Tau functions as Widom constants

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Motivation:

- isomonodromic deformations of linear ODEs with rational coefficients
- simplest cases described by Painlevé equations
- for example, Painlevé VI corresponds to rank 2 Fuchsian system with 4 regular singularities at $0, t, 1, \infty$:

$$\partial_z \Phi = \Phi A(z), \qquad A(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}$$

- monodromy representation

$$\rho : \pi_1\left(\mathbb{CP}^1 \backslash \{4 \text{ pts}\}\right) \to \mathrm{SL}(2, \mathbb{C})$$

- Riemann-Hilbert correspondence: système linéaire ↦ monodromie
- $A_{0,t,1}$ and $A_\infty := -A_0 - A_t - A_1$ are 2×2 matrices with eigenvalues $\pm\theta_{0,t,1,\infty}$, and isomonodromy equations are

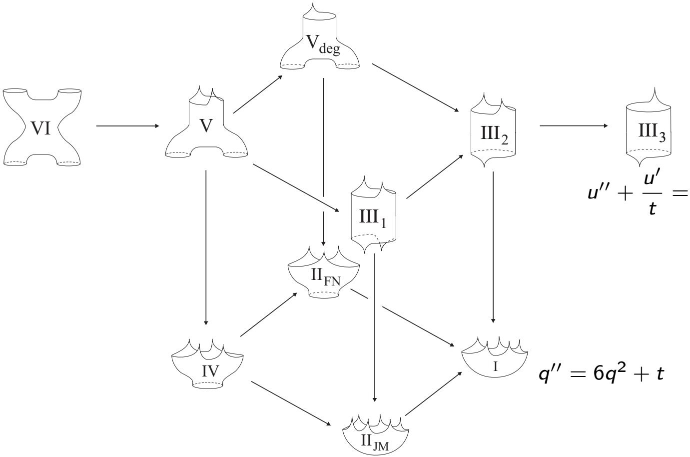
$$\frac{dA_0}{dt} = \frac{[A_0, A_t]}{t}, \qquad \frac{dA_1}{dt} = \frac{[A_1, A_t]}{t-1}, \qquad A_\infty = \text{const}\,.$$

Painlevé VI:

$$\left(t(t-1)\zeta''\right)^2 = -2\det\begin{pmatrix} 2\theta_0^2 & t\zeta'-\zeta & \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\zeta'-\zeta & 2\theta_t^2 & (t-1)\zeta'-\zeta \\ \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\zeta'-\zeta & 2\theta_1^2 \end{pmatrix}$$

- $\zeta(t) = (t-1)\,\mathrm{Tr}\,A_0A_t + t\,\mathrm{Tr}\,A_1A_t = t(t-1)\frac{d}{dt}\ln\tau$
- $\tau(t)$ is the Painlevé VI tau function
- résolution = inversion de l'application de Riemann-Hilbert

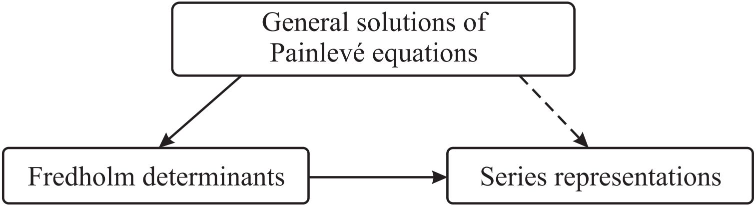
Geometric confluence diagram [Chekhov, Mazzocco, Rubtsov, '15]:



$$u'' + \frac{u'}{t} = \sin u$$

$$q'' = 6q^2 + t$$

	PVI	PV	$PIII_1$	$PIII_2$	$PIII_3$	PIV	PII	PI
♯(parameters)	4	3	2	1	0	2	1	0



General solutions of Painlevé equations

Fredholm determinants

Series representations

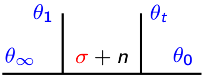
- block integrable kernels
- Widom's constants

- summation over partitions/Young diagrams

General solution of PVI [Gamayun, Iorgov, OL, '12]:

PVI tau function is a Fourier transform of $c = 1$ Virasoro conformal block:

