varepsilon}dx$$

- Y_ε converges in $C^\alpha(\mathbb{T}^2)$ for $\alpha<1$, the preserved mass implies a bound in $L^2(\mathbb{T}^2)$:

$$\|v_\varepsilon(t)\|_{L^2}^2 \le e^{2\|Y_\varepsilon\|_{L^\infty}}\widetilde{M}(v_\varepsilon(t)) = e^{2\|Y_\varepsilon\|_{L^\infty}}\widetilde{M}(v_\varepsilon(0)) \le e^{4\|Y_\varepsilon\|_{L^\infty}}\|u(0)\|_{L^2}^2.$$

- For the energy, we are now in a similar situation as in the one dimensional case. Take $\kappa>0$:

$$\begin{aligned}\int_{\mathbb{T}^2}|v_\varepsilon|^2 :|\nabla Y_\varepsilon(x)|:^2 e^{-2Y_\varepsilon}dx &\le \left\||v_\varepsilon|^2\right\|_{B^\kappa_{1,1}}\left\||\nabla Y_\varepsilon|^2 e^{-2Y_\varepsilon}\right\|_{B^{-\kappa}_{\infty,\infty}} \\ &\le c\|v_\varepsilon\|^2_{H^{\frac12+\frac\kappa2}} \\ &\le c\|v_\varepsilon\|_{L^2}^{1-\kappa}\|\nabla v_\varepsilon\|_{L^2}^{1+\kappa}\end{aligned}$$

- We get a bound in H^1. This is not sufficient to control $\nabla v_\varepsilon\cdot\nabla Y_\varepsilon$.

H^2 bound

$$i\partial_t v_\varepsilon = \Delta v_\varepsilon - 2\nabla v_\varepsilon \cdot \nabla Y_\varepsilon + v_\varepsilon : |\nabla Y_\varepsilon| :^2$$

Set $w_\varepsilon = \partial_t v_\varepsilon$ then it satisfies the same equation

$$i\partial_t w_\varepsilon = \Delta w_\varepsilon - 2\nabla w_\varepsilon \cdot \nabla Y_\varepsilon + w_\varepsilon : |\nabla Y_\varepsilon| :^2$$

- The mass is again preserved: $\widetilde{M}(w_\varepsilon(t)) = \widetilde{M}(w_\varepsilon(0))$. But

$$w_\varepsilon(0) = \Delta v_0 - 2\nabla v_0 \cdot \nabla Y_\varepsilon + v_0 : |\nabla Y_\varepsilon| :^2 \notin L^2$$

- We have

$$\mathbb{E}\left(\|\nabla Y_\varepsilon\|_{L^p}\right) \leq c_p |\ln \varepsilon|, \ \mathbb{E}\left(\left\|: |\nabla Y_\varepsilon| :^2\right\|_{L^p}\right) \leq c_p |\ln \varepsilon|^2$$

- We deduce $\widetilde{M}(w_\varepsilon(0)) \leq c\left(\|v_0\|_{H^2} + |\ln \varepsilon|^4\right)$ provided $v_0 = u_0 e^Y \in H^2(\mathbb{T}^2)$.
- It follows $\|w_\varepsilon(t)\|_{L^2} \leq c\left(\|v_0\|_{H^2} + |\ln \varepsilon|^4\right)$ and writing

$$\Delta v_\varepsilon = -iw_\varepsilon + 2\nabla v_\varepsilon \cdot \nabla Y_\varepsilon - v_\varepsilon : |\nabla Y_\varepsilon| :^2$$

We deduce a bound in $H^2(\mathbb{T}^2)$:

$$\|v_\varepsilon(t)\|_{H^2} \leq c\left(\|v_0\|_{H^2} + |\ln \varepsilon|^4\right).$$

H^γ bound, $\gamma < 2$

- Take $\varepsilon_1 > \varepsilon_2 > 0$ and set $r = v_{\varepsilon_1} - v_{\varepsilon_2}$:

$$i\partial r = \Delta r - 2\nabla r \cdot \nabla Y_{\varepsilon_1} + r :|\nabla Y_{\varepsilon_1}|:^2 - 2\nabla v_{\varepsilon_2} \cdot \nabla (Y_{\varepsilon_1} - Y_{\varepsilon_2})$$
$$+ v_{\varepsilon_1}(:|\nabla Y_{\varepsilon_1}|:^2 - :|\nabla Y_{\varepsilon_2}|:^2)$$