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta}\,\mathcal{B}(\vec{\theta}, \sigma + n, t) = \sum_{n\in\mathbb{Z}} e^{in\eta}$$



$$(t)$$

- $\mathcal{B}(\vec{\theta}, \sigma, t) = t^{\sigma^2-\theta_0^2-\theta_t^2} \sum_{k=0}^{\infty} B_k(\vec{\theta}, \sigma)\, t^k$
- B_k determined by commutation relations of Vir
- as $c \to \infty$ (Vir $\to \mathfrak{sl}_2$), conformal block $\mathcal{B}(t) \sim {}_2F_1(t)$
- all 4 parameters $(\theta_0, \theta_t, \theta_1, \theta_\infty)$
- 2 integrals of motion (σ, η)

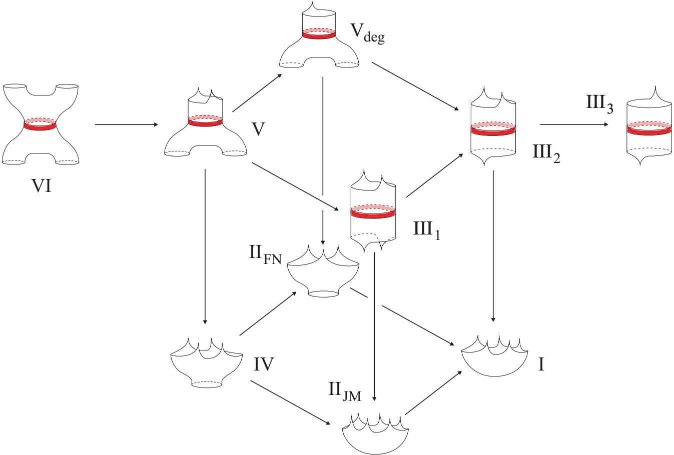
- AGT correspondence [Alday, Gaiotto, Tachikawa, '09]:

$$\mathcal{B}(t) = \mathcal{Z}_{\mathrm{inst}}(t) = \begin{array}{c}\text{sum over pairs}\\ \text{of Young diagrams}\end{array} \qquad \text{[Nekrasov, '04]}$$

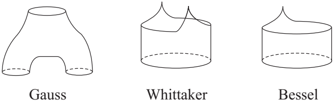
- series representation for PVI tau function

$$\tau\,(t) = \sum_{n\in\mathbb{Z}} e^{in\eta} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}(\vec{\theta},\sigma+n)\, t^{(\sigma+n)^{2}+|\lambda|+|\mu|}$$

- proof in [Gavrylenko, OL, '16]
- similar expansions in other cross-ratios: $1-t$, $\frac{1}{t}$, $\frac{1}{1-t}$, $\frac{t}{t-1}$, $\frac{t-1}{t}$
- expansions for PV and $\mathrm{PIII}_{1,2,3}$ at $t=0$ have very similar structure



- PVI, PV, $\mathrm{PIII}_{1,2,3}$ surfaces may be cut into solvable pieces



- More surprisingly, Fourier transform also appears in “irregular type” expansions for PI–PV at $t = \infty$.

Goals

“Painlevé project”:

- develop a general approach that would allow to derive systematically (asymptotic) series for PI-PV functions
- explicit expressions for coefficients of the series + connection formulas (in terms of monodromy of the associated linear problem)

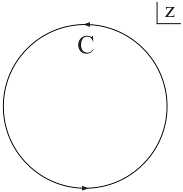
Beyond isomonodromic setting:

- give a “good” (non-log-derivative) definition of the tau function; in particular, define a tau function for arbitrary Riemann-Hilbert problem

Riemann-Hilbert setup

- let $\mathcal{C} \subset \mathbb{C}$ be a circle centered at the origin
- pick a loop $J(z) \in \mathrm{Hom}(\mathcal{C}, \mathrm{GL}_N(\mathbb{C}))$
- $J(z)$ continues into an annulus $\mathcal{A} \supset \mathcal{C}$

$$J(z) = \sum_{k\in\mathbb{Z}} J_k z^k,$$



Two Riemann-Hilbert problems:

$$\begin{aligned} \textcolor{blue}{\text{direct}} : \quad & J(z) = \Psi_-(z)^{-1}\Psi_+(z) \\ \textcolor{blue}{\text{dual}} : \quad & J(z) = \bar{\Psi}_+(z)\,\bar{\Psi}_-(z)^{-1} \end{aligned}$$