- L^2 estimate:

$$\frac{1}{2}\frac{d}{dt}{}_t \int_{\mathbb{T}^2} |r(x,t)|^2 e^{-2Y_{\varepsilon_1}} dx = -2Im\left(\int_{\mathbb{T}^2} \nabla v_{\varepsilon_2} \cdot \nabla (Y_{\varepsilon_1} - Y_{\varepsilon_2}) \bar{r} e^{-2Y_{\varepsilon_1}} dx\right)$$
$$+Im\left(\int_{\mathbb{T}^2} v_{\varepsilon_1}(:|\nabla Y_{\varepsilon_1}|:^2 - :|\nabla Y_{\varepsilon_2}|:^2)\bar{r} e^{-2Y_{\varepsilon_1}} dx\right)$$
$$\leq c\|\nabla v_{\varepsilon_2} \bar{r} e^{-2Y_{\varepsilon_1}}\|_{B^{\kappa}_{1,1}} \|\nabla (Y_{\varepsilon_1} - Y_{\varepsilon_2})\|_{B^{-\kappa}_{\infty,\infty}}$$
$$+c\|v_{\varepsilon_2} \bar{r} e^{-2Y_{\varepsilon_1}}\|_{B^{\kappa}_{1,1}} \|:|\nabla Y_{\varepsilon_1}|^2 - :|\nabla Y_{\varepsilon_2}|:^2\|_{B^{-\kappa}_{\infty,\infty}}$$
$$\leq c|\ln \varepsilon_2|^8 \varepsilon_1^{\kappa/2}$$

H^γ bound, $\gamma < 2$

- Take $\varepsilon_1 > \varepsilon_2 > 0$ and set $r = v_{\varepsilon_1} - v_{\varepsilon_2}$:
- L^2 estimate: $$|r(t)|_{L^2} \le c|\ln \varepsilon_2|^4 \varepsilon_1^{\frac{\kappa}{4}}$$
- H^2 estimate: $$\begin{aligned}|r(t)|_{H^2} &\le |v_{\varepsilon_1}(t)|_{H^2} + |v_{\varepsilon_2}(t))|_{H^2}^2 \\ &\le c|\ln \varepsilon_2|^4.\end{aligned}$$
- Interpolate: $$|r(t)|_{H^\gamma} \le c|\ln \varepsilon_2|^4 \varepsilon_1^{\frac{\kappa}{4}(1-\frac{\gamma}{2})}$$
- Take a sequence $\varepsilon_k = 2^{-k}\varepsilon_0$, the corresponding solutions (v_{ε_k}) is Cauchy in $C([0,T]; H^\gamma(\mathbb{T}^2))$.
- It is then easy to prove that the limit v satisfies

$$i\partial_t v = \Delta v - 2\nabla v \cdot \nabla Y + v :|\nabla Y|:^2$$

Uniqueness is easy.

Conclusion

We deduce that for any u_0 such that $v_0 = u_0 e^Y \in H^2(\mathbb{T}^2)$ there exists a unique $v \in C([0, T]; H^\gamma(\mathbb{T}^2))$ satisfying:

$$i\partial_t v = \Delta v - 2\nabla v \cdot \nabla Y + v : |\nabla Y| :^2, \; v(0) = v_0.$$

The nonlinear case

$$i\partial_t u = \Delta u + \lambda |u|^{2\sigma} u + \xi u,\ x \in \mathbb{T}^2,\ t \geq 0.$$

We assume $\sigma \leq 1$.

- We use the same transform: $v = ue^Y$:

$$i\partial_t v = \Delta v - 2\nabla v \cdot \nabla Y + v : |\nabla Y| :^2 + \lambda e^{-2\sigma Y} |v|^{2\sigma} v.$$

- We smooth the noise and use the mass and energy

$$\widetilde{M}(v_\varepsilon) = \int_{\mathbb{T}^2} |v_\varepsilon(x)|^2 e^{-2Y_\varepsilon} dx,$$
$$\widetilde{H}(v_\varepsilon) = \int_{\mathbb{T}^2} \left(|\nabla v_\varepsilon(x)|^2 + |v_\varepsilon|^2 : |\nabla Y_\varepsilon(x)| :^2 \right) e^{-2Y_\varepsilon} dx - \frac{\lambda}{\sigma+1} \int_{\mathbb{T}^2} |v_\varepsilon|^{2\sigma+2} e^{-2(\sigma+1)Y_\varepsilon} dx.$$

This gives bounds in L^2 and in H^1 for $\lambda \leq 0$ or $\sigma < 1$.