Main definition: The tau function of RHPs defined by $(\mathcal{C}, J)$ is defined as Fredholm determinant

$$\tau\left[J\right] = \det_{H_+}\left(\Pi_+ J^{-1}\Pi_+ J\,\Pi_+\right),$$

where $H = L^2\left(\mathcal{C}, \mathbb{C}^N\right)$ and Π_+ is the orthogonal projection on H_+ along H_-.

Properties:

- dual RHP is solvable iff the operator $P := \Pi_+ J^{-1}\Pi_+$ is invertible on H_+, in which case $P^{-1} = \bar{\Psi}_+\Pi_+\bar{\Psi}_-^{-1}\Pi_+$
- likewise, for direct RHP and $Q := \Pi_+ J\,\Pi_+$, with $Q^{-1} = \Psi_+^{-1}\Pi_+\Psi_-\Pi_+$
- if either direct or dual RHP is not solvable, then $\tau\left[J\right] = 0$

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Example: scalar case $(N = 1)$

- direct and dual factorization coincide
- $J\left(z\right) = \left(1 - t_1 z\right)^{\nu_1}\left(1 - t_2/z\right)^{\nu_2}$ with $|z| = 1$ and $|t_1|, |t_2| < 1$, then

$$\tau[J] = \left(1 - t_1 t_2\right)^{\nu_1 \nu_2}$$

Remark. $\tau[J]$ appears [Widom, '76] in the asymptotics of determinant of block Toeplitz matrix with symbol $J = \sum_{k\in\mathbb{Z}} J_k z^k$,

$$T_K[J] = \begin{pmatrix} J_0 & J_{-1} & \dots & J_{-K+1} \\ J_1 & J_0 & \dots & J_{-K+2} \\ \vdots & \vdots & \ddots & \vdots \\ J_{K-1} & J_{K-2} & \dots & J_0 \end{pmatrix}.$$

In this context, $\tau[J]$ is called Widom's constant.

- strong Szegő for $N = 1$: $\tau[J] = \exp \sum_{k=1}^{\infty} k\,(\ln J)_k\,(\ln J)_{-k}$
- no nice general formula for $N \geq 2$

Lemma: If the direct RHP is solvable, then $\tau[J]$ may also be written as

$$\tau[J] = \det_H (\mathbf{1} + K)\,, \qquad K = \begin{pmatrix} 0 & \mathsf{a}_{+-} \\ \mathsf{a}_{-+} & 0 \end{pmatrix} \in \mathrm{End}\,(H_+ \oplus H_-)\,,$$

where $\mathsf{a}_{\pm\mp} : H_\mp \to H_\pm$ are integral operators

$$(\mathsf{a}_{\pm\mp} f)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{a}_{\pm\mp}(z, z')\, f(z')\, dz',$$

with block integrable kernels

$$\mathsf{a}_{\pm\mp}(z, z') = \pm \frac{\mathbf{1} - \Psi_\pm(z)\, \Psi_\pm(z')^{-1}}{z - z'}.$$

Proof. We have $\mathsf{a}_{\pm\mp} = \Psi_\pm \Pi_\pm \Psi_\pm^{-1} - \Pi_\pm$. □

In our applications:

- $\Psi_\pm$ (direct factorization) are given and define the jump $J = {\Psi_-}^{-1}\Psi_+$
- $\Psi_\pm$ are expressed via classical special functions
  (Gauss, Kummer & Bessel for PVI, PV, PIII's)
- dual factorization ($\bar{\Psi}_\pm$ in $J = \bar{\Psi}_+ \bar{\Psi}_-^{-1}$) is a problem to be solved

Differentiation formula

Theorem: Let $(z,t) \mapsto J(z,t)$ be a smooth family of $\mathrm{GL}(N,\mathbb{C})$-loops which depend on an extra parameter t and admit both direct and dual factorization. Then

$$\partial_t \ln \tau[J] = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathrm{Tr}\left\{ J^{-1}\partial_t J \left[\partial_z \bar{\Psi}_- \, \bar{\Psi}_-^{-1} + \Psi_+^{-1}\, \partial_z \Psi_+ \right] \right\} dz.$$