The nonlinear case: H^γ bound

We introduce $w_\varepsilon = \partial_t v_\varepsilon$:

$$
\begin{aligned}
i\partial_t w_\varepsilon = \ & \Delta w_\varepsilon - 2\nabla w_\varepsilon \cdot \nabla Y_\varepsilon + w_\varepsilon : |\nabla Y_\varepsilon| :^2 + \lambda e^{-2\sigma Y_\varepsilon} |v_\varepsilon|^{2\sigma} w_\varepsilon \\
& + 2\lambda(\sigma - 1) e^{2\sigma Y_\varepsilon} Re(v_\varepsilon \bar{w}_\varepsilon) |v_\varepsilon|^{2\sigma-2} v_\varepsilon .
\end{aligned}
$$

- L^2 estimate:

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} |w_\varepsilon(x,t)|^2 e^{-2Y_\varepsilon} dx \\
& = 2\lambda(\sigma-1)\int_{\mathbb{T}^2} Re(v_\varepsilon \bar{w}_\varepsilon)|v_\varepsilon|^{2\sigma-2} Im(v_\varepsilon \bar{w}_\varepsilon) e^{-2(\sigma+1)Y_\varepsilon} dx \\
& \leq c\|v_\varepsilon\|_{L^\infty}^{2\sigma} \int_{\mathbb{T}^2} |w_\varepsilon(x,t)|^2 e^{-2Y_\varepsilon} dx
\end{aligned}
$$

- Brezis-Gallouet:

$$
\begin{aligned}
\|v_\varepsilon\|_{L^\infty} & \leq c\|v_\varepsilon\|_{H^1}(1 + \sqrt{\ln(1 + \|v_\varepsilon\|_{H^2})} \\
& \leq c(1 + \sqrt{\ln(1 + \|w_\varepsilon\|_{L^2} + |\ln \varepsilon|^4)})
\end{aligned}
$$

The nonlinear case: H^γ bound

$$i\partial_t w_\varepsilon = \Delta w_\varepsilon - 2\nabla w_\varepsilon \cdot \nabla Y_\varepsilon + w_\varepsilon :|\nabla Y_\varepsilon|:^2 + \lambda e^{-2\sigma Y_\varepsilon}|v_\varepsilon|^{2\sigma} w_\varepsilon$$

$$+2\lambda(\sigma-1)e^{2\sigma Y_\varepsilon} Re(v_\varepsilon \bar{w}_\varepsilon)|v_\varepsilon|^{2\sigma-2} v_\varepsilon.$$

- L^2 estimate:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} |w_\varepsilon(x,t)|^2 e^{-2Y_\varepsilon} dx \le c\|v_\varepsilon\|_{L^\infty}^{2\sigma} \int_{\mathbb{T}^2} |w_\varepsilon(x,t)|^2 e^{-2Y_\varepsilon} dx$$

- Brezis-Gallouet:

$$\|v_\varepsilon\|_{L^\infty} \le c(1+\sqrt{\ln(1+\|w_\varepsilon\|_{L^2} + |\ln\varepsilon|^4)})$$

- We obtain

$$\frac{d}{dt}\|w_\varepsilon(t)|_{L^2}^2 \le c(1+\ln(1+\|w_\varepsilon(t)\|_{L^2} + |\ln\varepsilon|^4))\|w_\varepsilon(t)\|_{L^2}$$

and

$$\|w_\varepsilon(t)\|_{L^2} \le e^{e^{ct}}\left(\|w_\varepsilon(0)\|_{L^2} + |\ln\varepsilon|^4\right)$$

The nonlinear case: conclusion

- We get a similar, ε dependent, H^2bound and use the same argument to get a H^γ bound.
- We deduce existence and uniqueness for $\lambda \leq 0$ and $\sigma \leq 2$ or $\lambda \geq 0$ and $\sigma < 2$ of solution in $C([0,T]; H^\gamma(\mathbb{T}^2))$ if $v_0 \in H^2(\mathbb{T}^2)$ for

$$i\partial_t v = \Delta v - 2\nabla v \cdot \nabla Y + v :|\nabla Y|:^2 + \lambda e^{-2\sigma Y}|v|^{2\sigma} v.$$

(AD, H. Weber)

- Extension to dimension 3 (local in time) by (Gubinelli, Ugurcan, Zachhuber)
- It is also possible to study the equation on $\mathbb{R}^2$ (AD, J. Martin).
- It would be interesting to investigate scattering properties, soliton behavior, blow-up ...

Thanks for your attention.