Proof.

$$\partial_t \ln \det_{H_+} PQ = \mathrm{Tr}_{H_+} \left(\partial_t P\, P^{-1} + Q^{-1} \partial_t Q \right) =$$

$$= \mathrm{Tr}_H \Big(\Pi_+ J^{-1}\partial_t J \, \left(\bar{\Psi}_- \Pi_- \bar{\Psi}_-^{-1} - \Pi_- \right) + \left(\Psi_+^{-1} \Pi_+ \Psi_+ - \Pi_+ \right) J^{-1} \partial_t J \Big)$$

Given $\tilde{d}(z,z') = \dfrac{\Psi_+(z)^{-1}\,\Psi_+(z') - \mathbf{1}}{z-z'}$, we have $\tilde{d}(z,z) = \Psi_+^{-1}\,\partial_z \Psi_+$. □

- due to [Widom, '74]; rediscovered by [Its, Jin, Korepin, '06]

Corollary: in isomonodromic RHPs,

Widom's constant $\tau[J]$ $\simeq$ Jimbo-Miwa-Ueno tau function

Isomonodromic example

Fuchsian system with 4 regular singularities at $0, t, 1, \infty$:

$$\partial_z \Phi = \Phi A(z), \qquad A(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}$$

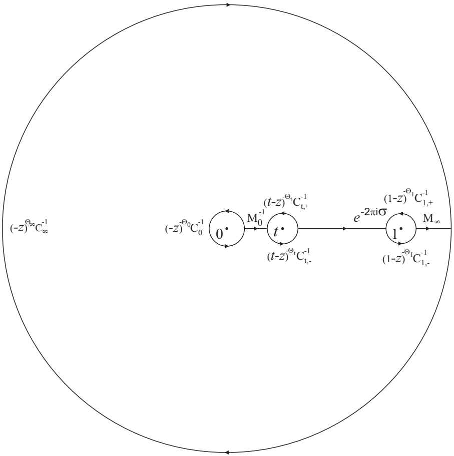
- arbitrary rank: $A_{0,t,1} \in \mathrm{Mat}_{N\times N}(\mathbb{C})$
- generic case: $A_{0,t,1}$ and $A_\infty := -A_0 - A_t - A_1$ are diagonalizable
- fix the diagonalizations $A_\nu = G_\nu^{-1}\Theta_\nu G_\nu$ with diagonal Θ_ν
- eigenvalues of A_ν are assumed distinct mod $\mathbb{Z}$

There exist unique fundamental solutions $\Phi^{(\nu)}(z)$, holomorphic on the universal covering of $\mathbb{C}\setminus\{0, t, 1\}$ and such that

$$\Phi^{(\nu)}(z) = \begin{cases} (\nu - z)^{\Theta_\nu}\, G^{(\nu)}(z), & \text{for } \nu = 0, t, 1, \\ (-z)^{-\Theta_\infty}\, G^{(\infty)}(z), & \text{for } \nu = \infty, \end{cases}$$

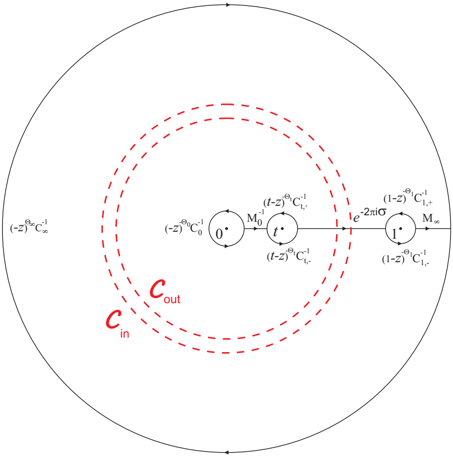
where $G^{(\nu)}(z)$ is holomorphic and invertible in a finite open disk around $z = \nu$ and satisfies $G^{(\nu)}(\nu) = G_\nu$.

Dual RHP$_1$ for $\tilde{\Psi}$



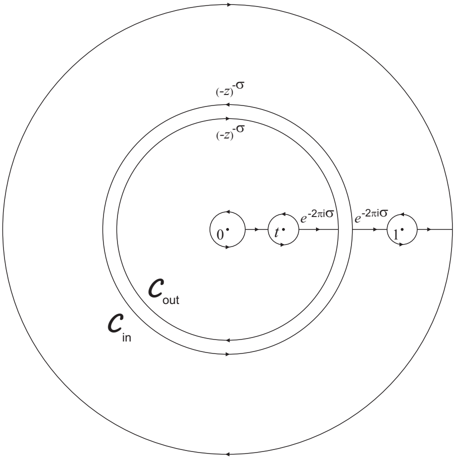
$$\tilde{\Psi}(z) = \begin{cases} G^{(\nu)}(z), & z \in D_\nu, \\ \Phi(z), & z \notin \mathbb{R}_{\geq 0} \cup \bar{D}_0 \cup \bar{D}_t \cup \bar{D}_1 \cup \bar{D}_\infty. \end{cases}$$

Dual RHP$_1$ for $\tilde{\Psi}$

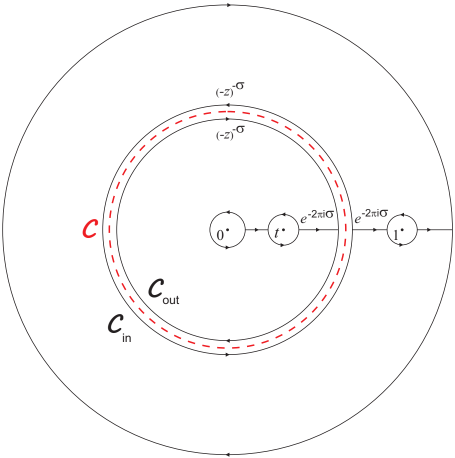


$$\hat{\Psi}(z) = \begin{cases} (-z)^{-\mathfrak{S}}\,\tilde{\Psi}(z), & z \in \mathcal{A}, \\ \tilde{\Psi}(z), & z \notin \bar{\mathcal{A}}. \end{cases}$$

Dual RHP$_2$ for $\hat{\Psi}$

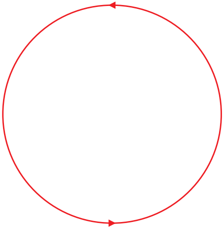


Dual RHP$_2$ for $\hat{\Psi}$



$$\bar{\Psi}(z) = \begin{cases} \Psi_+(z)^{-1}\hat{\Psi}(z), & \text{outside } \mathcal{C}, \\ \Psi_-(z)^{-1}\hat{\Psi}(z), & \text{inside } \mathcal{C}. \end{cases}$$

Dual RHP$_3$ for $\bar{\Psi}$



$$\bar{\Psi}(z) = \begin{cases} \Psi_+(z)^{-1}\hat{\Psi}(z), & \text{outside } \mathcal{C}, \\ \Psi_-(z)^{-1}\hat{\Psi}(z), & \text{inside } \mathcal{C}. \end{cases}$$

- contour $\mathcal{C}$ (single circle !), smooth jump $J : \mathcal{C} \to \mathrm{GL}(N, \mathbb{C})$ given by

$$J(z) = \Psi_-(z)^{-1}\Psi_+(z) = \bar{\Psi}_+(z)\,\bar{\Psi}_-(z)^{-1}$$

- we are in the previously described setup!

Widom's differentiation formula

$$\partial_t \ln \tau\left[J\right] = \frac{1}{2\pi i}\oint_{\mathcal{C}} \operatorname{Tr}\left\{J^{-1}\partial_t J\left[\partial_z\bar{\Psi}_-\,\bar{\Psi}_-^{-1} + \Psi_+^{-1}\,\partial_z\Psi_+\right]\right\} dz.$$

implies that

$$\partial_t \ln \tau\left[J\right] = \underbrace{\frac{\operatorname{Tr} A_0 A_t}{t} + \frac{\operatorname{Tr} A_t A_1}{t-1}}_{\partial_t \ln \tau_{\mathrm{JMU}}(t)} - \frac{\operatorname{Tr} A_0^+ A_t^+}{t},$$

so that in turn

$$\tau_{\mathrm{JMU}}\left(t\right) = t^{\frac12 \operatorname{Tr}\left(\mathfrak{S}^2-\Theta_0^2-\Theta_t^2\right)}\tau\left[J\right].$$

- Recall that

$$\tau\left[J\right] = \det\left(\mathbf{1}+K\right), \qquad K = \begin{pmatrix} 0 & \mathsf{a}_{+-} \\ \mathsf{a}_{-+} & 0 \end{pmatrix},$$

$$\mathsf{a}_{\pm\mp}\left(z,z'\right) = \pm\frac{\mathbf{1}-\Psi_\pm\left(z\right)\Psi_\pm\left(z'\right)^{-1}}{z-z'}.$$

- $\tau_{\mathrm{JMU}}\left(t\right)$ for 4-point system written via auxiliary 3-point solutions
- hypergeometric representations for $N=2 \Longrightarrow$ PVI tau function !

For $N = 2$:

$$\mathsf{a}_{+-}(z,z') = \frac{(1-z')^{2\theta_1}\begin{pmatrix} K_{++}(z) & K_{+-}(z) \\ K_{-+}(z) & K_{--}(z) \end{pmatrix}\begin{pmatrix} K_{--}(z') & -K_{+-}(z') \\ -K_{-+}(z') & K_{++}(z') \end{pmatrix} - \mathbf{1}}{z-z'},$$

$$\mathsf{a}_{-+}(z,z') = \frac{\mathbf{1} - \left(1-\frac{t}{z'}\right)^{2\theta_t}\begin{pmatrix} \bar{K}_{++}(z) & \bar{K}_{+-}(z) \\ \bar{K}_{-+}(z) & \bar{K}_{--}(z) \end{pmatrix}\begin{pmatrix} \bar{K}_{--}(z') & -\bar{K}_{+-}(z') \\ -\bar{K}_{-+}(z') & \bar{K}_{++}(z') \end{pmatrix}}{z-z'}$$

with

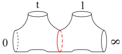
$$K_{\pm\pm}(z) = {}_2F_1\left[\begin{matrix} \theta_1+\theta_\infty\pm\sigma, \theta_1-\theta_\infty\pm\sigma \\ \pm 2\sigma \end{matrix} ; z\right],$$

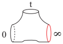
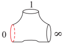
$$K_{\pm\mp}(z) = \pm\frac{\theta_\infty^2-(\theta_1\pm\sigma)^2}{2\sigma(1\pm 2\sigma)}\, z\, {}_2F_1\left[\begin{matrix} 1+\theta_1+\theta_\infty\pm\sigma, 1+\theta_1-\theta_\infty\pm\sigma \\ 2\pm 2\sigma \end{matrix} ; z\right],$$

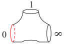
$$\bar{K}_{\pm\pm}(z) = {}_2F_1\left[\begin{matrix} \theta_t+\theta_0\mp\sigma, \theta_t-\theta_0\mp\sigma \\ \mp 2\sigma \end{matrix} ; \frac{t}{z}\right],$$

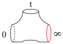
$$\bar{K}_{\pm\mp}(z) = \mp\, t^{\mp 2\sigma} e^{\mp i\eta}\frac{\theta_0^2-(\theta_t\mp\sigma)^2}{2\sigma(1\mp 2\sigma)}\,\frac{t}{z}\, {}_2F_1\left[\begin{matrix} 1+\theta_t+\theta_0\mp\sigma, 1+\theta_t-\theta_0\mp\sigma \\ 2\mp 2\sigma \end{matrix} ; \frac{t}{z}\right].$$

Schematically,

$\tau_{\mathrm{JMU}}\Bigg($  $\Bigg) =$

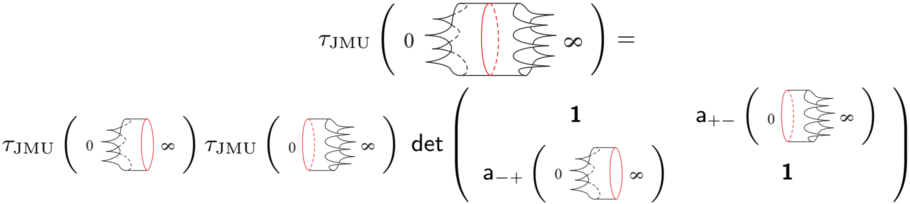
$\tau_{\mathrm{JMU}}\Big($  $\Big)\;\tau_{\mathrm{JMU}}\Big($  $\Big)\;\det\Bigg($

$\mathbf{1}\qquad \mathsf{a}_{+-}\Big($  $\Big)$

$\mathsf{a}_{-+}\Big($  $\Big)\qquad \mathbf{1}$

$\Bigg)$

Similarly, for a linear system with 2 irregular singularities



Series representations

Given $K \in \mathbb{C}^{\mathfrak{X}\times\mathfrak{X}}$, we can expand Fredholm determinant

$$\det\left(\mathbf{1}+K\right) = \sum_{\mathfrak{Y}\in 2^{\mathfrak{X}}} \det K_{\mathfrak{Y}} = 1 + \sum_{m\in\mathfrak{X}} K_{mm} + \frac{1}{2!}\sum_{m,n\in\mathfrak{X}} \begin{vmatrix} K_{mm} & K_{mn} \\ K_{nm} & K_{nn} \end{vmatrix} + \ldots$$

- in the Fourier basis,

$$\mathsf{a}_{\pm\mp}\left(z,z'\right) = \sum_{p,q\in\mathbb{Z}'_+} \mathsf{a}^{\pm p}_{\mp q} z^{-\frac{1}{2}\pm p} z'^{-\frac{1}{2}\pm q},$$

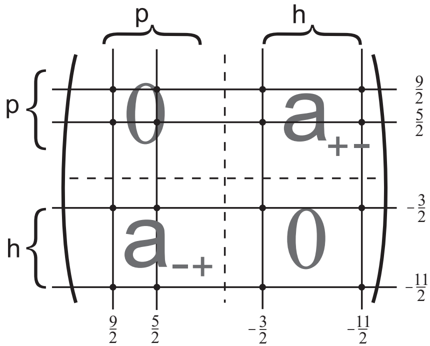
  with $\mathsf{a}^{\pm p}_{\mp q} \in \mathrm{Mat}_{N\times N}\left(\mathbb{C}\right)$.

- multi-indices $m, n, \ldots$ of principal minors

$$\det K_{\mathfrak{Y}} = \det \begin{pmatrix} 0 & \mathsf{a}^{\mathbf{p}}_{\mathbf{h}} \\ \mathsf{a}^{\mathbf{h}}_{\mathbf{p}} & 0 \end{pmatrix}$$

  incorporate color indices $\alpha = 1, \ldots N$ and (half-)integer Fourier indices

$N = 1$ **case**:

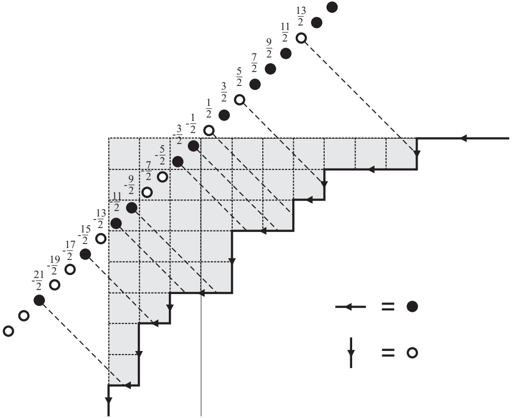


- combinatorial expansion

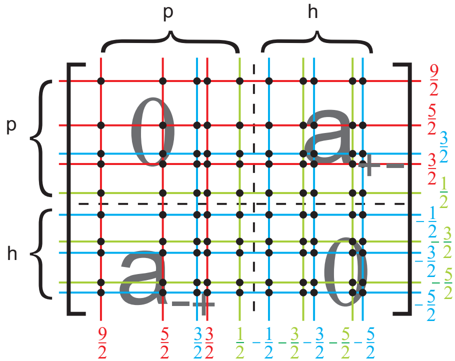
$$\det\left(\mathbf{1}+K\right)=\sum_{(\mathsf{p},\mathsf{h})}(-1)^{|\mathsf{p}|}\det \mathsf{a}^{\mathsf{p}}_{\mathsf{h}}\det \mathsf{a}^{\mathsf{h}}_{\mathsf{p}},$$

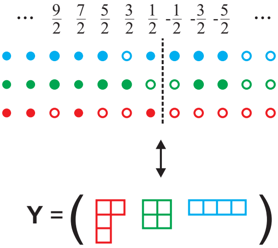
  with balance condition $|\mathsf{p}| = |\mathsf{h}|$

- Fourier indices in p and h are resp. positive and negative



- A Maya diagram is a map $\mathrm{m} : \mathbb{Z}' \to \{-1, 1\}$ subject to the condition $\mathrm{m}\,(p) = \pm 1$ for all but finitely many $p \in \mathbb{Z}'_{\pm}$ (positions of particles and holes)
- Maya diagram = charged partition/Young diagram
- charge(m) = ♯(particles) − ♯(holes) (zero for balanced diagrams)





Q = (3,-1,-2)

- balanced configurations (p, h) are in bijection with N-tuples of Young diagrams of zero total charge
- $\mathbb{M}_0^N \cong \mathbb{Y}^N \times \mathfrak{Q}_N$, where $\mathfrak{Q}_N$ denotes the A_{N-1} root lattice:

$$\mathfrak{Q}_N := \left\{ \vec{Q} \in \mathbb{Z}^N \,\middle|\, \sum_{\alpha=1}^{N} Q_\alpha = 0 \right\}.$$

- in the case $N = 2$, Widom's constant

$$\det(\mathbf{1} + K) = \sum_{(\lambda,\mu;n)\in\mathbb{Y}^2\times\mathbb{Z}} (-1)^{|\mathsf{p}|} \det(\mathsf{a}_{+-})_{\lambda,\mu,n} \det(\mathsf{a}_{-+})_{\lambda,\mu,n}$$

Isomonodromic examples

Explicit computation of elementary determinants $\det \mathsf{a}_{\mathsf{h}}^{\mathsf{p}}$, $\det \mathsf{a}_{\mathsf{p}}^{\mathsf{h}}$:

- a variant of Tracy-Widom conditions

  $$\partial_z \Psi_\pm(z) = \Psi_\pm(z) A_\pm(z) + z^{-1} \Lambda_\pm(z) \Psi_\pm(z),$$

  with $A_\pm(z)$ rational in z and $\Lambda_\pm(z)$ polynomial in $z^{\pm 1}$.
- acting with $\mathcal{L}_0 = z\partial_z + z'\partial_{z'} + 1$ e.g. on

  $$\frac{\mathbf{1} - \Psi_+(z) \Psi_+(z')^{-1}}{z - z'} = \sum_{p,q \in \mathbb{Z}'_+} \mathsf{a}_{-q}^{\;p} z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}+q}$$

  yields a system of linear matrix equations on Fourier modes $\mathsf{a}_{-q}^{\;p}$ thanks to the fact that $\mathcal{L}_0 \frac{1}{z-z'} = 0$.
- PVI, V, III semisimple cases $(N = 2) \Longrightarrow$ Cauchy determinants

  $$\det \frac{f_{p,\alpha} g_{q,\beta}}{p + q + \sigma_\alpha + \sigma_\beta}$$

Conclusions

1. A tau function (= Widom's constant) can be assigned to "any" RHP.
2. Given the direct factorization of the jump matrix, $\tau\,[J]$ may be written as a Fredholm determinant with a block integrable kernel.
3. Principal minor expansion of this determinant in the Fourier basis leads to combinatorial series over tuples of partitions.
4. In RHPs of isomonodromic origin, $\tau\,[J] \simeq \tau_{\mathrm{JMU}}$
5. Integral kernels and coefficients of combinatorial series can be computed explicitly when auxiliary solutions from the direct factorization have hypergeometric representations; in particular, for PVI, PV and PIIIs.
6. Results can be generalized to many-oval contour (e.g. Garnier system)

References

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